What Changes When Gravity Changes (Tides, part 4)

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Let us consider what happens to a pendulum clock when gravity changes by a small amount due to tides. Given $T \approx 2\pi\sqrt{L/g}$ we know what happens to period. What effect, if any, is there on amplitude and velocity?

This question arose at the 2004 Ward Francillon Time Symposium in Portland where Bryan Mumford and I heard Timothy Treffry read Philip Woodward's assessment of the Littlemore clock. This initiated many discussions about pendulum clocks and gravity, none of which reached a satisfying conclusion. Gravity, or changing gravity, appears to be a very tricky subject.

Thought Experiments

To recall some of our confusion at the time, consider this scenario. Suppose gravity decreases just as the bob is at *center swing* – there would be no change in velocity and no change in kinetic energy but at the end of the swing the height or amplitude would be greater in the now reduced gravitational field. That implies energy and velocity are the same but amplitude increases.

On the other hand if gravity decreases while the bob is at *end of swing* – amplitude would remain the same but the now reduced potential energy would translate to less velocity when the bob reached center. So which is it; less velocity, or greater amplitude, or neither or both? Is energy not conserved? How can that be? Furthermore, in the real world gravity doesn't change suddenly at one point; it is gradual over time, but that only makes the problem of changing gravity more intractable.

I suggested we experiment with a pendulum in an elevator to simulate changing acceleration of gravity. Bryan, ever more practical, used his MicroSet timer and a repulsive variable magnet to simulate more or less downward force on the bob. The published results (HSN 2004-5) indicate that when his pseudo gravity was weakened, the pendulum rate decreased, velocity decreased, and amplitude increased, perhaps contrary to intuition. I understand others in the horological community have also considered the conundrum of changing gravity but without a satisfying or simple explanation.

To the rescue came my physicist brother who suggested we look into *adiabatic invariants*, an old topic in classical mechanics. According to this treatment, the key to understanding our problem is not to worry about what changes – but instead, consider *what stays the same*!

Solvay Conference of 1911

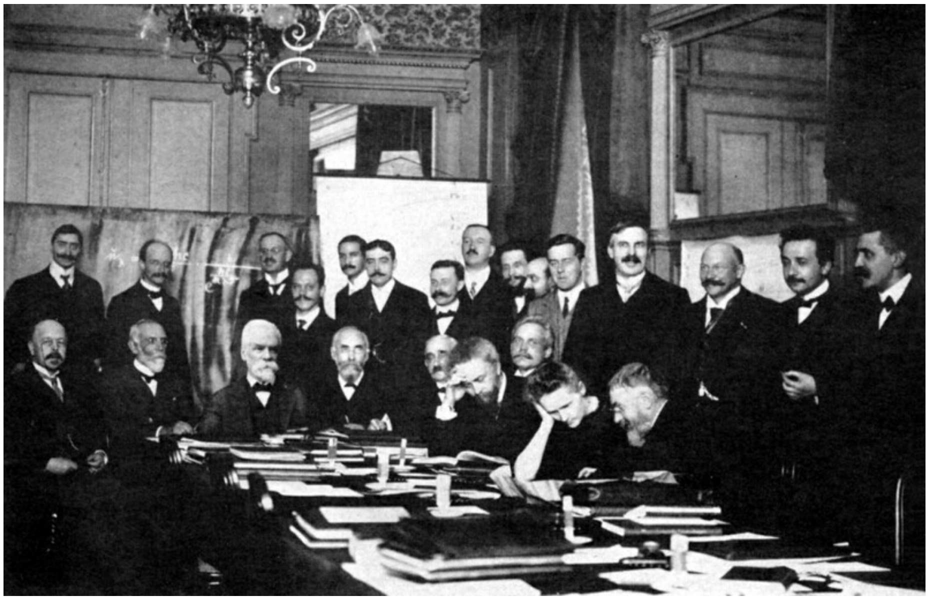
We go back in history to the first Solvay conference, where issues of quantum mechanics were debated. You may recognize many of the participants in ***photo 4a***. To illustrate a problem with energy conservation Lorentz asked how the amplitude of a simple pendulum would vary if its period was slowly changed by shortening its string. Would the number of quanta of its motion change?

It was Einstein, having read the work of Ehrenfest, who immediately answered that E/ω (where E = energy and ω = frequency) would remain constant and thus the number of quanta would remain unchanged as long as the rate of the pulling was very small compared to the period of the pendulum.

In other words, if the string is pulled slowly enough while the pendulum is busy swinging, even though the quantities E and ω would change, the quantity E/ω would not. Under conditions of slowly changing parameters and periodic motions such a quantity is called an *adiabatic invariant*. Likewise, since T (pendulum period) is proportional to $1/\omega$, the quantity $E \times T$ is also an adiabatic invariant.

Apparently the principle of adiabatic invariance holds for any SHO (Simple Harmonic Oscillator) where any parameter of the system is slowly changed. Lorentz slowly pulled a string (change of L) but in our case we want to slowly change gravity (change of g). The principle of adiabatic invariance is ideally suited to a pendulum clock and tides since the pendulum period is mere seconds and the changes in gravity occur over many hours – easily satisfying Einstein's stated requirement that the rate of change is much smaller than the rate of oscillation.

So having established the applicability of adiabatic invariants to the pendulum and tides problem, what conclusions can we draw?



Energy Conservation

Before we worry about gravity, let's try a simpler case. Consider a normal pendulum with *constant* mass m, length L, and acceleration of gravity, g. From simple geometry, given half-angle θ, half-amplitude A, and height h, we know $A = \sin(\theta)$ and $h = L(1 - \cos(\theta))$. For small angles (measured in radians) recall that $A \approx \theta$ and $h \approx \frac{1}{2}L\theta^2$.

Now since $\frac{1}{2}mv^2 = KE = PE = mgh$ we know that if energy E were to change by a tiny amount (a relative amount, say 1 percent, due to impulse or friction) then height h would also change by 1 percent and velocity v would change by ½ percent; A would also change by ½ percent. Note that period, T, does not change (except for second-order circular error) since L and g are constant. In summary, if E changes by one small relative amount (denoted by $\Delta E = 1$), then the corresponding changes to other parameters are:

$$\Delta E = 1,\ \Delta v = ½,\ \Delta h = 1,\ \Delta A = ½.$$

The above is the familiar relationship among energy, velocity, amplitude that we know and use in the context of energy gain and loss in a typical pendulum system.

Adiabatic Invariant, E×T

But now, what happens if gravity is *not* constant? When g itself changes, the constant is no longer energy, but energy times period; E×T. Consider Δg, a very slow and small relative change in g (e.g., 0.1 ppm, due to tides). From $T \approx 2\pi\sqrt{L/g}$, we see $\Delta T = -½\Delta g$, meaning the period changes half as much as gravity and in the *opposite* direction; i.e., if gravity decreases by 0.1 ppm, period will increase by ½ as much, or 0.05 ppm. Since rate or frequency f is 1/T, we have also $\Delta f = ½\Delta g$, meaning rate changes half as much as gravity changes and in the *same* direction.

To continue, if T increases then E must decrease since E×T is invariant. So $\Delta E = -\Delta T = ½\Delta g$. And finally since $½mv^2 = E = mgh$, we can determine three additional relationships: $\Delta v = ¼\Delta g$, $\Delta h = -½\Delta g$, and $\Delta A = -¼\Delta g$. In summary, when g changes by one small unit (denoted by $\Delta g = 1$), then the corresponding changes to other parameters are:

$$\Delta g = 1,\ \Delta f = ½,\ \Delta T = -½,\ \Delta E = ½,\ \Delta v = ¼,\ \Delta h = -½,\ \Delta A = -¼.$$

This should settle the question about gravity and tides and pendulum clocks. We now know what happens to a free pendulum under the influence of tides. It finally explains Bryan’s artificial gravity results (where amplitude changed opposite of velocity and rate) and further gives the specific relative magnitude for each change. It is also very interesting to see Einstein’s name come up in the context of pendulum clocks!

What Happened to Energy Conservation

When we start looking into changes to gravity, one has to be careful about the concept of *energy conservation* in pendulum clocks. Usually all we think about is the bob: its velocity and amplitude or height. True, there is a continuous equal exchange between kinetic energy (velocity at center of swing) and potential energy (height or amplitude at end of swing). But when gravity itself changes what happens to the pendulum bob’s KE and PE is not the *whole* picture.

The whole picture is to consider the combined sun-moon-earth-pendulum system. If, for example, the moon moves slightly closer to the earth, the net gravitational force acting on the pendulum bob decreases. From the pendulum’s isolated perspective energy seems to have been lost. But from a solar system perspective all energy is conserved when you consider planetary orbital velocities and distances, gravitational potential, pendulum velocity and amplitude. A bob may “own” its own kinetic energy but its potential energy is really a function of the shared earth-sun-moon-bob system not just the bob alone.

Each time something happens to one of the bodies (earth, moon, sun, bob) they all react in harmony. No parameters stay the same; but it is a graceful dance among all the bodies. And so the fractional relationship among all pendulum parameters seen above makes sense. By knowing the exact magnitude and sign, we can better model the internal behavior of precision pendulum clocks under the influence of slow external changes. One caution: for any of this to matter the pendulum clock must itself be quiet enough compared to the fractional changes induced by tides. Only the very best pendulum clocks appear to satisfy this condition.

Conclusion

I am still pondering what all of this means to a precision clock such as Shortt or Littlemore. It's difficult enough to fully appreciate how minor changes in g cause instability in timekeeping. But when we realize that not only is the period changing but also amplitude and velocity then the complexity increases. This, because at the lowest level of detail, changes in velocity and changes in amplitude do have an indirect effect on period, due to non-linearity of friction or drag or circular error, etc.

On the other hand, if g changes only by 0.1 ppm due to tides we see that velocity or amplitude change by an even smaller fraction. And it is my hunch that even in the best pendulum clocks the amplitude or velocity is nowhere near this level of constancy from period to period.

It would be helpful if someone could confirm, either mathematically or experimentally, the nice set of fractional relationships derived above.