Analysis of the Design and Performance of Harpsichords

by N. H. Fletcher

Department of Physics, University of New England, Armidale N.S.W. 2351, Australia

Summary

The mechanical and acoustic behaviour of a plucked string coupled to a soundboard is investigated in relation to the principles underlying harpsichord design. Radiated sound energy, acoustic spectrum and sound decay time are all considered in relation to string length, wire gauge and soundboard properties, and simple scaling rules giving satisfactory musical balance are derived. These ab initio design principles are related to a typical harpsichord and found to be in general accord with building practice.

The measured musical properties, which show a nearly constant sound pressure level over the whole compass, increased harmonic development in the bass, and a decay time varying closely as the inverse 0.4 power of fundamental frequency, are close to what is expected from the analysis.

Analyse des Entwurfs und des Betriebsverhaltens von Cembalos

Zusammenfassung

Unter Berücksichtigung der dem Cembalo-Entwurf zugrundeliegenden Prinzipien wird das mechanische und akustische Verhalten einer gezupften, an einen Resonanzboden angekoppelten Saite untersucht. Die abgestrahlte Schallenergie, das akustische Spektrum und die Schall-Abklingzeit werden in ihrer Abhängigkeit von Saitenlänge, Drahtstärke und von den Eigenschaften des Resonanzbodens betrachtet und es werden einfache Skalierungsregeln abgeleitet, die eine befriedigende musikalische Balance gewährleisten. Diese Entwurfsprinzipien ab initio werden zu einem typischen Cembalo in Beziehung gesetzt. Man findet, daß sie generell mit der Praxis des Cembalo-Baus übereinstimmen.

Die gemessenen musikalischen Eigenschaften, nämlich ein über den gesamten Bereich nahezu konstanter Schalldruckpegel, ein verstärktes Anwachsen der Harmonischen im Baß und eine ungefähr mit der inversen 0,4. Potenz der Grundfrequenz sich ändernde Abklingzeit stimmen eng mit dem nach der Analyse zu Erwartenden überein.

L'établissement de projets de clavecins et la performance de ceux-ci

Sommaire

On étudie le comportement mécanique et acoustique d'une corde pincée couplée à une table de résonance, en relation avec les principes d'établissement d'un projet de clavecin. On considère l'énergie sonore rayonnée, le spectre acoustique et l'amortissement du son, en fonction de la longueur des cordes, du diamètre des fils et des proportions de la table; on déduit des règles simples pour les rapports qui donnent un équilibre musical satisfaisant. Ces principes d'établissement ab initio sont appliqués à un clavecin type: on constate qu'ils sont en accord général avec la pratique des facteurs. Les propriétés musicales mesurées sont voisines de ce à quoi on s'attendait d'après la théorie: niveau de pression sonore quasi-constant dans toute la gamme, augmentation du développement harmonique aux basses, et durée d'amortissement variant à très peu près comme l'inverse de la puissance 0,4 de la fréquence fondamentale.

1. Introduction

Several types of musical instruments, of which the guitar, harpsichord and piano are representative examples, employ as their basic element one or more freely vibrating strings. In many respects such instruments are much more easily accessible to physical analysis than are their relatives the bowed-string instruments, which have the additional complication of a highly non-linear exciting force.

Among the free-string instruments the harpsichord is particularly suitable for analysis for several reasons. It has a considerable compass over which each string is required to produce one note only, it has a simple mechanism which excites each string by plucking in an invariant manner, and it has a well documented design history and widely accepted tonal objectives. This is not to say that there is in any sense a standard instrument, but variations are moderate and their effects can be reasonably understood.

The freely vibrating string is itself one of the most thoroughly studied systems in classical physics and its detailed analysis is included in all standard texts on vibrations [1]. When examined in detail, however, it is very far from being a simple system; the string is appreciably non-linear in its behaviour and its interaction with the surrounding medium is complex [2], [3].

The other acoustically important part of the harpsichord, or indeed of any stringed instrument, is the soundboard. This is a complicated mechanical structure, sometimes associated with an air cavity, and coupled to the vibrating string through a relatively rigid bridge. Analysis of soundboard behaviour must necessarily be much less complete than that of string behaviour unless we are dealing with a particular instrument.

The purpose of the present paper is to analyse the elements of harpsichord design, particularly in relation to the string choirs, to isolate the important design parameters and to see how they were manipulated, on the basis of tradition and intuition, by the master craftsmen of the past.

2. String behaviour

The first-order equations describing the behaviour of a plucked string are well known but we set them out here for later reference. Suppose we have an ideally flexible string of length L, radius r and density ϱ_s stretched with tension T between rigid supports as shown in Fig. 1. Then its normal mode frequencies are

$$\nu_n = \frac{n}{2L}\sqrt{\frac{T}{\pi r^2 \varrho_s}} = \frac{n}{2L}\sqrt{\frac{S}{\varrho_s}} \qquad (1)$$

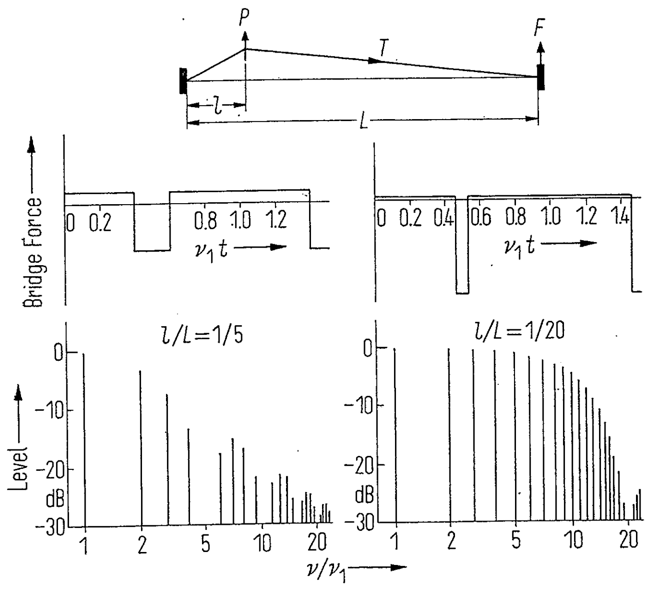

Fig. 1. Essential features of a string under tension T and plucked with transverse force P as in a harpsichord; time variation of the transverse force F on the bridge and spectral analysis of F for $l/L = 1/5$ and $1/20$.

where S is the elastic stress in the string material. Clearly the normal modes are in exact harmonic relation in this approximation.

If such a string is plucked by deflecting it with a sharp quill at a distance l from one end, as shown, and then releasing it, the subsequent behaviour can readily be determined. What we are interested in, however, is not so much the behaviour of the string but rather the force on the string supports, for one of these will be a relatively rigid bridge which communicates the string motion to the soundboard to which it is fixed. The string itself, being an acoustic dipole source, is an inefficient radiator of sound, and most of the acoustic energy is radiated from the soundboard.

Elementary considerations easily show that the transverse force F on the bridge has the form of a rectangular wave, as shown in Fig. 1, the durations of the positive and negative segments being proportional to $(L-l)$ and l respectively. Spectral analysis of this force gives

$$F(t) = \sum_{n=1}^{\infty} (P/n\pi\nu_1) \sin(n\pi l/L) \cos 2\pi n \nu_1 t \qquad (2)$$

where P is the transverse plucking force. Typical spectra for $l/L=1/5$ and $1/20$ are shown in Fig. 1. The first zero in the spectrum occurs at frequency

$$\nu^* = n^* \nu_1 = (L/l)\,\nu_1 \qquad (3)$$

and either ν^* or n^* represents a convenient parameter to characterize the sound. If L/l is an integer, the n^* harmonic is absent in this approximation.

The behaviour of a real string is more complicated than this in several ways. The string has a certain stiffness, depending on its radius r and Young's modulus Q_s, so that the equation of motion contains a quartic term. As is well known [1], this introduces inharmonicity so that the normal mode frequencies are raised to

$$\nu_n \cong \frac{n}{2L}\sqrt{\frac{S}{\varrho_s}}\left[1 + \frac{r}{L}\sqrt{\frac{Q_s}{S}} + \left(1 + \frac{n^2\pi^2}{8}\right)\frac{Q_s}{S}\frac{r^2}{L^2}\right] \cong$$
$$\cong n\,\nu_1'(1 + \varepsilon n^2) \qquad (4)$$

where the inharmonicity parameter ε, given by

$$\varepsilon = \frac{\pi^2 Q\, r^2}{8 S L^2} \qquad (5)$$

is small if the strings are thin and under high tensile stress S. For practical strings the fundamental frequency ν_1' is only slightly greater than the ideal-string value ν_1.

While the string behaviour remains linear this inharmonicity has little effect, except for introducing a slightly bell-like quality into the sound [4]. The same is true of the additional inharmonicity introduced by the reactive component of the impedance of the supporting bridges, which will raise the normal mode frequencies for a mass-like impedance and lower for a spring-like impedance [1]. These inharmonicities, particularly that due to string stiffness, are well known in the case of the piano. They are much less pronounced for the thinner strings of the harpsichord.

We must recognise also that the string is non-linear, because a wave of any amplitude necessarily lengthens the string and increases its tension. This causes coupling between longitudinal and transverse string waves [2] and hence between transverse waves of different frequencies and polarizations [4]. The effects of this coupling are twofold.

In the first place, because the tension term is quadratic in the mode amplitudes, the major coupling is between three modes, say ν_j, ν_k, ν_l, giving a driving term with frequency close to the mode $\nu_{j\pm k\pm l}$. This has the effect of filling in gaps in the string spectrum so that the sharp minima shown in Fig. 1 do not occur.

The second effect relates to inharmonicity. Because ν_n is not exactly $n\nu_1$, the various non-linear driving terms differ slightly in frequency and produce beat-like effects among the upper partials of the string and, indeed, sometimes for the fundamental as well. These effects, which have been discussed by Lieber [3], are easily examined if the individual partials of a plucked string are isolated by use of filters. It is worthy of note, in passing, that non-linear effects should be much less apparent for gut strings, which have a relatively small Young's modulus, than for steel strings under the same tension, because the change in string tension for a given vibration amplitude is clearly proportional to this modulus.

The final effect which must be considered is the damping of the string vibration by internal friction, air viscosity, sound radiation, and energy transfer to the soundboard through the bridge. The most important of these for simple thin metal strings is usually viscous loss to the air. For gut strings internal friction may also be important, specially for high-frequency modes. Only a small fraction of the string energy is generally passed on to the soundboard and either dissipated in its losses or radiated as sound, while the direct sound radiation from the string is usually negligible.

The complex problem of viscous loss from a vibrating string was solved long ago by Stokes [5] in connection with his study of pendulums. He showed that, for the range of wire diameters and vibration frequencies encountered in musical instruments, the retarding force R (i.e. the component of the total force which is π out of phase with the velocity) experienced by a cylinder of length L and radius r moving with frequency ν and velocity v is given by

$$R \cong 2\pi^2 \varrho_a \nu r^2 L \left(\frac{\sqrt{2}}{M} + \frac{1}{2M^2}\right) v \tag{6}$$

where ϱ_a is the density of air ($\varrho_a \cong 0.0012$ g cm^{-3}) and

$$M = \frac{r}{2} \sqrt{\frac{2\pi\nu}{\mu_a}} \tag{7}$$

μ_a being the kinematic viscosity of air ($\mu_a \cong 0.15$ cm^2 s^{-1}). For typical harpsichord strings $0.3 < M < 1.0$, which is within the range of validity of eq. (6), namely $M > 0.3$. For smaller values of M Stokes gives a different approximation.

From eq. (6) we see that $R \propto v$ so that the rate of energy loss varies as v^2, which is proportional to the kinetic energy. For a simple oscillation at a single frequency, therefore, we expect an exponential decay of amplitude. If we write the energy decay factor as $\exp(-t/\tau_1)$, then we readily deduce that

$$\tau_1(\nu) = (\varrho_s / 2\pi \varrho_a \nu) \left(\frac{\sqrt{2}}{M} + \frac{1}{2M^2}\right)^{-1}. \tag{8}$$

The decay time is thus proportional to the wire density, as we should expect from simple considerations, and depends in a rather more complicated way on wire radius and on vibration frequency.

Because of the non-linear coupling between the modes of a string and because the mode frequencies are not in exact harmonic relationship, we should not be surprised to find much more complex decay behaviour for the individual modes of a real string than is suggested by this discussion. Our results should provide, however, a reasonable approximation to the average behaviour.

3. Soundboard behaviour

Analysis of soundboard behaviour is complicated, as we have mentioned before, by the complex soundboard geometry. A normal harpsichord soundboard consists of a sheet of wood, perhaps 2 to 3 mm thick and of roughly triangular shape, clamped at its edges and divided into smaller panels by ribs glued to its under side. It behaves therefore essentially like a series of four to eight coupled flexible plates with areas typically ranging from about 0.04 m^2 to 1 m^2, though in a small instrument the largest area might not exceed 0.3 m^2. The bridge usually couples the strings to the largest of these panels and the smaller panels are coupled to this across the main rib as shown in Fig. 2.

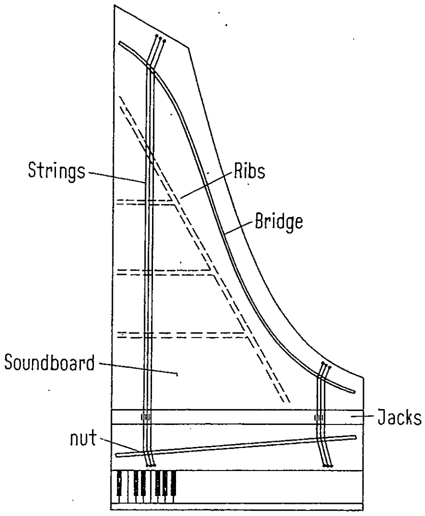

Fig. 2. General arrangement of mechanical features in a typical harpsichord. The dashed lines show soundboard ribs.

Precise analysis of such a system is clearly out of the question unless we confine ourselves to a specific instrument. Instead of doing this we shall be content with a more superficial but more general discussion.

The vibrations of a regular elastic plate are well understood [6]. For a peripherally clamped circular plate of radius a and thickness d, constructed from material with Young's modulus Q_p, Poisson's ratio σ_p and density ϱ_p, the normal mode frequencies are, to a sufficient approximation for our present purpose,

$$\nu_{mn} \cong \frac{\pi d}{2a^2} \sqrt{\frac{Q_\mathrm{p}}{3\varrho_\mathrm{p}(1-\sigma_\mathrm{p}^2)}} \left(n + \frac{m}{2}\right)^2. \tag{9}$$

These frequencies are not harmonically related but are, none the less, approximately uniformly distributed in frequency.

The individual panels in a harpsichord soundboard are not circular in shape, but it will be an adequate approximation here to neglect this and perform a calculation for circles of equivalent area. For a typical panel 2 mm in thickness and with area $\pi a^2 \mathrm{m}^2$, we find for the fundamental mode

$$\nu_{01} \cong 50\, a^{-2} \tag{10}$$

so that we may expect to find fundamental panel resonances ranging from 4 kHz down to perhaps 150 Hz in a large instrument. Because of the large damping associated with individual resonances and the coupling between them, they will overlap very considerably in frequency and should produce a smooth frequency response. The differences between different instruments, as far as soundboard resonances are concerned, may often be subtle, but there will generally be a clear difference between large and small instruments as far as the frequencies of the lowest few resonances are concerned, the low-frequency cut-off being approximately inversely proportional to soundboard area.

Most harpsichord soundboards are built with the ribs not extending completely to the case walls. This leaves the soundboard free to vibrate also as a complete unit, stiffened and loaded by the ribs. The frequency of this vibration mode depends on design details but it is probably below the lowest panel resonance and thus extends the bass response.

In most harpsichords of classical design the soundboard forms the top of a box which communicates with the outside air through a rose opening. In other designs there may be additional holes in the base or the base plate may be omitted entirely.

A closed, vented box acts as a Helmholtz resonator at low frequencies and has a simple resonance at

$$\nu_0 = \frac{c}{2\pi} \sqrt{\frac{A}{l_\mathrm{e} V}} \tag{11}$$

where c is the velocity of sound in air, V the enclosed volume, A the area of the opening and l_e the equivalent length of the opening port, generally approximately equal to $0.8\sqrt{A}$ for the geometry found here. This air cavity resonance will generally lie below 100 Hz for simple rose-vented designs and the fret-work of the rose will add resistance and broaden the resonance.

4. String-soundboard coupling

In a harpsichord the string is coupled to the soundboard by passing tightly beside a pin set in the bridge which is fixed to the soundboard. The angle of the string in relation to pin and bridge ensures tight coupling for any polarization of the string vibration. The string vibrations are, however, initially perpendicular to the plane of the soundboard so that no directional change, such as is achieved by the rocking of a violin bridge on its soundpost, is required.

The mechanical impedance presented to the strings by the bridge has a complex character determined largely by the properties of the soundboard. The impedance presented by a single panel can be calculated for simple geometries [7] and its inverse, the admittance, has a form like

$$Y(\nu) = -\frac{\mathrm{i}\,\nu}{a^2 \varrho_\mathrm{p} d} \sum_{m,n} \frac{A_{mn}}{\nu_{mn}^2 - \nu^2 - \mathrm{i}\,\nu\,\delta_{mn}} \tag{12}$$

where the A_{mn} are constants of order 10^{-1} or less, determined by the driving-point geometry. δ_{mn} is the damping coefficient for the (m, n) resonance, and the other symbols have their previous meanings. In a soundboard δ_{mn} is contributed to both by internal friction in the panel material and by losses to acoustic radiation. It seems likely, though the question deserves closer study, that in most cases the internal friction will dominate, though clearly the builder will attempt to minimize its magnitude.

As we have said, the harpsichord soundboard consists of a set of panels with different resonant frequencies more or less closely coupled together. An equivalent-circuit analysis suggests that $Y(\nu)$ will be dominated by the behaviour of the large panel on which the bridge is supported and from the form of eq. (9) there is always a resonance of this panel reasonably close to any given string frequency ν. If this resonance is heavily enough damped, as seems likely to be the case, then the real part of the admittance $Y(\nu)$ will be approximately

$$G(\nu) = (a^2\, d\varrho_{\mathrm{p}}\, \delta_{mn})^{-1} \tag{13}$$

where (m, n) is the resonance involved. This conductance should be approximately constant with changing frequency and radiation should contribute an approximately constant fraction to it. The radiated sound spectrum should thus have the same form as the bridge force spectrum (2) over the whole range above the lowest resonance frequency.

If now we calculate the total energy stored in a string of radius r, length L and density ϱ_{s}, vibrating at its n-th harmonic, and the rate of loss of this energy to the soundboard, we find an exponential decay of stored energy with time constant

$$\tau_2(n\,\nu_1) = [8\pi r^2\, \varrho_{\mathrm{s}}\, L\, \nu_1^2\, G(n\,\nu_1)]^{-1}\,. \tag{14}$$

We have now discussed the time constants associated with energy loss from the string by the two principal mechanisms — viscous loss from the string to the air and loss through the bridge to the soundboard. The resultant combined time constant τ is given by

$$\tau^{-1} = \tau_1^{-1} + \tau_2^{-1}\,. \tag{15}$$

5. Instrument design

The classical harpsichord design [8] combines all the variables we have discussed to produce an instrument with satisfying tonal balance, adequate loudness and good mechanical properties. In this section we now make use of our earlier analysis to see how this is accomplished.

The compass of the fundamental frequencies for an "8-foot" (normal pitch) string choir on a harpsichord is from about 50 Hz to 1.5 kHz. A 4-foot string choir would extend the upper range to 3 kHz, while the unusual and non-classical 16-foot choir would extend the lower range to 25 Hz. When overtones are considered, there is no strict upper limit to the frequency range radiated but there is considerable energy up to 5 kHz and audible components occur up to 10 kHz or more.

6. Soundboard

Because the soundboard is the fundamental radiator for the instrument, it is necessary that it should be efficient at least over the range 5 Hz to 3 kHz and this can be achieved by dividing it into panels whose fundamental resonances evenly span this range. The fundamental resonances are important here because they radiate more efficiently than do the higher panel modes.

The difficulties of soundboard design involve, by eqs. (9) and (12), the achievement of a sufficiently low fundamental resonance for the largest panel and of a smooth distribution of adequately broad resonances over the remainder of the range. By eq. (9) the first objective is achieved by using a main panel as large and as thin as possible, made from wood or other material with a small value of $Q_{\mathrm{p}}/\varrho_{\mathrm{p}}$. A thin panel also has broad resonance peaks since the reactances decrease with decreasing mass and stiffness while the resistive losses due to radiation loading remain constant. An even distribution of resonance frequencies can be achieved by arranging that the soundboard panels decrease in area by a constant factor from one to the next and cover the whole frequency range. The smaller panels might also be made thinner to further broaden their resonances.

This is almost exactly the solution adopted in soundboard design. Panel resonances from about 150 Hz to 4 kHz are fairly readily obtainable in a reasonably large instrument and, by not carrying the dividing ribs quite to the edge, an additional whole-board resonance of low frequency can also be introduced. If the quality factor (Q-value) for individual resonances does not exceed about 3, then five or six panels should provide an adequately smooth response.

There is, however, considerable difficulty in extending the fundamental resonance much below 100 Hz, because of the sheer size of soundboard required, though large instruments will clearly have more extended bass response than small ones. Below the lowest soundboard resonance the spring-like reactance at the bridge will become important and response will fall at an ultimate rate of 12 dB per octave. Because of the great extent of the harmonic development in harpsichord sound, particularly in the bass, this intensity loss in the low components of bass sounds will affect tone colour more significantly than total loudness.

7. Strings

In a harpsichord the energy input to each string is fixed by the nature of the plucking mechanism. The loudness of the note is therefore determined by the efficiency with which string energy is transferred to the soundboard. Because sound radiation contributes a constant fraction of τ_2^{-1}, this efficiency is proportional to τ_2^{-1} and so, by eq. (14), other things being equal, loud sound is favoured by using long thick strings of dense material. The length of string for a given frequency is, by eq. (1), limited by the stress which its material can support, independent of its thickness, but in practice harpsichords have always been built with their strings stressed to near their elastic limit. Metal strings are clearly indicated.

Thickening the strings to increase loudness brings with it difficulties because of the large total stress exerted on the wooden case by the whole string choir. In addition, the use of very thick strings is ultimately self-defeating, for the tensioned string becomes so stiff that the amount of energy that can be given to it by the harpsichord plucking mechanism falls. Indeed if the plucking force rather than the maximum string deflection becomes the limitation, then initial loudness becomes independent of string length or radius.

Developments in the direction of thicker strings ultimately led to the iron-framed pianoforte with its more efficient percussive method of transferring energy to the string from the keyboard.

Fairly clearly an optimum harpsichord design is likely to involve graded string diameters and it is reasonable to examine a scaling rule of the form

$$r = \nu_1^{-x} \tag{16}$$

where we expect x to be positive, leading to thicker strings in the bass. We expect that the value of x may affect the balance of both loudness and decay time, and this we now examine. We write the plucking distance ratio as

$$\beta = l/L \tag{17}$$

and defer consideration of possible variation of this ratio to later.

If the plucking force P is constant over the keyboard, which it must nearly be to make the instrument playable, then it is easy to show that the total energy given to a string varies as

$$E(\nu_1) \propto \beta\, r^{-2} L^{-1} \nu_1^{-2} P^2 . \tag{18}$$

Now this energy is transferred to the soundboard at a rate τ_2^{-1} and a roughly constant fraction of the soundboard energy is radiated as sound. The initial sound intensity from a string with fundamental ν_1 is then, using eq. (14),

$$I(\nu_1) \propto E(\nu_1)\, \tau_2^{-1}(\nu_1) \propto \beta\, G\, \varrho_s P^2 \tag{19}$$

and, rather surprisingly, the length and radius of the string do not enter.

Subjective loudness, however, also depends upon decay time, since the brain tends to integrate sensation to some extent. Decay time also determines the slowest speed at which a legato melody can be played, so it is of independent musical importance. Most serious music requires a decay time which is longer in the bass than in the treble, and a harpsichord is designed with this sort of repertoire in mind, rather than the reversed situation often found in contemporary popular music.

It will help with our discussion to simplify eq. (8) and consider only its asymptotic behaviour for small and large M, the dividing point occurring for $M \simeq 0.4$. For a string vibration of frequency ν the appropriate forms are, for r in centimetres,

$$\tau_1 \propto r^2 , \qquad \nu \ll 10^{-2} r^{-2} \tag{20a}$$

$$\tau_1 \propto r\, \nu^{-1/2} , \qquad \nu \gg 10^{-2} r^{-2} . \tag{20b}$$

Similarly from relations (13) and (14) we have, for a vibration of frequency ν on a string of fundamental frequency ν_1,

$$\tau_2 \propto r^{-2} L^{-1} \nu_1^{-2} . \tag{21}$$

From eqs. (8) and (14) applied to a typical mid-compass metal string of length 50 cm and radius 0.1 mm sounding a fundamental of 300 Hz and coupled to a typical soundboard of area 0.5 m^2, we find $\tau_1 \sim 1$ s while $\tau_2 \sim 10$ s. We therefore conclude that viscous damping is the major loss mechanism for this string and probably for most harpsichord strings. (The opposite conclusion applies to the uppermost strings of a piano.)

Using these simplified forms (20) and (21), Fig. 3 shows the results of three plausible string scaling rules. τ_2 is shown as a broken line for the upper part of the compass in which $L \propto \nu_1^{-1}$ and as a dotted line for the low range in which $L \simeq$ constant. The resulting curve for τ in the two cases is essentially a smoothed lower envelope to appropriate τ_1, τ_2 curves.

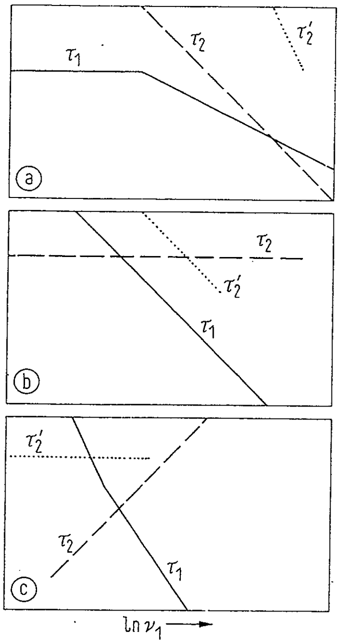

Fig. 3. Decay times τ_1, due to viscous damping, and τ_2, due to losses to the soundboard, for three possible string scaling laws: (a) constant radius, (b) $r \propto \nu_1^{-1/2}$, (c) $r \propto \nu_1^{-1}$, where ν_1 is the fundamental frequency of the string. The dashed curves for τ_2 are based on the assumption that string length varies as ν_1^{-1} and the dotted curves on the assumption that string length is constant. The net decay time, τ is a smoothed lower envelope to these asymptotic curves.

From Fig. 3a, it is clear that a given string radius will be satisfactory, as far as decay time is concerned, over only a limited range. At both ends of this range τ is decreased and the range can only be extended towards higher frequencies by using thinner wire and towards lower frequencies by using thicker wire. This is, in fact, the practical solution but we can approximate it for theoretical discussion by a rule of the form (16) with $x>0$ as expected.

From relations (20b) and (21) the high frequency cut-off due to intersection of τ_2 with τ_1 can be avoided by taking $x \geqq 1/6$, giving a gradual reduction in wire radius towards the treble. The low frequency cut-off, as shown in Fig. 3a, is initially due to the change in behaviour of τ_1 and, from the inequalities in (20), this can be eliminated by taking $x=1/2$ in the bass, giving a behaviour like Fig. 3b. In the bass τ_2 now appears to become the limiting factor if the string lengths are still in the proportional range. Once the string length becomes constant however this limitation no longer occurs and an x value approaching 1 could be used.

Thus we should expect, from linear acoustic criteria, that harpsichord string radii should vary roughly in accord with eq. (16), with $x \cong 1/6$ in the extreme treble, $x \cong 1/2$ in the lower range and perhaps $1/2 < x < 1$ in the region of constant string length.

Actually this scaling is also determined by nonlinear considerations, since unpleasant effects enter if the vibration amplitude of any string is too large, and it may be these considerations rather than our simple linear ones which determine string gauge, particularly in the extreme bass.

From this discussion and Fig. 3 it seems that we should expect to find a decay time τ varying roughly as $\nu_1^{-1/2}$ over the middle and high range for which x is small, and perhaps changing to a rather steeper law like ν_1^{-1} in the bass where $x > 1/2$. Calculation of the actual behaviour is complicated by the large number of overtones and by the variation of plucking point.

8. Plucking point

Finally we examine the variation in plucking point across the compass of the instrument. The main reason for any variation at all is to achieve tonal coherence for the string choir so that the tone quality is matched across the compass. In part this is achieved through the steady gradation in decay time we have already discussed, and in part through a similar coherence in attack time produced by use of similar plectra. The plucking point is, however, of prime importance.

Much of the characteristic quality of a sound is contained in those partials occupying the normal speech range, say 300 Hz to 3 kHz, the reasons for this being physiological and psychological. Thus a variety of male, female and juvenile voices singing the same vowel sound are tonally coherent to some extent for just this reason, while different vowels are not. To maintain coherence it is therefore necessary to increase the harmonic development of the lower notes relative to those of higher pitch. This can be achieved if the plucking ratio varies as

$$l/L = \alpha\, \nu_1^{y} \tag{22}$$

where $y>0$ and α determines the general timbre of the string choir.

We noted before that the plucking ratio also influences the energy $E(\nu_1)$ given to the string and, through this and the influence on n^*, hence the initial intensity $I(\nu_1)$. For ease of playing, the keyboard force and key drop must be constant over the whole compass and this generally leads to constant plucking force P. It is then easily verified that, rather than the simple form (18), the initial string energy $E(\nu_1)$ now varies as

$$E(\nu_1) \propto r^{-2} L^{-1} \nu_1^{y-2} P^2 . \tag{23}$$

Thus the analog of relation (19) becomes

$$I(\nu_1) \propto \beta G \varrho_s P^2 \nu_1^y \tag{24}$$

which, as before, is independent of wire radius but increases with wire density ϱ_s and with soundboard efficiency βG. A thicker wire, however, allows use of a larger plucking force P before other difficulties intervene.

To achieve a satisfactory tonal balance, as we have already discussed, we require $y>0$ and an average value of $y \simeq 1/2$ might be reasonable. This leads to an expected rise in peak sound intensity of 1.5 dB per octave towards the treble. Subjective loudness depends, however, both on frequency and on decay time, if this latter is short. Because of the increased harmonic development in the bass, the change in sensitivity of the ear with frequency is not greatly involved, but decay time certainly is. We have already discussed the expectation that τ should vary roughly as ν_1^{-z} with $1/2<z<1$ and this leads to the conclusion that the effective loudness $\bar{I}$ should vary as

$$\bar{I}(\nu_1) \simeq I(\nu_1)\,\tau(\nu_1) \propto \nu_1^{y-z} \tag{25}$$

which should show little apparent variation over the compass of the instrument.

Of course a more complete analysis must involve the behaviour of all the partials of the sound and this is complicated by their interactions as we discussed before. It is, however, unlikely that the conclusions reached after such a more complete analysis would differ much from those set out above.

9. Practical example

Detailed comparison between theory and experiment is beyond the scope of the present paper but it is helpful to relate the discussion to reality by reporting relevant measurements on a typical harpsichord. That used was a modern single-manual instrument constructed by the author to the design of W. Zuckermann of New York. Comparison of general design and wire gauges with those for classical instruments [8] shows very considerable similarity so that we have no reason to be specially cautious about interpretation of the results.

The instrument in question is of small dimensions with a triangular soundboard of area 0.6 m^2 divided into six panels with areas ranging from 0.2 to 0.04 m^2. The 8-foot bridge lies entirely on the largest of these panels and the soundboard forms the top of a totally enclosed box of volume about 0.07 m^3. (The design allows vents but these were not included.)

There are two choirs of strings of 8- and 4-foot pitch respectively, but our measurements refer only to the 8-foot choir. Details of string lengths and radii are shown in Fig. 4, from which we can

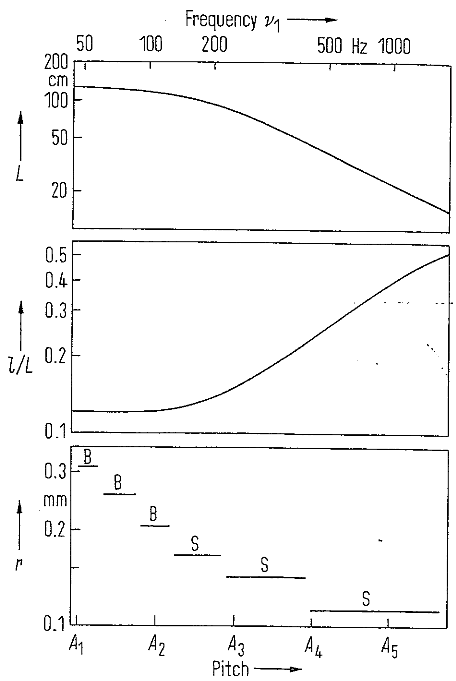

Fig. 4. Physical characteristics of string courses in the measured harpsichord. L is a string length, l plucking distance, r wire radius and ν_1 the fundamental frequency of the string involved. S indicates steel and B brass strings.

see a general progression towards finer wire gauges in the treble together with a steepening of the gauge variation in the bass where the strings are of less than proportional length. The general trend of gauge variation is between $\nu_1^{-0.3}$ in the treble and $\nu_1^{-1.0}$ in the bass. The mid-range and treble strings are of steel while the bass strings are of brass. L varies as $\nu_1^{-1.0}$ over the top half of the compass and l/L as $\nu_1^{0.6}$. Both become constant in the extreme bass.

Fig. 5 shows measurements of the most readily accessible acoustic parameters: the peak A-weighted and C-weighted sound pressure levels I_A and I_C at ordinary listening distance, the damped ("slow" meter) A-weighted level $\bar{I}_A$ and the decay time to inaudibility τ_0.

The sound pressure levels vary little across the compass of the instrument and the similarity of I_A and I_C reflects the fact that much of the energy of even the extreme bass strings extends into the region near 1000 Hz. All the I measurements rise slightly towards the treble: I_A by about 1.5 dB(A)/octave, $\bar{I}_A$ by about 0.5 dB(A)/octave, reflecting the influence of the fall in τ_0 with rising frequency, and

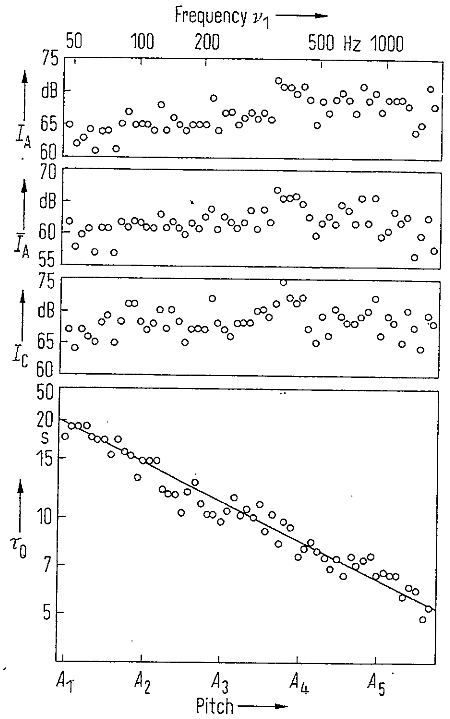

Fig. 5. Measured acoustical properties of the experimental harpsichord:
I_A — measured A-weighted SPL at 2 m with fast meter response,
$\bar{I}_A$ — same with slow response,
I_C — C-weighted SPL with fast response,
τ_0 — estimated decay time to inaudibility in seconds; the slope of the line is $\nu_1^{-0.4}$.

I_C by perhaps 0.5 dB(C)/octave. These results are in agreement with expectations from a musical point of view and our analysis shows that this behaviour is to be expected from the acoustic design.

The aural decay time τ_0 was determined as the time for decay to inaudibility, representing a drop in loudness from about 70 phons to the background level near 30 phons. Thus τ_0 corresponds to a drop in level of about 40 dB while our theoretical τ corresponds to a drop in level by a factor e or about 4 dB. We therefore conclude that $\tau_0 \simeq 10\,\tau$ so that the mid-range value of τ is about 1 s, in agreement with our theoretical expectations. The measured variation of τ_0 as $\nu_1^{-0.4}$ is also very close to the expected behaviour.

10. Conclusions

Our analysis of harpsichord design parameters was admittedly based in part on a knowledge of actual building practice, but it lends credibility to our assumptions and analysis to find that the actual design practices arise naturally from the theory and that the quantitative behaviour of a real instrument conforms well with our expectations.

The analysis as set out here is incomplete in many of its details, and some major aspects of soundboard behaviour require further investigation. Such studies might well proceed simultaneously using experimental and theoretical approaches and our analysis leads us to expect that a relatively complete first-order study should be quite feasible. After that it may be possible to examine some of the subtleties which distinguish the work of master craftsmen of the past and present. It may also be useful to examine the relations between harpsichord design objectives and the rather similar considerations which apply to pipe organs [9].

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