Nonlinearities in Musical Acoustics

Neville H. Fletcher
Institute of Physical Sciences, CSIRO
Limestone Avenue
Canberra ACT 2602

ABSTRACT: The role of nonlinearity in the behaviour of musical instruments is discussed, with particular reference to the clarinet, the trumpet, the flute, the violin, and certain percussion instruments.

Nearly all the acoustical theory to which we have all been exposed over the years has been linear — twice the excitation gives twice the response — though we probably recall that at large amplitudes, as in shock waves, the situation is much more complex and nonlinear equations are involved. But this is too complicated for most of us to worry about, and surely shock waves are restricted to explosions and supersonic aircraft anyway!

Against this background it may come as something of a shock to realise that the behaviour of musical instruments such as the violin and the clarinet is dominated by nonlinearity and we have no hope of understanding their behaviour without considering it quite explicitly. In these few pages I would like to give a gentle introduction to nonlinearity and its importance in musical instruments. It is not a subject to which many people have given explicit attention, and I apologise in advance for the limited reference list, but a recognition of its importance is quite fundamental.

LINEAR AND NONLINEAR OSCILLATORS

All the instruments of music, and indeed almost all physical vibrating systems, behave as linear or harmonic oscillators if their amplitude of vibration is small enough. There are a few singular cases which we meet later, but this assertion really derives from the mathematical process of neglecting all but the most important terms or from the physical process of assuming all elastic forces to be reasonably described by Hooke's law provided the displacements are small.

Suppose that the coordinate x_n represents in a general way some oscillatory quantity. It could be the physical displacement, as a function of time, of a mass hanging from a spiral spring, or the displacement of a point on a vibrating string, or the velocity of air flow in a wind instrument. Whatever it is, the equation describing its behaviour can be written in the form

$$\ddot{x}_n + 2k_n\dot{x}_n + \omega_n^2 x_n = F(t) + G(x_n, \dot{x}_n) \qquad (1)$$

where a dot implies differentiation with respect to time, and on the right-hand side of (1) we have collected all the terms not explicitly accounted for on the left side. k_n and ω_n are constants chosen so that there are no terms linear in $\dot{x}_n$ or x_n on the right side. F(t) is a forcing function that represents the external force on the system while $G(x_n, \dot{x}_n)$ collects all the terms in x_n or $\dot{x}_n$ of higher than first power [1].

The linear approximation arises from the observation that, if x_n is small compared with some characteristic dimension of the system, then x^2_n is very small and we can neglect $G(x_n, \dot{x}_n)$. Further, if we are interested in the system only when it is not being acted upon by external forces other than steady ones, for example a piano string after the hammer impact or a trumpet playing a steady note, then we can neglect F(t) or incorporate it as a change of origin for x_n. As we all know, the solution to equation (1) then has the simple form

$$x_n = a_n \cos(\omega_n t + \phi_n) \exp(-k_n t) \qquad (2)$$

where the amplitude a_n and phase ϕ_n are determined from the way in which the motion was started.

For an "extended" system, like a bell or a violin string or the air column in a trumpet, which has several possible vibration modes, we have a set of equations like (1) and solutions like (2) for each mode x_n. The mode frequencies ω_n for a real system do not ever have exact integer ratios. The stiffness of real strings makes

$$\omega_n \cong n\omega_1(1 + \alpha n^2) \qquad (3)$$

for example in pianos, and there is a similar sharpening of upper mode frequencies in simple open pipes, while bells and gongs have mode frequencies distributed in a quite complex way. All this, however, is still quite linear. If we excite a bell with its clapper then each mode sounds out by itself and dies away according to an expression like (2).

The complication begins to arise, for simple systems like plucked strings or hammered gongs, when the initial amplitude a_n becomes so large that we can no longer neglect $G(x_n, \dot{x}_n)$ in (1). The simplest case is that of a plucked string. At large amplitude the tension T of the string is increased, in proportion to x_n^2, every time the string moves away from equilibrium. The restoring force is proportional to the product $T\nabla^2 x_n$ and so there is an extra term in G varying like x_n^3. Now x_n^3 involves $a_n^3\cos^3(\omega_n t + \phi_n)$, which can be expressed as terms in $a_n^3\cos(\omega_n t + \phi_n)$ and $a_n^3\cos 3(\omega_n t + \phi_n)$, and the first of these is at the correct frequency to influence the left-hand side of the equation. The resulting solution [1] for x_n has a frequency ω which varies like

$$\omega \cong \omega_n + \beta a_n^2 \exp(-2k_n t) \qquad (4)$$

so that the string emits a note with a descending "twang".

In gongs, particularly those used in China, the shape can be arranged so that the pitch glides either downward (for a flat-faced gong) or upward (for a slightly domed gong), the glide being as much as several semitones [2].

SELF-EXCITED OSCILLATORS

Of far more interest than the free oscillators discussed above, from both physical and musical points of view, are the self-excited oscillators which can produce a steady sound when supplied with a constant source of power. Such oscillators are, of course, common in the communications industry as well, but the design objectives in the two fields are completely different. The woodwind, brass and bowed-string instruments all belong to this category and for all of them nonlinearity is vital rather than merely incidental.

From a mathematical point of view F(t) in (1) is still constant, so can be neglected, but the physical system has been so arranged that $G(x_n, \dot{x}_n)$ now feeds back part of the acoustic output as a driving force, so that the system oscillates of its own accord. We examine a few typical systems in turn. In all of them we have a very nearly linear system — the air column or the string — coupled to a highly nonlinear feedback-controlled generator — a reed, an air jet, or a friction-regulated bow.

In (1) we can now suppose that the damping coefficient k_n is that which refers to the resonant system alone, without its generator, and that G may now contain terms in x_n and in $\dot{x}_n$ contributed by the generator. From (2), the condition for an oscillation to begin and build up is clearly that G should contain a term in-phase with $\dot{x}_n$ and larger than $2k_n\dot{x}_n$. Conversely if this term in G is smaller than $2k_n\dot{x}_n$, the oscillation will decay, while if equality prevails it will remain steady in amplitude.

THE CLARINET

One of the simplest systems to analyse is the clarinet [3, 4]. It has a light, responsive elastic reed closing one end of a cylindrical tube as shown in Figure 1(a). The player's mouth provides a blowing pressure p_o tending to close the reed, while the acoustic pressure p inside the mouthpiece (which we take as our variable x_n) tends to force it open. Since the resonant frequency of the reed is arranged to be much higher than the playing frequency ω_n the reed position responds like a simple spring valve and lets more or less air into the mouthpiece from the player's mouth. It is this flow of air, U, which drives the instrument oscillation.

The actual form of U(p), which is the same as that of $G(x_n)$ except for a phase shift of 90°, is shown in Figure 1(b). If the internal pressure p is equal to the blowing pressure p_o, which is typically about 3 kPa above atmospheric (i.e. 30 cm water gauge), then there is no air flow into the mouthpiece and we are at point D. As p is decreased the flow increases by Bernoulli's law as $(p_o - p)^{1/2}$ but soon this begins to be balanced by the fact that the pressure difference $p_o - p$ is forcing the reed closed. This closing effect dominates to the left of C and at B the reed is forced completely closed. The normal operating point is near A with the acoustic pressure swinging back and forth along the curve BC. This actually represents a negative resistance because the flow U is measured towards the driving pressure rather than away from it. In the region BC, $G(x_n)$ is in phase with $\dot{x}_n$ and the clarinet sounds.

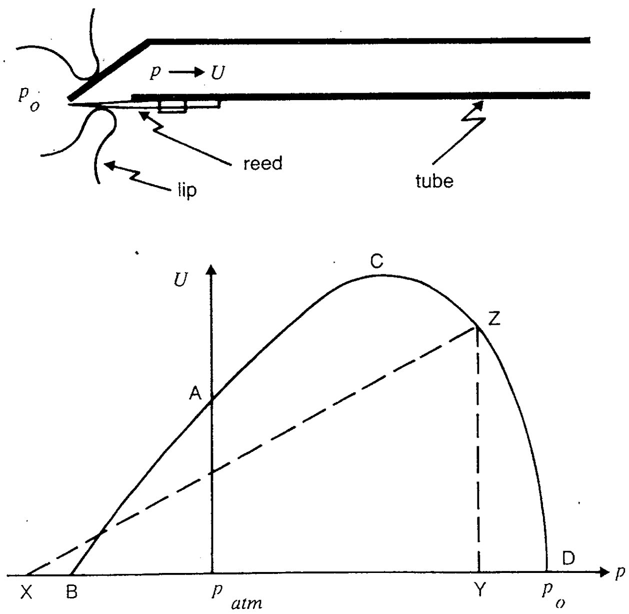

Figure 1: (a) Schematic diagram of a clarinet mouthpiece and tube. The reed is blown closed by the blowing pressure p_o in the player's mouth but this is resisted by the acoustic pressure p inside the instrument. The volume flow is U. (b) The static nonlinear relation between U and p. The normal operating point is close to A.

It is clear that the very functioning of the clarinet and similar reed instruments depends upon the nonlinearity of the flow relation U(p) for the reed but this nonlinearity also has other effects. The curve in Figure 1(b) is invariant if we change the blowing pressure p_o — it simply slides along the axis — while the operating point always oscillates about atmospheric pressure. Clearly if p_o is too small then A will lie on the curve between C and D, and G will be dissipative rather than generating, while if p_o is too large A will be to the left of B and G will be zero.

If the operating pressure makes small excursions about A the coefficient of $\dot{x}$ that it contributes is essentially the slope of the curve at A. If this is greater than 2k the instrument will play — if not then we must remake the reed. As the pressure excursion increases, however, the curvature of the section BC begins to introduce harmonics of ω_n — precisely phase locked — into the flow U. These appear in drive terms not only for x_n but also for other modes x_m through equations like (1) and, since the cylindrical air column has nearly harmonic resonances at about $(2n-1)\omega_1$ the odd harmonics are preferentially reinforced. Quite generally, since the nonlinearity can be expanded as a power series in p, the amplitude of the sth harmonic of mode n will vary initially as a_n^s.

As the pressure amplitude grows larger and swings from X to Y say, the effective value of the coefficient of $\dot{x}$ in G decreases to roughly the slope of the line XZ, and growth stops when this is equal to the tube loss coefficient 2k. For a loudly blown reed — the amplitude is controlled by changing the geometry of the reed with the lips — the pressure excursion is typically like XY in the figure and the flow waveform is something like a square wave. This produces many harmonics, no longer with the simple a_n^s amplitude relation, and the sound is rich and reedy.

THE TRUMPET

In the case of lip-blown instruments such as the trumpet, the situation is rather different because the blowing pressure forces the lip valve open rather than closed [4–6]. This reversal in sign requires a compensating phase change of 180° somewhere else in the system, and this is introduced by adjusting muscle tension so that the resonant frequency for buzzing of the lips is just below the sounding frequency rather than well above as in the case of the clarinet reed. It turns out that the exact lip resonance frequency is critical to the operation of the system, so the player can (and must) adjust his lips to select just the pipe mode required. There is no blowing pressure limit for brass instruments as there is with reeds, and the dynamic nonlinearity is of the form shown in Figure 2. The limit to the sound power output is set only by the blowing pressure and volume flow that can be applied by the player.

FLUTES AND ORGAN PIPES

Among the gentler-toned instruments the flue organ pipe, flute and recorder are alike in that their sound generating mechanism relies upon a nearly plane jet of air emerging from a flue (or from the lips), traversing a mouth-hole cut in the pipe near one end, and then impinging on a more or less sharp

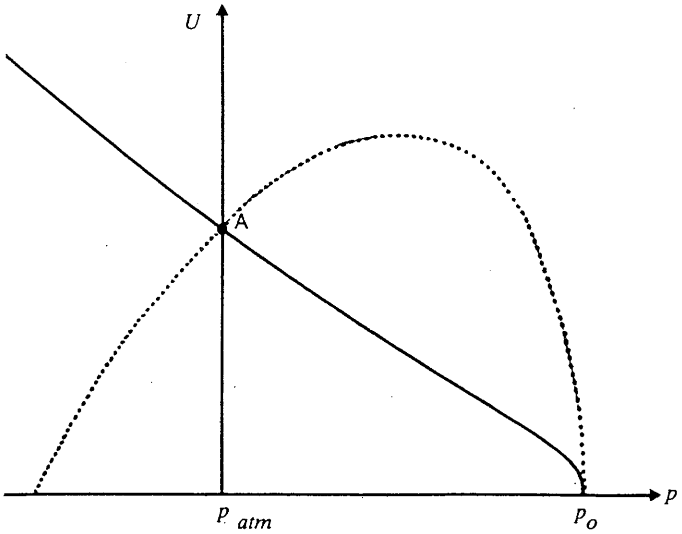

Figure 2: The static nonlinear relationship between airflow U and mouthpiece pressure p for a lip-excited (brass) instrument blown with pressure p_o. The dynamic flow relationship (dotted curve) is changed in sign because the operating frequency is above the lip resonance. The normal operating point is near A.

upper lip as shown in Figure 3(a). The jet can be deflected by the acoustic flow through the pipe mouth so that either more or less of it enters the pipe at the lip to drive increased pipe flow. The whole situation is complicated by the fact that the interaction between the mouth flow and the jet takes place at the flue where it induces transverse waves on the jet which take an appreciable time to reach the lip. Detailed study [7–9] shows that the phase relations are appropriate for regeneration when this wave transit time, which is close to twice the jet flow transit time, is very nearly half a period of the oscillation in the pipe.

Assuming now that this phase shift has been appropriately adjusted by varying blowing pressure and flue-to-lip distance, it is clear that the flow into the pipe at the lip, and hence the interaction function G, has the general form shown in Figure 3(b). Here U_m is the acoustic flow in the mouth and U_j is the jet flow into the pipe. The curve saturates for large positive or negative values of U_m when the jet is flowing entirely into or entirely out of the pipe. The operating point A is generally set asymmetrically so that the jet flows predominantly outside the pipe in its undeflected state. If the slope at A (allowing for other factors in G) is greater than 2k, then the oscillation will grow, to stabilise eventually (in fact after 20-40 cycles) to a sweep such as XY for which the slope of XZ is equal to 2k (again allowing for any deviation of the phase shift from the optimal value).

This nonlinearity both limits the amplitude and generates harmonics which can interact with the higher modes of the pipe. The relative strengths of even and odd harmonics depend critically upon the placing of the operating point A on the curve, and this is one of the operations carried out in pipe voicing, or one of the performance variables available to flute players [9, 10]. The relatively gentle nature of the nonlinearity gives a much smaller degree of harmonic development to the tone than is the case for reed pipes or lip-blown instruments.

STRINGS

As a quite different form of nonlinearity we consider now the bowed-string instruments. The string is itself a nearly linear vibrator and the interaction at the bow involves a stick-slip motion derived from the fact that static friction is greater than dynamic friction, which is itself velocity dependent. The speed v_o of the bow is constant and, if we take the oscillatory variable x to be the velocity v of the string at the bow position, then the curve relating the transverse force F on the string to

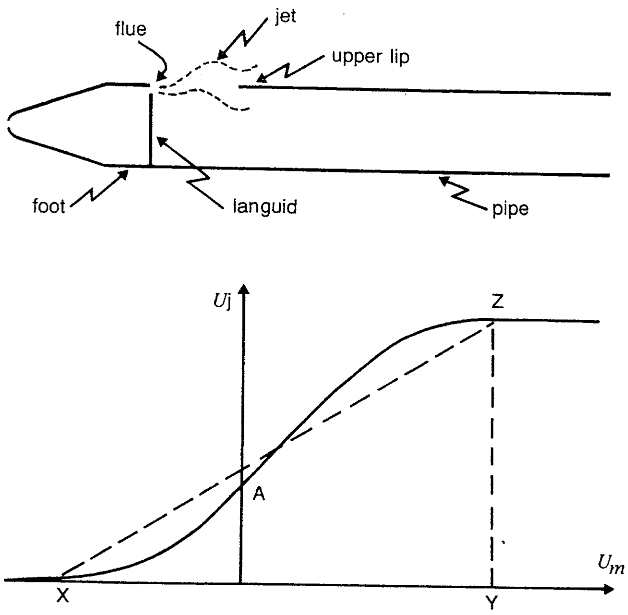

Figure 3: (a) Schematic diagram of an organ flue pipe. The air jet emerges from the flue, crosses the mouth, and strikes the upper lip. The jet has waves induced upon it by acoustic flow through the mouth and these deflect it into or out of the pipe at the lip. (b) The nonlinear relationship between jet flow U_j into the pipe at the upper lip and acoustic flow U_m through the mouth. The normal operating point is near A.

the velocity v has the form shown in Figure 4. The nonlinearity is obviously pathological, with a discontinuity at $v = v_o$.

The nonlinearity of the frictional characteristic implies that we could have self-sustained nearly sinusoidal vibrations about an operating point such as A, but in reality this can occur only for the case of a large oscillating mass driven by a small frictional force. The case of the bowed string is at the opposite extreme — the string mass is small and the rosined bow exerts a large frictional force. The motion then turns out to be one in which the string moves for a large part of each cycle in the sticking position B and then makes a switching transition for a small part of the cycle to the slipping position C. This stick-slip motion is so highly, and indeed essentially, nonlinear that it is quite inappropriate to attempt to analyse it by considering growth from the nearly linear situation. The string motion has however been analysed in detail, beginning with the studies of Helmholtz and Raman, and we now have a very good appreciation of most of the subtleties of its behaviour [11, 12].

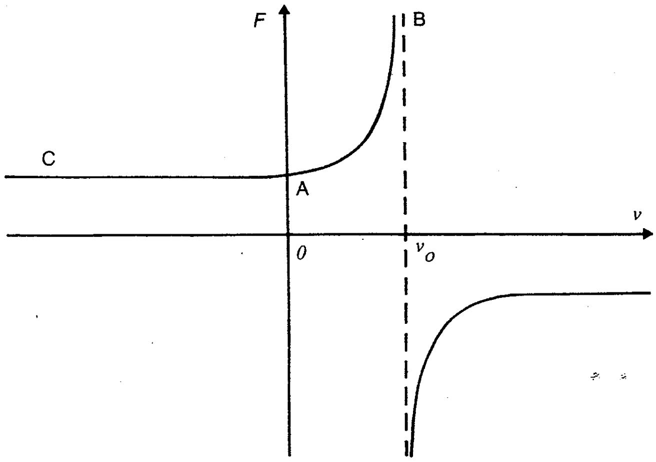

Figure 4: The relation between frictional force F and string velocity v for a bow drawn with velocity v_o across the string.

jump of nearly two octaves in the vibration pitch — can, in fact, be heard in the decaying vibrations of some dented flat-sided tin cans. It is saved from physical unreality as $\phi \rightarrow 0$ by the fact that thickness can no longer be neglected if it is comparable with L sin ϕ.

Nonlinearities are real, nonlinearities are often extremely important and, if we treat them right, they do not get out of hand.

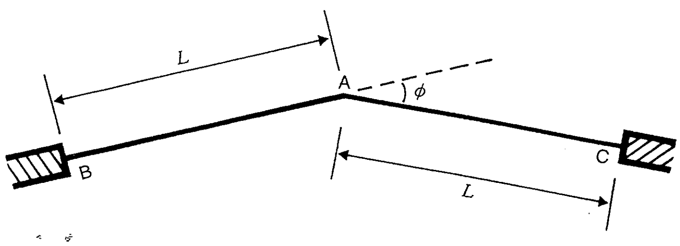

Figure 5: A thin bar (or plate) kinked (or creased) at a small angle ϕ.

EPILOGUE

As a final pathological nonlinearity let me return to consider a passive vibrating system consisting of either a very lightly creased thin plate or a slightly kinked thin bar held between clamps as shown in Figure 5. If the oscillation amplitude is very small, then simple analysis shows that the point A must remain fixed to first order so that the first and second modes will be respectively the antisymmetric and symmetric vibrations of the two half-bars of length L. However, as the vibration amplitude increases, nonlinear effects of the kind we have discussed before will move the point A downwards by an amount proportional to $a_n^2(1 + \cos 2\omega_n t)$ where ω_n is the frequency and a_n the amplitude of the mode involved. When a_n becomes large enough, the point A will approach the line BC and the bar will then be able to sustain an additional mode in which it vibrates as a whole as a bar of length 2L. The frequency of this mode is only about one quarter of that of the previous fundamental.

All this is not surprising when the angle ϕ is reasonably large. What does give cause for wonder is that this transition behaviour is confined to an amplitude range of order L sin ϕ, which clearly approaches zero as ϕ approaches zero, giving us another pathological or essential nonlinearity. This effect — a jump of nearly two octaves in the vibration pitch — can, in fact, be heard in the decaying vibrations of some dented flat-sided tin cans.

REFERENCES

1 Morse, P.M. and Ingard, K.U. "Theoretical Acoustics", McGraw-Hill, New York 1968, Chapter 14.
2 Fletcher, N.H. "Nonlinear frequency shifts in quasispherical-cap shells: Pitch glide in Chinese gongs", J. Acoust. Soc. Amer. *78* 2069-2073 (1985).
3 Backus, J. "Small vibration theory of the clarinet", J. Acoust. Soc. Amer. *35* 305-313 (1963).
4 Fletcher, N.H. "Excitation mechanisms in woodwind and brass instruments", Acustica *43* 63-72 (1979).
5 Benade, A.H. and Gans, D.J. "Sound production in wind instruments", Ann. N.Y. Acad. Sci. *155* 247-263 (1968).
6 Backus, J. and Hundley, T.C. "Harmonic generation in the trumpet", J. Acoust. Soc. Amer. *49* 509-519 (1971).
7 Coltman, J.W. "Sounding mechanism of the flute and organ pipe", J. Acoust. Soc. Amer. *44* 983-992 (1968).
8 Thwaites, S. and Fletcher, N.H. "Acoustic admittance of organ pipe jets" J. Acoust. Soc. Amer. *74* 400-408 (1983).
9 Fletcher, N.H. "Nonlinear interactions in organ flue pipes", J. Acoust. Soc. Amer. *56* 645-652 (1974).
10 Fletcher, N.H. and Douglas, L.M. "Harmonic generation in organ pipes, recorders,and flutes", J. Acoust. Soc. Amer. *68* 767-771 (1980).
11 Schelleng, J.C. "The bowed string and the player" J. Acoust. Soc. Amer. *53* 26-41 (1973).
12 Cremer, L. "The Physics of the Violin" MIT Press, Cambridge Mass. 1984.

Terminal 4 — Heathrow

The opening of Terminal Four (T4) at Heathrow Airport in April 1986 will have been treated by regular air travellers with a great sigh of relief, as it will no doubt decrease the congestion experienced in the other terminals during the past few years. The prestigious new building on the south-eastern perimeter of the airport will cater for up to eight million passengers per annum right from the start, and will service all British Airways overseas flights as well as their Paris and Amsterdam routes, and all KLM and Air Malta flights. The opening of the terminal may, however, have been greeted with rather less enthusiasm by the residents in nearby Bedfont and Stanwell, except in so far as it meant the completion of several years major construction work.

During the Public Inquiry on the planning application for T4, held between May and December 1979, the most important single issue related to noise. There was concern firstly, about any increase in air traffic which might result, and secondly, about ground movements and activities around T4, which unlike the existing terminals was to be situated at the edge of the airport and close to a built-up area. The Inspector recognised the potential problem as presented to him in the evidence given by the GLC and other local authorities, and made a number of recommendations, most of which were accepted, some in a modified form, by the Secretary of State and included in the planning conditions.

Regarding total air traffic he recommended that annual Air Transport Movements (ATMS) should be limited to 260,000 once T4 was opened. The Secretary of State changed this to 275,000 but following the inquiry on **Terminal Five** the government decided in the White Paper on Airport Policy to scrap the limit altogether. The latest figures indicate over 285,000 ATMS per annum at Heathrow.

Many positive steps have however been taken to implement the Inspector's recommendations. These include the construction of 7m high concrete noise barriers, with a total length of 1.2km, covering the two aprons on the 'land' side of the airport, and some of the taxiways. Furthermore all maintenance work involving the running of aircraft engines at T4 throughout the day was prohibited. On the operational side conditions were set for T4 banning aircraft movements, the running of aircraft engines, and the use of APUs (auxiliary power units) between 23.30 and 6.30 hours. Subsequently however, following an appeal by the British Airports Authority and a further local inquiry, these conditions were relaxed to exclude aircraft on or taxing to and from the apron on the airport side of the terminal building. This relaxation was agreed on the basis of a three year experiment during which time the BAA and local authorities would monitor night time aircraft noise in the residential area. Continuous monitoring of aircraft noise is now taking place at sites in East and West Bedfont, and this is supported by occasional sampling at other locations in that area.

(Extracted from "T4 Up and Running" by George Vulkan, published in London Environmental Bulletin, Vol 3, No 4, 1986.)