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THE NONLINEAR ACOUSTICS OF MUSICAL INSTRUMENTS

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ABSTRACT. While linear considerations suffice to determine many of the design and performance parameters of musical instruments, such as the shapes of horns, the placing of finger holes, and the resonances of vibrating structures, it is essential to take into account the nonlinearity of the system in order to understand such things as radiated sound power, relative intensity of harmonics, and attack transients. This paper discusses some aspects of nonlinear behaviour, with particular reference to wind instruments.

1 INTRODUCTION

Musical instrument makers have a practical tradition which defines the structural features of their instruments in a completely adequate way so that they have good intonation and tone quality, though there are subtle differences between those produced by different craftsmen. In the case of percussion instruments and plucked-string instruments nonlinear effects are not usually of great importance, though exceptions to this remark occur for certain Chinese gongs and for instruments like the sitar. In these cases the sound quality and even the pitch depends on the amplitude of the vibration, so that nonlinearity is showing its presence.

Instruments with sustained tones, however, such as violins, trumpets or flutes, rely very directly upon nonlinearity to determine the loudness and harmonic content of their sound, and indeed also to lock together their various possible vibrational modes into a coherent and harmonic whole. The violin, in which Raman was so interested, relies for its sound production mechanism upon the nonlinear relation between frictional force and relative velocity, giving in its extreme case the stick-slip motion and characteristic waveforms which have become so well known. Not so well known are the nonlinearities of wind instruments, and it is upon these that my paper will concentrate.

In its essentials, a musical wind instrument consists of a very nearly linearly behaved resonator, the air column in the horn of the instrument with its associated finger holes or valves, closely coupled to a nonlinear acoustic generator, the reed, lips or air-jet produced by the player(1). To be complete we must include the player as well, as shown in Fig. 1, but rather surprisingly the acoustic output, which is the whole reason for the exercise, appears as only a small perturbation which consumes only about 1 percent of the input pneumatic energy. Complex feedback loops, some of them involving the player, are an essential feature of the system and largely determine its performance.

In most of the discussion below we will concern ourselves with the internal acoustics of the instrument, since radiation is such a small component of the energy balance. The transfer function from internal to external spectrum and waveform involves an emphasis on high frequencies, to the extent of 6 dB/octave, up to the radiation cut-off frequency, above which the internal and external spectra are the same. This simply reflects the behaviour of the resistive component of the radiation impedance at the open horn or open holes of the instrument, the cut-off frequency being determined by the size of these openings.

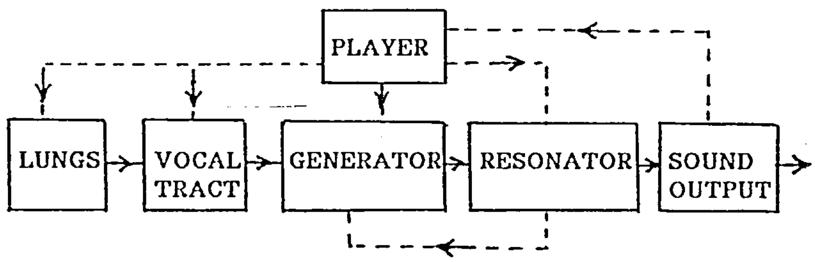

Fig. 1 The system diagram for a musical wind instrument, showing the couplings and feedback loops.

Table 1 analyses the behaviour of the system in terms of the relative influence of its various components, with the influence of the generator being separated into linear and nonlinear aspects. It is clear that the loudness and tone quality of the acoustic output are largely controlled by nonlinear effects, and these quantities are second in importance only to the pitch of the note being played. It is the purpose of this paper to examine, in a fairly general way, the particular aspects of nonlinearity that are important in this context and to show how the acoustic behaviour to be expected from a particular type of instrument can be calculated.

TABLE 1 Importance of Various mechanisms

	Resonator	Generator		Vocal Tract
	(Linear)	Linear	Nonlinear	(Linear)
Pitch of note	S	S W		W
Amplitude	W	S	S	W
Spectrum	W		S	W
Transients	W	W	W	W

S = strong influence; W = weak influence

2 REED GENERATORS

2.1 Linear Theory

Pressure-driven reed generators, such as those found in woodwind instruments of the clarinet, saxophone, oboe and bassoon families, are perhaps the simplest of all to analyse. The clarinet reed has received most attention because its geometry is particularly simple and easily reproducible, which is less true of double reeds. Let us first outline the linear theory as a background(2).

Fig 2a shows the essentials of a clarinet-like reed generator. A thin flexible cane reed is clamped against an opening in a slightly curved mouthpiece in such a way as to leave a narrow opening through which the air can pass. The player's lips seal the mouthpiece and reed and the blowing pressure p_0 in the player's mouth tends to force the reed closed against the lay of the mouthpiece. The acoustic pressure p inside the mouthpiece, conversely, tends to force the reed open. The double reed generator of an oboe or bassoon operates similarly, but the two leaves of the reed close against each other rather than against a rigid mouthpiece. The reed itself is essentially a tapered elastic plate and can be excited to vibrate in one of its normal modes. In all ordinary playing, only the lowest cantilever mode is excited, with the reed tip free and its base clamped against the mouthpiece. The natural frequency of this mode is ordinarily much higher than the frequency of the fundamental of the note being played. It is this feature which makes the first-order analysis particularly simple, for the displacement of the reed is essentially proportional to the difference in the pressures on its two sides and there is very little phase shift.

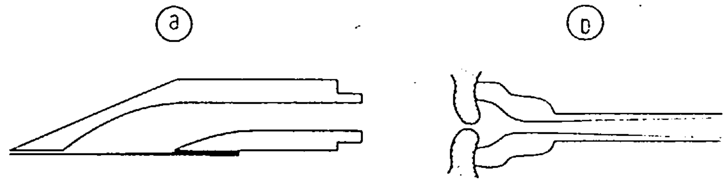

Fig. 2 The physical arrangement of (a) the reed in a clarinet, blown closed by the mouth pressure p_0, and (b) the physical arrangement of the lips and mouth cup in a brass instrument, in which the lips are blown open.

If x measures the opening at the reed tip, then the volume flow U from the mouth into the instrument through the reed is essentially

$$U \approx Bx(p_0 - p)^{1/2} \approx B[x_0 - s(p_0 - p)](p_0 - p)^{1/2} \qquad (1)$$

where B is a constant proportional to the width of the reed, x_0 is the equilibrium opening when $p_0 = p = 0$, and s is the elastic compliance of the reed for cantilever deflection under a pressure difference. All these quantities can be fixed in magnitude and refinements added to account for things such as the curvature of the lay of the mouthpiece, but this need not concern us here. The important thing is the general shape of the flow curve, as shown in Fig. 3. The flow first increases as the pressure difference is increased, goes through a maximum at the point A, and then decreases to zero at C where the reed is completely closed. The acoustic conductance of the generator as seen from inside the instrument is just the slope of this flow curve, and so is negative in the region AC. If the blowing pressure p_0 is adjusted to bring the operating point close to O in the middle of the negative-resistance region then playing conditions will be optimal and the generator will feed energy into the vibrational modes of the air column of the pipe resonator.

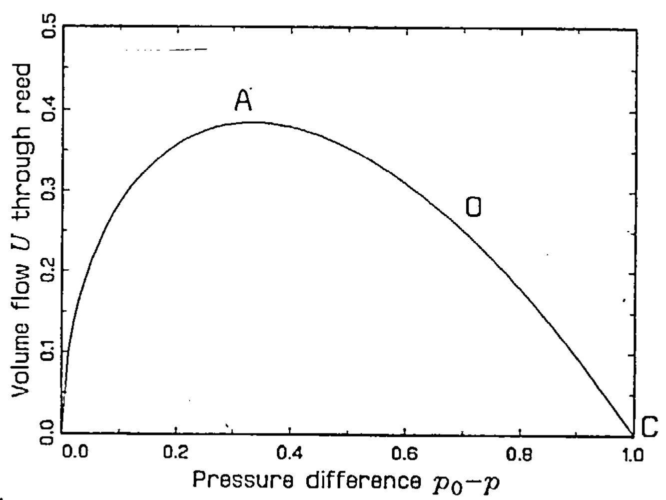

Fig. 3 The steady flow U through a clarinet-like reed system as a function of blowing pressure p_0 and mouthpiece pressure p.

When we take the resonance frequency of the reed into account, the situation is a little more complex and the reed opening x is described by an equation of the form

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega_r^2 x = \frac{S}{m}(p - p_0) \qquad (2)$$

where m is the effective mass of the moving part of the reed, S is its area, and k and ω_r are respectively the damping constant and resonance frequency of the reed. To be complete we must add to the right-hand side of this equation a term representing the aerodynamic Bernoulli force on the reed, while the flow equation must be modified by adding a term describing the inertia of the air in the reed passage and another term for the flow associated with the physical displacement of the reed surface. None of these refinements makes a significant difference to the general behaviour.

The acoustic conductance for a blowing pressure in the region AC of Fig. 3 then has the form shown in Fig. 4. It is still negative at all frequencies below the reed resonance, but is positive and therefore dissipative above(3). Just below the resonance there is a maximum in the negative conductance, which is actually used as the operating point for organ reed pipes, but in woodwind instruments this resonance lies well above the fundamental of the note being played. There is evidence, however, that players tune the reed resonance by changing the tension of the lips so that it coincides with a harmonic of the note being played, thus stabilising its production and emphasising the harmonic concerned(4).

Both these techniques are important for playing high notes, since the air column generally has several lower resonances available to it(5), and the reed can potentially interact with any of them. It is here that nonlinearity also becomes important, as we shall see presently.

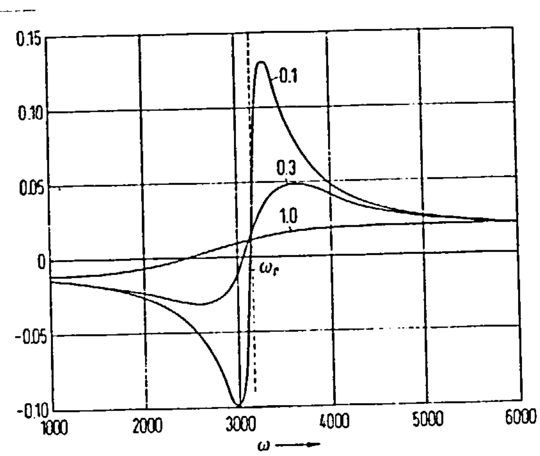

Fig. 4 Acoustic conductance of a clarinet-like reed system as a function of frequency.

2.2 Nonlinearity

While the conductance characteristic shown in Fig. 3 is linear if the pressure p makes only small excursions around the operating point O, this is clearly no longer true if p becomes an appreciable fraction of p_0. Fig. 5 shows this explicitly by giving the flow waveforms for sinusoidal pressure excursions p of various magnitudes. It is clear that the flow is limited at both ends of the pressure excursion, though the clipping is "hard" when the reed beats against the mouthpiece and much "softer" when it simply opens past point A on the curve of Fig. 3. While the waveforms in Fig. 5 do have a close resemblance to those produced inside a real clarinet, the calculation is more involved than we have suggested so far, since the flow waveform must be allowed to act upon the acoustic impedance of the resonator to produce the pressure waveform inside the mouthpiece which then drives the reed.

Before we go on to calculate the behaviour in detail, we can see immediately that the nonlinearity does two things. It certainly produces harmonics of the fundamental frequency – all harmonics are present but there is a tendency for odd harmonics to be emphasised because of the nearly symmetrical nature of the clipping. The final amplitude of these harmonics will, of course, be considerably influenced by the nature of the resonances of the air column of the instrument.

The second feature of the nonlinearity is that it imposes a limit on the maximum amplitude of the flow waveform, and can therefore be expected to determine in large measure the maximum amplitude of the pressure wave in the air column and thus the sound power radiated from the instrument.

Two different approaches have been developed to describe in detail the nonlinear behaviour of the instrument, and these will be outlined below. When taken to completion they are mathematically equivalent, and so give the same answers, but each gives a rather different physical picture of the processes involved, and different approximations are more easily made. The first approach treats everything in the frequency domain, where the behaviour of the pipe resonator is simple and that of the reed generator complex; the second uses the time domain, where the situation is reversed. We discuss these in turn.

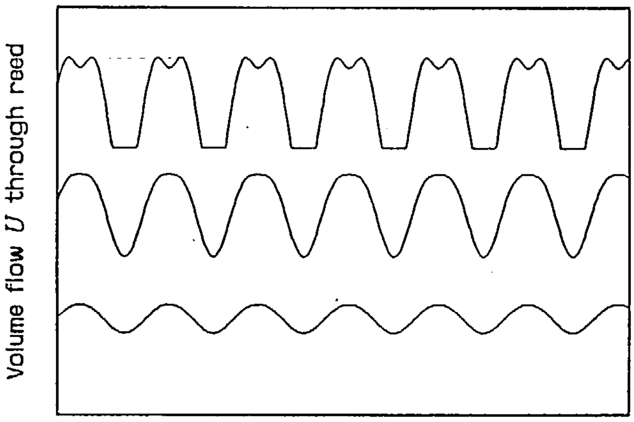

Fig. 5 Flow waveform for a clarinet-like system with an assumed sinusoidal mouthpiece pressure, for various pressure amplitudes.

2.3 Nonlinear Theory in the Frequency Domain

The behaviour of the linear air column resonator can be expressed in terms of its pressure eigenfunctions (or normal modes, or standing waves), which we write in the form $\Psi_n(y,t)$ where y is a coordinate measuring distance along the air column from the reed at $y = 0$. All these modes have a pressure maximum near to the reed end of the instrument, and the pressure acting on the reed is essentially just the sum of their individual contributions. If we assume that the nth mode has pressure amplitude a_n at the reed, natural frequency ω_n and phase ϕ_n, then we can show(6,9) that the flow $U(p)$ drives these modes away from their initial states according to the equations

$$\frac{da_n}{dt} \approx \frac{\mu_n}{2\pi\omega_n}\int_0^{2\pi}\frac{dU}{dt}\cos\theta_n d\theta_n - ka_n \qquad (3)$$

$$\frac{d\phi_n}{dt} \approx -\frac{\mu_n}{2\pi a_n \omega_n}\int_0^{2\pi}\frac{dU}{dt}\sin\theta_n d\theta_n \qquad (4)$$

where $\mu_n \approx \rho c \omega_n / S$, S being the pipe cross section, and we have written for convenience

$$\theta_n = \omega_n t + \phi_n . \tag{5}$$

The integrals in (3) and (4) are intended to remove all high frequency terms so that the amplitudes and phases of the normal modes vary only slowly with time and quickly settle down to steady values. Since U in the integrals depends on the total pressure and thus on the sum of all the mode pressures, there is a reasonable amount of complication about the solution, but the final result is a set of steady modes with frequencies

$$\omega(n) = \omega_n + d\phi_n / dt \tag{6}$$

which are all locked into exact harmonic relationship(10). It is this consequence of the nonlinearity that gives to wind instruments their exactly repetitive harmonic waveforms and hence their usefulness for ordinary musical performance. It is only in the case of air column resonances that are very far from integral relationship that this coupling breaks down, giving the peculiar "multiphonic" tones now being exploited by contemporary composers.

2.4 Nonlinear Theory in the Time Domain

The solution in the frequency domain, outlined above, has a counterpart in the time domain which is simpler in concept though usually more difficult in application to wind instruments(11). Description of the motion of the reed and of flow through it is relatively simple in the time domain, since this description is given by the equations (1) and (2) which are formulated in terms of the time-domain functions $p(t)$ and $U(t)$. To treat the air-column resonator in the time domain we need to know its impulse response, or Green function, $G(t-t')$ which describes the pressure response at time t when a sharp impulsive flow is injected at time t'. If the injected flow has the more general form $U(t')$, then the pressure response is

$$p(t) = \int_{-\infty}^{t} G(t-t')U(t')dt' . \tag{7}$$

The difficulty arises from the fact that, since G is defined with the mouthpiece end of the air column rigidly closed except at the instant $t = t'$, so that U will be zero, the pressure pulse produced by the impulse is reflected back and forth between the open and closed ends of the tube for a very considerable time. Stated in another way, the impulse response is actually just the Fourier transform of the input impedance, and if the impedance function has sharp peaks in the frequency domain, then its transform has a large extent in the time domain. The convolution integral (7) is therefore very extended and laborious to evaluate.

Schumacher(12) has suggested a way out of this dilemma by recognising that there are initially no reflections when a flow pulse is injected into the air column, and the initial pressure response is just that due to the characteristic impedance Z_0 of the air column at its input. We can therefore write

$$p(t) = Z_0 U(t) + \int_0^{\infty} r(t')[Z_0 U(t-t') + p(t-t')]dt' \qquad (8)$$

where $r(t)$ is the impulse response when the horn input is blocked with a non-reflecting termination after the initial pulse is injected. In general $r(t-t')$ will have a much shorter time span than $G(t-t')$, though it may still be complicated by multiple reflections from finger holes and other irregularities in the bore. For a simple smooth cylindrical tube, $r(t)$ has the form of an inverted and slightly attenuated pulse, delayed by the transit time along the pipe and back - the integral is very simple to evaluate in this case.

3 LIP GENERATORS

3.1 Linear Theory

There are important similarities between the behaviour of lip-driven instruments of the brass family, the generator mechanism of which is shown in Figure 2b, and reed-driven instruments of the woodwind family. There are also important differences which arise from the fact that the blowing pressure tends to force the player's lips open, while in woodwinds it tends to force the reed closed. This appears as a change in the sign of (p_0-p) on the right-hand side of both equations (1) and (2) above. This has the consequence that the static flow characteristic $U(p_0-p)$ simply rises with (p_0-p) and has no negative-resistance region. The form of the acoustic conductance at non-zero frequency is similarly inverted compared with the reed case shown in Figure 4, and now has the shape shown in Figure 6. The region of negative conductance associated with operation as an acoustic generator is limited to a narrow frequency range just above the resonance frequency of the mechanical vibration of the lips(3).

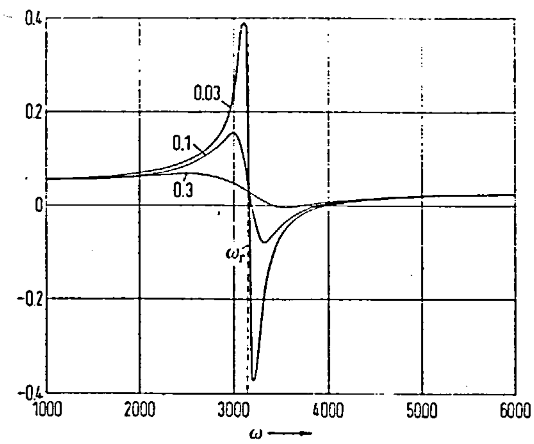

Fig. 6 Acoustic conductance of a vibrating-lip generator, as in the trumpet, as a function of frequency.

The consequences of this behaviour are immediate. Clearly the player must adjust the lip resonance frequency to coincide quite closely with the frequency of the note to be played, and this frequency must also be very close to an impedance maximum of the instrument air column, or rather of the instrument in series with the player's vocal tract, in order that an adequate mouthpiece pressure can be built up to control the reed. Balance between the reactive parts of the lip and instrument impedances will control the exact sounding frequency. .The lip resonance can fortunately be made very narrow, so that notes lying only a semitone apart, and represented by adjacent resonances in the upper register of an instrument such as the French horn, can be selected with good reliability by a skilled player.

The input impedance of a typical brass instrument is determined by the length and profile of the horn itself and by the Helmholtz resonance of the cavity and back-bore of the mouthpiece cup(13). The exact frequencies of the horn resonances depend upon the flare at its bell end and are generally adjusted to produce an accurate integer-harmonic series, though with the first-mode resonance unavoidably very much below its nominal frequency. The player selects any one of these resonances as the fundamental of the note to be played, and intermediate notes can be produced by shifting the whole frequency scale downwards by adding lengths of tube at the mouthpiece end of the instrument using valves or a slide.

There is an interesting feature of brass instrument performance which arises from the considerable length of the horn - several metres in a typical case. The travel time for the first pulse of a note to propagate from the player's lips to the open end of the bell and then reflect back to the lips may be many periods of the lip vibration when a high note is being played. The player must thus adjust lip and vocal-tract resonances to produce a self-sustaining vibration during this time, and a characteristic attack transient will evolve.

Once again, the radiated power, frequency spectrum and transient behaviour of the instrument are all dominated by nonlinear effects, and these can be treated either in the frequency domain or in the time domain. The formal development is identical with that outlined above for reed-driven instruments, so that all that is required is a set of comments on the results.

3.2 Nonlinear Behaviour in the Frequency Domain

Description of the behaviour of the instrument mouthpiece and horn, together with the player's vocal tract, is relatively simple in the frequency domain, as we have outlined above. The behaviour of the player's lips is described by a driven harmonic oscillator equation (2), as for a reed but with the sign of the driving term reversed. The resulting motion is now much simpler, however, since the driving pressure is close to the reed resonance, giving large amplitude, while all harmonics are well above this resonance frequency and produce very little disturbance to the motion. This has been verified by stroboscopic study, the lips being found typically to just close once in each cycle(14).

With the lip motion $x(t)$ determined in frequency and form, though not in amplitude, it is now simple to write down the flow U in terms of the pressure difference (p_0-p), using the modified form of equation (1) appropriate for a lip generator. The flow $U(t)$ can be decomposed into harmonic frequency components, each of which lies very close to a resonance of the instrument and therefore interacts with a nearly purely resistive input impedance R, which we can assume to be nearly the same for the first few harmonics. All this leads to an equation of the form

$$p \approx RU \approx \frac{(R\beta x)^2}{2}\left\{\left[1+\frac{4p_0}{(R\beta x)^2}\right]^{1/2}-1\right\} \qquad (9)$$

where β is a constant related to the mass, width and compliance of the lips. If the blowing pressure is moderate and the lip opening large, as is usually the case for moderately loud playing,then $4p_0 \ll (R\beta x)^2$, and the approximate solution to (9) is

$$p \approx RU \approx p_0 - p_0^2/[R\beta x(1+\sin\omega t)]^2 \qquad (10)$$

The shape of this waveform is shown in Figure 7, and is very similar to that actually observed for a trombone(15). The waveform is clearly non-sinusoidal and contains overtones in significant amplitude, though they are not as marked as in the case of a reed instrument.

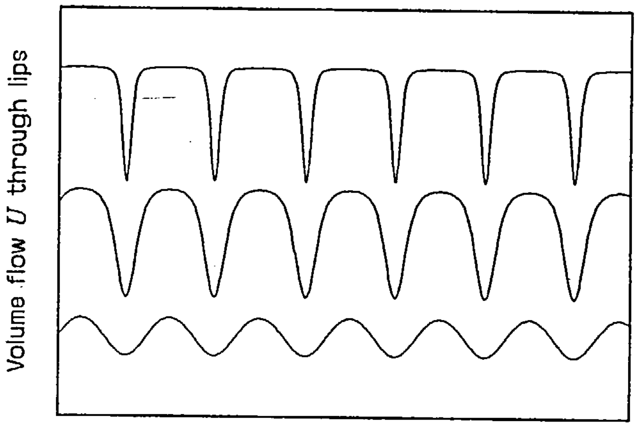

Fig. 7 Calculated flow waveforms for a brass instrument at several amplitudes of motion of the player's lips.

This treatment is only a first approximation. To derive a more accurate waveform and amplitude we must solve the pressure and flow equations in the frequency domain properly, as was indicated for reed instruments. The process has a large computational load, but is essentially straightforward. This treatment will give the form of the initial transient as well as that of the steady state.

3.3 Nonlinear Behaviour in the Time Domain

An instrument of the brass family is in many ways more suitable for treatment in the time domain than is a woodwind instrument, since the smooth bore of the horn produces mostly a reflection associated with the sharp flare and open end of the bell. The nature of this bell reflection is, however, quite complicated, because different frequencies are reflected by the flare at different places, giving a complex shape to the reflected pulse. This corresponds, in the frequency domain, to the adjustment of mode frequencies accomplished by the bell(16).

The form of the analysis is very similar to that for a woodwind instrument, except that explicit note must be taken of the autonomous nature of the lip vibration. Clearly the time-domain approach gives a very direct description of the initial transient, since the integral in (8) is zero until the first part of the pulse returns from the bell, and during this time the mouthpiece pressure is simply $Z_0'U(t)$, where Z_0' is the characteristic impedance of the horn at its input, modified by the Helmholtz resonance effect of the mouthpiece. No waveforms for brass instruments calculated in this way have been published, to our knowledge, but it is not expected that they would lead to any surprises.

4 AIR-JET GENERATORS

4.1 Linear Theory

To complete this survey of wind instruments, we must look at those of the flute family, which are excited by an air jet blown across an aperture and impinging on a lip at the opposite side. Such a mechanism relies upon the deflection of the jet by acoustic flow at the aperture and is therefore described as a flow-controlled generator, in distinction from the pressure-controlled reed and lip generators.

When a plane jet (either laminar of turbulent) emerges from a slit into a transverse acoustic flow field, it is deflected as shown in Figure 8 and this deflection propagates along the jet with a phase velocity equal to about half the jet velocity. The amplitude of the disturbance grows, provided that its wavelength is greater than about 5 times the jet thickness; for shorter wavelengths the disturbance is attenuated as it propagates(17). The growth is exponential for small amplitudes, but becomes linear when the amplitude becomes comparable with the wavelength, the wave-like disturbance then breaking up into a complex vortex street.

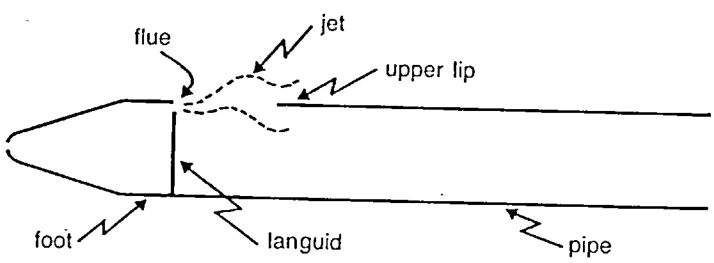

Fig. 8 *Geometry of the mouth of a flute-like instrument with a fixed windway, showing the sinuous disturbance growing on the jet as a result of the influence of transverse acoustic flow past the slit form which the jet emerges.*

The interaction of such a sinuous jet with an edge forming part of a resonator has been widely studied, and the phenomena are now moderately well understood(18-21). The details of this drive mechanism are too complex to set out here, but the essence of the driving force is the volume flow of the jet into the pipe, which has a form like that shown in Figure 9, saturating both when the flow is completely into the pipe and completely outside the pipe. Detailed consideration shows that the shape of this curve is fairly closely a hyperbolic tangent function, though any similarly shaped curve could be assumed. We return to the nonlinearity in the next section - for the present we simply note that the driving force is linear for small excursions about the operating point O.

The important thing about the jet mechanism is that it has a built-in time delay because of the transit time of wave-like displacements from the slit or flue to the edge or lip. For a typical blowing pressure of a few kilopascals, the jet velocity is about 50 m/s and the wave speed about half this, so that the transit time from the slit to the lip, typically a distance of 5-10 mm, is a fraction of a millisecond. This represents a very significant, and blowing-pressure dependent, phase shift in the excitation function. When all details are taken into account, it turns out that the linear acoustic conductance of the jet, as seen by the air column, is negative when this phase shift is around 180°. The player can therefore select the pipe mode to be excited by varying the blowing pressure and, in the case of lip-blown flutes, the length of the jet - this latter adjustment is not available, of course, in instruments such as the recorder(22).

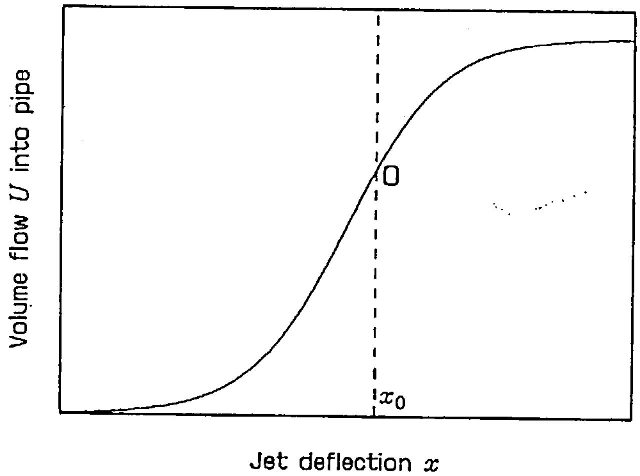

Fig. 9 Form of the acoustic flow into the mouth of a flute-like instrument as a function of jet displacement x away from the static offset position x_0.

4.2 Nonlinear Behaviour

We can use the same techniques as outlined in Sections 2 and 3 to treat the nonlinear sound production mechanism in air-jet driven instruments. We shall not go into this in detail, but rather just look at the first approximation. To do this we note that, because of the phase shifts and attenuation of high frequencies on the jet, only the fundamental of the note being played is driven directly, with higher harmonics being produced by the nonlinearity. Detailed consideration of this situation shows that the relative strength of the harmonics is strongly influenced by the extent to which the pipe lip is offset relative to the centre-plane of the jet(23). Such an offset moves the operating point O of Figure 12 away from its symmetrical condition, and the result then follows from expansion of the flow function

$$U(t) = B\tanh\left[x_0 + a\sin(\omega t)\right] \tag{11}$$

where x_0 defines the operating-point offset from the centre-plane of the jet and a is the amplitude of the jet displacement at the lip. The amplitudes of the upper harmonics produced by the driving function increase with increasing jet deflection amplitude, and their relative strengths vary with jet offset as shown in Figure 10. To convert this driving function to response of the resonator we need to multiply by the air column admittance at each frequency. In most flute-like instruments the resonances are well aligned, and the upper partials are well developed, though the "softness" of the nonlinearity in Figure 9 means that they are much less fully developed than in reed or lip-driven instruments.

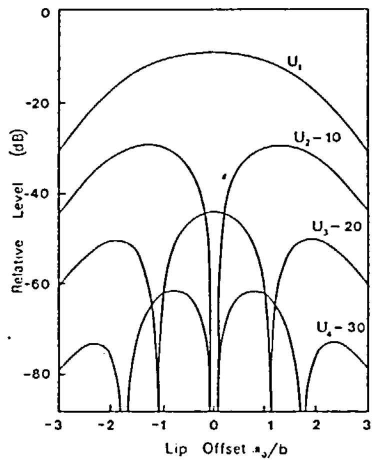

Fig. 10 *Relative amplitude of the harmonics produced by a driven air-jet as a function of the offset of the lip from the centre plane of the jet.*

A full treatment of the nonlinearity and its interaction with the pipe modes, including transient effects, can be carried out essentially as outlined in Section 2, but in a dual way, in the sense that flows replace pressures and admittances replace impedances. Initial transients in jet-driven instruments typically occupy 20 to 40 periods of the fundamental of the note being played.

5 CONCLUSION

In this review I have had time only to touch briefly upon the important role played by nonlinearity in determining the sound output of musical wind instruments, and to show in outline how one may reach a quantitative understanding of the steady waveforms and transient effects that are so important in giving each instrument its characteristic auditory effect.

In concentrating on nonlinearity, I have not meant to belittle the role of linear effects or of linear theory in determining instrument behaviour - indeed, since nonlinearity is always with us, do what we may, it is not unreasonable to take the approach of looking after linear effects in instrument design and leaving nonlinearities to take care of themselves! Be that as it may, it is the nonlinearities, as I have tried to show, that win in the end.

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