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ASYMPTOTIC ANALYSIS OF THE FREE IN-PLANE VIBRATIONS OF BEAMS WITH ARBITRARILY VARYING CURVATURE AND CROSS-SECTION

T. Tarnopolskaya, F. de Hoog

CSIRO Division of Mathematics and Statistics, GPO Box 1965, *Canberra ACT* 2601, *Australia*

AND

N. H. Fletcher and S. Thwaites

CSIRO Divison of Applied Physics, PO Box 218, *Lindfield NSW* 2070, *Australia*

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An asymptotic analysis is carried out for the equations of free vibrations of a beam having varying curvature and cross-section. The effect of splitting the asymptotic limit for eigenvalues into two families is revealed and its connection with boundary conditions is discussed. The analysis of the properties of the asymptotic solution explains the phenomenon of transformation of mode shape with change in curvature and provides a method for predicting the spectrum of curved beams. The asymptotic solution obtained also gives a simple approximation for high mode number extensional vibrations of curved beams which are difficult to analyse by other means. The asymptotic behaviour of the solution is illustrated numerically for different types of curvature including antisymmetric curvature. An experimental verification of the asymptotic behaviour of mode frequencies is presented.

1. INTRODUCTION

Despite the long history and large number of publications on beam vibrations, studies of beams of variable curvature and cross-section are limited. In fact, the work in this area relates to the last two decades, and references [1–7] provide a useful starting point to the literature. A particularly useful reference is [1] which contains a detailed overview of the literature on the topic.

Several authors [1, 3, 6, 8, 9] have reported on a transformation phenomenon which is characterised by the sharp increase in frequencies of some modes that occurs at certain combinations of curvature and length of the beam and is accompanied by a significant change in the mode shape. As a result, the usual mode sequence in the spectrum can be changed. The understanding of this phenomenon is very important for correct selection of the supporting or connecting points for beam-type structures in practical applications. The possibility of the interpretation of the behaviour of frequency with change in curvature in terms of two separate approximate theories (membrane and bending theories) has been suggested in [9], where the vibration of a particular case of an S-shaped strip of uniform cross-section has been studied. There is, however, still no comprehensive analysis of the transformation phenomenon for even the simplest geometry of a beam, and, as a consequence, there are no proper explanations and methods for prediction for the spectrum of a curved beam. This is possibly due to the fact that numerical simulations,

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commonly employed for the analysis, provide little analytical insight into the vibrational problem. Asymptotic methods, which prove to provide a very useful insight into the physics of the phenomenon, seem, however, not to have been applied so far.

In the present paper, the theory of the transformation phenomenon in vibrational behaviour of a beam of varying curvature and cross-section with change in curvature is developed on the basis of the asymptotic analysis of the equations of free vibrations. A non-dimensional parameter proportional to the squared ratio of the characteristic thickness of the beam to its length is considered to be small. There is a feature of the problem that puts it beyond the scope of the traditional singular perturbation analysis and does not allow the application of a conventional expansion. Specifically, the eigenfunctions of the perturbed problem differ from those of the unperturbed problem by an oscillatory term which does not vanish in the limit as the perturbation parameter tends to zero. The analysis of a beam of constant curvature and cross-section proves to give a useful insight into the structure of the approximate solution, which is then used for the analysis of beams of arbitrarily varying curvature and cross-section. The asymptotic analysis reveals that the structure of the solution together with certain types of boundary conditions create a special feature of the eigenvalue problem, namely the splitting of the asymptotic limit for eigenvalues into two families of asymptotic limits. The analysis of the properties of eigenvalues and eigenfunctions gives an explanation of the transformation phenomenon as well as a method for predicting the spectrum of curved beams. Several numerical examples illustrating the asymptotic behaviour of the eigenvalues and the accuracy of the asymptotic approximation for beams of thickness to length ratio up to 0·01 are given in the paper. They embrace different types of curvature variation, including curvature changing sign. The effect of variation of the cross-section on the beam spectrum is illustrated by the examples of linearly varying thickness and width. Experimental verification of the asymptotic behaviour of mode frequencies is also presented.

2. GOVERNING EQUATIONS

In this section the equations of in-plane vibrations of a curved beam of varying cross-section are derived on the basis of Hamilton's principle and the assumptions of classical beam theory. An outline of the derivation is given below.

A curvilinear orthogonal system of reference associated with the undeformed beam geometry is introduced so that the position of an arbitrary point on the beam in the plane of initial curvature (which is assumed to be the plane of symmetry) is uniquely specified by the length s of the arc along the center line and the co-ordinate z along the normal to the center line (positive outward). A local Cartesian co-ordinate system $(\boldsymbol{\tau}, \mathbf{n})$ is chosen so that the unit tangent vector $\boldsymbol{\tau}$ is co-directional with increasing s and the unit vector pair $\boldsymbol{\tau}$, $\mathbf{n}$ is counterclockwise. The curvature of the center line $\kappa(s)$ is considered to be positive if $\mathbf{n}$ is directed towards the center of curvature. The position vector of an arbitrarily point in the plane of symmetry before deformation is given by

$$\mathbf{x}(s, z) = \mathbf{r}_0(s) + z\mathbf{n}, \tag{1}$$

where $\mathbf{r}_0(s)$ is a position vector of the point on the center line. The metric tensor $\mathbf{G}$ in this system of reference is $G_{11} = (1 - \kappa z)^2$, $G_{22} = 1$, $G_{12} = G_{21} = 0$. The position vector $\hat{\mathbf{x}}$ of the point after deformation at time t is given by

$$\hat{\mathbf{x}}(s, z, t) = \mathbf{x}(s, z) + u(s, z, t)\boldsymbol{\tau}(s) + v(s, z, t)\mathbf{n}(s), \tag{2}$$

where u and v are tangential and normal displacements respectively. The components of the strain tensor ε_{ij} are given by [10]

$$\varepsilon_{11} = \frac{1}{(1-\kappa z)}\left(\frac{\partial u}{\partial s} - \kappa v\right), \qquad \varepsilon_{22} = \frac{\partial v}{\partial z}, \qquad \varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{1}{1-\kappa z}\left(\frac{\partial v}{\partial s} + \kappa u\right)\right]. \quad (3)$$

Using the Bernoulli-Euler hypothesis, the displacements of the arbitrary point can be expressed in terms of the displacements of the point on the center line. So, to leading order

$$u = u_0 - z(\partial v_0/\partial s + \kappa u_0), \qquad v = v_0,$$

where u_0 and v_0 are the displacements of the point on the center line. The components of strain tensor can now be expressed as

$$\varepsilon_{11} = \frac{1}{1-\kappa z}\left[\frac{\partial u_0}{\partial s} - \kappa v_0 - z\frac{\partial}{\partial s}\left(\frac{\partial v_0}{\partial s} + \kappa u_0\right)\right], \qquad \varepsilon_{22} = 0, \qquad \varepsilon_{12} = 0. \quad (4)$$

In what follows the index will be omitted in the notations for displacements keeping in mind that the displacements are taken on the center line.

Hamilton's principle states that

$$\delta \int_{t_0}^{t_1} (T - W)\,\mathrm{d}t = 0, \quad (5)$$

where T and W are kinetic and strain energies. They are given by [10], [8]

$$W = \frac{E}{2}\int_V \varepsilon_{11}^2 (1-\kappa z)\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}y, \qquad T = \frac{\rho}{2}\int_0^l A\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2\right]\mathrm{d}s, \quad (6, 7)$$

where E is Young's modulus and ρ is the density. Rotary inertia effects have been neglected in the expression for the kinetic energy of the beam. Substituting (4) in (6) produces

$$W = \frac{E}{2}\int_0^l \left\{ A\left(\frac{\partial u}{\partial s} - \kappa v\right)^2 + Q\left[\left(\frac{\partial u}{\partial s} - \kappa v\right)^2 \right.\right.$$

$$\left.\left. - \frac{2}{\kappa}\left(\frac{\partial u}{\partial s} - \kappa v\right)\frac{\partial}{\partial s}\left(\frac{\partial v}{\partial s} + \kappa u\right) + \frac{1}{\kappa^2}\left(\frac{\partial}{\partial s}\left(\frac{\partial v}{\partial s} + \kappa u\right)\right)^2\right]\right\}\mathrm{d}s, \quad (8)$$

where $A(s)$ is the cross-sectional area, l is the length of a beam and

$$Q(s) = \int_{A(s)} \frac{\kappa^2 z^2}{1-\kappa z}\,\mathrm{d}A(s).$$

Denoting by h_0, d_0 the characteristic dimensions of the cross-section in the plane of initial curvature and in the normal plane respectively, a transformation to non-dimensional variables $\bar{s} = s/l$, $\bar{z} = z/h_0$, $\bar{A} = A/(h_0 d_0)$, $\bar{\kappa} = \kappa l$ can be made. Hamilton's principle (5) in non-dimensional form becomes

$$\delta \int_{t_0}^{t_1} \int_0^1 \left\{ C\bar{A}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2\right] - \bar{A}(u' - \bar{\kappa}v)^2 - \bar{\varepsilon}I[\bar{\kappa}^2(u' - \bar{\kappa}v)^2 \right.$$

$$\left. - 2\bar{\kappa}(u' - \bar{\kappa}v)(v' + \bar{\kappa}u)' + (v' + \bar{\kappa}u)'^2] \right\} \mathrm{d}\bar{s}\, \mathrm{d}t = 0, \tag{9}$$

where

$$C = \rho l^2/E, \quad \bar{\varepsilon} = h_0^2/l^2, \quad I = \int_{\bar{A}} \bar{z}^2\, \mathrm{d}\bar{A}.$$

In the expression for strain energy only terms of first order in $\bar{\varepsilon}$ are retained and differentiation with respect to $\bar{s}$ is denoted by a prime.

The equations of vibration can be obtained by taking the first variation of the integral in (9) with respect to the displacements. Making use of the fact that the displacements are harmonic functions of time with frequency ω, the equations of free vibrations can be written in the form

$$-\bar{\varepsilon}\{-[I\bar{\kappa}^2(u' - \bar{\kappa}v)]' + \bar{\kappa}[\bar{\kappa}I(u' - \bar{\kappa}v)]' + [\bar{\kappa}I(v' + \bar{\kappa}u)']' - \bar{\kappa}[I(v' + \bar{\kappa}u)']'\}$$

$$+ [\bar{A}(u' - \bar{\kappa}v)]' + \lambda\bar{A}u = 0,$$

$$-\bar{\varepsilon}\{-I\bar{\kappa}^3(u' - \bar{\kappa}v) - [\bar{\kappa}I(u' - \bar{\kappa}v)]'' + \bar{\kappa}^2 I(v' + \bar{\kappa}u)' + [I(v' + \bar{\kappa}u)']''\}$$

$$+ \bar{A}\bar{\kappa}(u' - \bar{\kappa}v) + \lambda\bar{A}v = 0, \tag{10}$$

where λ is a non-dimensional eigenvalue

$$\lambda = \rho l^2 \omega^2/E. \tag{11}$$

Here and henceforth u and v denote the non-dimensional amplitudes of the displacements. The boundary conditions to be considered are:

$$u = 0, \qquad v = 0, \qquad v' = 0 \qquad \text{at a clamped end,} \tag{12}$$

$$u = 0, \qquad v = 0, \qquad v'' = 0 \qquad \text{at a hinged end.} \tag{13}$$

Equations (10–13) form the basis of the present analysis.

3. ASYMPTOTIC ANALYSIS

In this section, the asymptotic analysis of the equations of vibrations of a beam with arbitrarily varying curvature and cross-section is performed under the assumption that the beam is thin. A parameter proportional to the squared ratio of the characteristic thickness and the length of the beam is considered to be a small parameter.

3.1. BEAM OF CONSTANT CURVATURE AND CROSS-SECTION

A beam of constant curvature and cross-section is first considered as it gives a useful insight into the general properties of the eigenvalue problem of interest.

3.1.1. *Structure of the solution*

First, one looks for a leading approximation of the general solution of the set of differential equations (10). Consider a beam of constant curvature and non-varying rectangular cross-section with width d_0 and thickness h_0. Equations (10) take the form

$$(u' - \bar{\kappa}v)' + \lambda u = 0, \qquad -\varepsilon(\bar{\kappa}^4 v + 2\bar{\kappa}^2 v'' + v'''') + (u' - \bar{\kappa}v)\bar{\kappa} + \lambda v = 0, \tag{14}$$

where $\varepsilon = \bar{\varepsilon}/12 = h_0^2/(12l^2)$. In the following analysis ε will be considered to be a small parameter.

One looks for a solution of (14) of the form

$$u = \gamma_1\, \mathrm{e}^{\alpha(\bar{s} - 0\cdot5)}, \qquad v = \gamma_2\, \mathrm{e}^{\alpha(\bar{s} - 0\cdot5)},$$

where α is determined by the characteristic equation

$$-\alpha^6\varepsilon - \alpha^4\varepsilon(2\bar{\kappa}^2 + \lambda) + \alpha^2(-2\varepsilon\bar{\kappa}^2\lambda - \varepsilon\bar{\kappa}^4 + \lambda) - \varepsilon\lambda\bar{\kappa}^4 + \lambda^2 - \lambda\bar{\kappa}^2 = 0. \tag{15}$$

This equation has two regular (α_1, α_2) and four singular ($\alpha_3, \alpha_4, \alpha_5, \alpha_6$) roots. The leading approximation for regular roots can be found from

$$\alpha_0^2\lambda_0 + \lambda_0^2 - \lambda_0\bar{\kappa}^2 = 0,$$

where λ_0 is the leading approximation to the eigenvalue.

In further analysis one investigates the region of the spectrum satisfying the condition

$$\lambda_0 \geqslant \bar{\kappa}^2. \tag{16}$$

Under this condition the leading approximations to regular roots are given by

$$\alpha_1 = \mathrm{i}\sqrt{\lambda_0 - \bar{\kappa}^2}, \qquad \alpha_2 = -\mathrm{i}\sqrt{\lambda_0 - \bar{\kappa}^2}.$$

Singular roots can be obtained on the basis of the principle of least degeneracy [11] using the change of variable $\alpha = \xi/\varepsilon^{1/4}$. The leading approximations for singular roots are given by

$$\alpha_3 = -\mathrm{i}\lambda_0^{1/4}\varepsilon^{-1/4}, \quad \alpha_4 = \mathrm{i}\lambda_0^{1/4}\varepsilon^{-1/4}, \quad \alpha_5 = -\lambda_0^{1/4}\varepsilon^{-1/4}, \quad \alpha_6 = \lambda_0^{1/4}\varepsilon^{-1/4}.$$

A fundamental set of real solutions can be rewritten as

$$\tilde{u}^{(1)} \sim -q \sin[q(\bar{s} - 0\cdot5)] \qquad \tilde{v}^{(1)} \sim \bar{\kappa}\cos[q(\bar{s} - 0\cdot5)] \qquad \tilde{u}^{(2)} \sim q\cos[q(\bar{s} - 0\cdot5)]$$

$$\tilde{v}^{(2)} \sim \bar{\kappa}\sin[q(\bar{s} - 0\cdot5)] \qquad \tilde{u}^{(3)} \sim (\bar{\kappa}/\beta)\sin[\beta(\bar{s} - 0\cdot5)] \qquad \tilde{v}^{(3)} \sim \cos[\beta(\bar{s} - 0\cdot5)]$$

$$\tilde{u}^{(4)} \sim (\bar{\kappa}/\beta)\cos[\beta(\bar{s} - 0\cdot5)] \qquad \tilde{v}^{(4)} \sim -\sin[\beta(\bar{s} - 0\cdot5)]$$

$$\tilde{u}^{(5)} \sim (\bar{\kappa}/\beta)(-\mathrm{e}^{-\beta\bar{s}} + \mathrm{e}^{-\beta(1-\bar{s})}) \qquad \tilde{v}^{(5)} \sim \mathrm{e}^{-\beta\bar{s}} + \mathrm{e}^{-\beta(1-\bar{s})}$$

$$\tilde{u}^{(6)} \sim -(\bar{\kappa}/\beta)(\mathrm{e}^{-\beta\bar{s}} + \mathrm{e}^{-\beta(1-\bar{s})}) \qquad \tilde{v}^{(6)} \sim \mathrm{e}^{-\beta\bar{s}} - \mathrm{e}^{-\beta(1-\bar{s})}$$

where $q = \sqrt{\lambda_0 - \bar{\kappa}^2}$, $\beta = \lambda_0^{1/4}\varepsilon^{-1/4}$. The general solution is given by

$$u = \sum_{m=1}^{6} c_m \tilde{u}^{(m)}, \tag{17}$$

$$v = \sum_{m=1}^{6} c_m \tilde{v}^{(m)}. \tag{18}$$

It can be seen that the fundamental solutions can be separated in two groups with different properties. The first group consists of the first and second solutions of the fundamental set, whose leading approximations do not depend on ε. They represent a slowly varying part of the eigenmode. The third to sixth solutions form a second group. The leading approximations of the solutions of this group contain $\varepsilon^{1/4}$ as a divisor of the arguments of the functions involved and therefore this is a group of fast variations. While the first two solutions of this group (third and fourth solutions of the fundamental set) are oscillatory functions with the region of fast variation including the whole length of the beam, the other two (fifth and sixth solutions of the fundamental set) are the boundary terms. Another property of this group is that all tangential displacements are of order $\varepsilon^{1/4}$. In view of all these considerations it is convenient to represent the general solution in the following way

$$u = u_0 + \varepsilon^{1/4} u_1, \qquad v = v_0 + v_1, \tag{19}$$

where

$$u_0 = \mathbf{u}_0 \mathbf{c}_1, \qquad u_1 = \mathbf{u}_1 \mathbf{c}_2, \qquad v_0 = \mathbf{v}_0 \mathbf{c}_1, \qquad v_1 = \mathbf{v}_1 \mathbf{c}_2,$$

$$\mathbf{c}_1^{\mathrm{T}} = (c_1 \quad c_2), \qquad \mathbf{c}_2^{\mathrm{T}} = (c_3 \quad c_4 \quad c_5 \quad c_6), \qquad \mathbf{u}_0 = (\tilde{u}^{(1)} \quad \tilde{u}^{(2)}), \qquad \mathbf{v}_0 = (\tilde{v}^{(1)} \quad \tilde{v}^{(2)}),$$

$$\mathbf{u}_1 = (1/\varepsilon^{1/4})(\tilde{u}^{(3)} \quad \tilde{u}^{(4)} \quad \tilde{u}^{(5)} \quad \tilde{u}^{(6)}), \qquad \mathbf{v}_1 = (\tilde{v}^{(3)} \quad \tilde{v}^{(4)} \quad \tilde{v}^{(5)} \quad \tilde{v}^{(6)}).$$

The form of the solution (19) is the basis of the subsequent analysis. It will be seen later that this form of the solution is valid for the general case of vibration of the beam with arbitrarily varying curvature and cross-section.

3.1.2. *Properties of eigenvalues and eigenfunctions*

In this section it is shown that the form of the solution (19) together with the specified boundary conditions yield specific properties of the solution to the eigenvalue problem. Note that the analysis of this section is applicable for a beam with varying curvature and cross-section, as the form of the solution (19) is valid for this general case (see Section 3.2).

The boundary conditions corresponding to clamped or hinged ends contain two conditions in terms of tangential displacement and four conditions in terms of normal displacement and its derivatives. They can be written in the form

$$\mathbf{D}\mathbf{c} = 0, \tag{20}$$

where

$$\mathbf{c}^{\mathrm{T}} = (c_1 \quad c_2 \quad c_3 \quad c_4 \quad c_5 \quad c_6),$$

$$\mathbf{D} = \begin{pmatrix} \mathbf{u}_0(0) & \varepsilon^{1/4}\mathbf{u}_1(0) \\ \mathbf{u}_0(1) & \varepsilon^{1/4}\mathbf{u}_1(1) \\ \mathbf{F}_0 & \mathbf{F} \end{pmatrix}. \tag{21}$$

The matrices $\mathbf{F}_0$ and $\mathbf{F}$ contain the boundary values of normal displacement and their derivatives involved in the specified boundary conditions. So the 4×4 matrix $\mathbf{F}$ contains the boundary values of normal displacements in terms of $\mathbf{v}_1$, while the rectangular 4×2 matrix $\mathbf{F}_0$ contains the boundary values of normal displacements in terms of $\mathbf{v}_0$.

The leading approximation to the eigenvalues can be found from the condition that equation (20) has a non-trivial solution

$$\det \mathbf{D}(\lambda_0) = 0.$$

As ε tends to zero, $\lambda_0(\varepsilon)$ tends to a root of the equation

$$\det \mathbf{M}(\lambda_0)\det \mathbf{F}(\lambda_0) = 0,$$

where

$$\mathbf{M} = \begin{pmatrix} \mathbf{u}_0(0) \\ \mathbf{u}_0(1) \end{pmatrix}.$$

Therefore, there are two asymptotic limits for eigenvalues, defined by the equations

$$\det \mathbf{M}(\lambda_0) = 0 \tag{22}$$

and

$$\det \mathbf{F}(\lambda_0) = 0. \tag{23}$$

Note, however, that this splitting of the asymptotic limit does not occur in the case of a beam with simply supported ends, as the boundary condition $u' = 0$ gives terms of the same order of magnitude in the first two rows of the matrix **D**.

The leading approximations for the eigenvalues defined by (22) do not depend on ε. They coincide with the eigenvalues of free vibrations of a "membrane" described by the equation obtained from (14) by setting $\varepsilon = 0$

$$u'' + (\lambda - \bar{\kappa}^2)u = 0, \tag{24}$$

with boundary conditions in terms of tangential displacements

$$u(0) = u(1) = 0.$$

The eigenvalues of this problem (called "membrane eigenvalues") are given by

$$\lambda_n^{(m)} = \bar{\kappa}^2 + \pi^2 n^2 \quad n = 0, 1, \ldots. \tag{25}$$

The first two solutions of the fundamental set coincide with the eigenfunctions of the vibration of a membrane. These solutions represent the extensional part of the eigenfunction as they originate from the extensional energy of deformation. The solution defined by (23) gives the eigenvalues of flexural vibrations of a straight beam

$$\varepsilon v'''' - \lambda v = 0 \tag{26}$$

with boundary conditions coinciding with those of the curved beam in terms of normal displacement. The last four solutions of the fundamental set coincide with the eigenfunctions of this problem and represent the inextensional (flexural) part of the eigenmode.

The following approximations for the eigenvalues of (26) (denoted by $\lambda^{(fl)}$) are valid for $n \geqslant 2$:

$$\lambda^{(fl)} = \varepsilon(\pi/4 + \pi n)^4 \quad \text{clamped-hinged beam,} \tag{27}$$

$$\lambda^{(fl)} = \varepsilon(\pi/2 + \pi n)^4 \quad \text{clamped-clamped beam,} \tag{28}$$

$$\lambda^{(fl)} = \varepsilon(\pi + \pi n)^4 \quad \text{hinged-hinged beam.} \tag{29}$$

3.1.3. *Interaction of flexural and extensional modes*

Two sets of asymptotic curves for the eigenvalues of (14) are given by the eigenvalues of membrane (25) and one of the expressions (28), (29) or (27) for the eigenvalues of flexural vibrations of a straight beam with clamped-clamped, hinged-hinged and

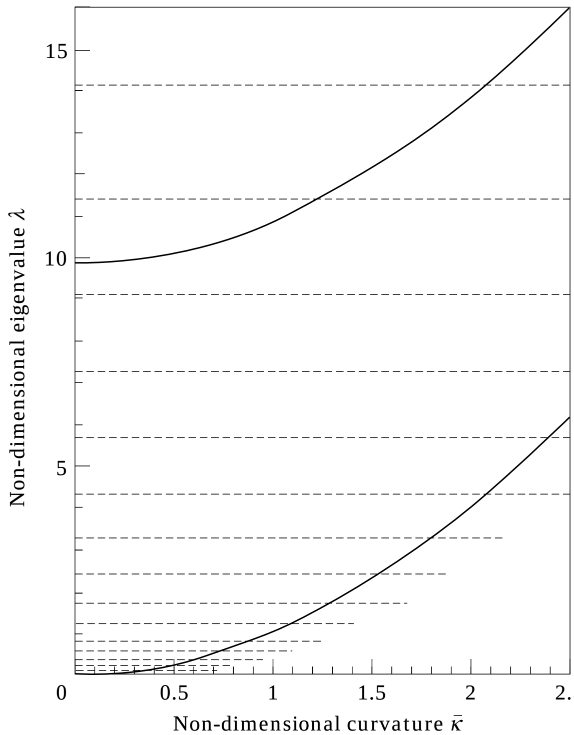

Figure 1. Asymptotic curves for the eigenvalues of a beam of constant curvature and cross-section with slenderness parameter $\varepsilon = 10^{-6}$. Note that non-dimensional eigenvalue λ is proportional to squared frequency ω^2. The broken lines show the asymptotic curves for inextensional vibration.

clamped-hinged ends respectively. As an example, the asymptotic curves for a beam of slenderness parameter $\varepsilon = 10^{-6}$ with clamped ends are shown in Figure 1 (the ‘‘membrane’’ asymptotic curves are shown with full lines, while the flexural asymptotic curves are shown with dashed lines). It can be seen that the asymptotic curves of the two families intersect at certain values of non-dimensional curvature. One would expect that the transition between flexural and extensional vibrations happens at the values of non-dimensional curvature corresponding to the intersection points. In order to understand this phenomenon the behaviour of the extensional and flexural parts of the eigenfunctions is examined as the eigenvalues approach the asymptotic curves of one or another family. The normalization condition

$$\| u \|^2 + \| v \|^2 = 1 \tag{30}$$

is introduced to define the eigenfunctions uniquely (up to a sign). It is convenient to consider the modes with symmetric and antisymmetric component v_0 separately. First consider the eigenmode with symmetric v_0. In this case $c_2 = 0$ and the order of the system (20) is reduced by one. If $\mathbf{c}_1$ is chosen to be found later from condition (30), then the remaining coefficients $\mathbf{c}_2$ can be found from the system

$$\mathbf{F}\mathbf{c}_2 = -\mathbf{c}_1\mathbf{f}_{01},$$

where the vector $\mathbf{f}_{01}$ contains the first column of the matrix $\mathbf{F}_0$. The solution for the coefficients $\mathbf{c}_2$ can be represented as

$$\mathbf{c}_2 = c_1\mathbf{c}_2^0, \tag{31}$$

where $\mathbf{c}_2^0$ is a solution of the equation

$$\mathbf{F}\mathbf{c}_2^0 = -\mathbf{f}_{01}. \tag{32}$$

It is seen that the components of $\mathbf{c}_2^0$ increase infinitely as det $\mathbf{F}\to 0$, i.e., when the eigenvalues tend to the eigenvalues of the flexural vibrations of a straight beam $\lambda^{(fl)}$

$$\mathbf{c}_2^0 \to \infty \qquad \text{as} \quad \lambda \to \lambda^{(fl)}.$$

In view of (31), the solution (19) can be represented in the form

$$v = c_1\tilde{v}^{(1)} + c_1\mathbf{v}_1\mathbf{c}_2^0, \qquad u = c_1\tilde{u}^{(1)} + \varepsilon^{1/4}c_1\mathbf{u}_1\mathbf{c}_2^0.$$

From (30)

$$\lim_{\lambda\to\lambda^{(fl)}} c_1 = \lim_{\lambda\to\lambda^{(fl)}} 1/\sqrt{\|\tilde{u}^{(1)} + \varepsilon^{1/4}\mathbf{u}_1\mathbf{c}_2^0\|^2 + \|\tilde{v}^{(1)} + \mathbf{v}_1\mathbf{c}_2^0\|^2} = 0.$$

Similarly,

$$\lim_{\lambda\to\lambda^{(fl)}} c_2 = 0.$$

This means that the extensional terms of the general eigenmode representing the eigenfunctions of a membrane disappear in the vicinity of the point of the intersection of the eigenvalues of the membrane $\lambda^{(m)} = \bar{\kappa}^2 + \pi^2 n^2$ with the eigenvalues of flexural vibrations of a straight beam, and the vibrations convert into flexural ones. This also means that $\lambda^{(m)}$ is not valid as the approximation to the eigenvalues of extensional vibrations at these points.

In the case of a beam with identical boundary conditions on both ends, the general symmetry of the problem implies that the eigenmodes can be separated into two groups—symmetric and antisymmetric. Considering them separately in the way described above, one comes to the conclusion that the extensional terms of the eigenmode, symmetric in v_0, of a curved beam disappear in the vicinity of the point of intersection of the eigenvalues of a membrane $\lambda^{(m)}$ with the eigenvalues of symetric modes of flexural vibrations of a straight beam, while the extensional terms of the eigenmode, antisymmetric in v_0, of a curved beam disappear in the vicinity of the point of intersection of eigenvalues of membrane $\lambda^{(m)}$ with the eigenvalues of antisymmetric modes of flexural vibrations of a straight beam.

Whilst the slowly varying extensional terms of eigenfunctions disappear as the eigenvalues approach the eigenvalues of flexural vibrations of a straight beam, the oscillatory terms originating from the bending energy are always present in the eigenfunctions of a beam of constant curvature. As the eigenvalues approach the eigenvalues of a membrane, the eigenfunctions represent the combination of the eigenfunction of a membrane and the eigenfunction of flexural vibrations of a straight beam with the boundary values of normal displacements and their derivatives taken as those of a membrane with the negative sign. It is possible to investigate the conditions under which the relative contribution of extensional terms is maximum, i.e. the quantities $|c_1|/|c_3|$, $|c_1|/|c_4|$, $|c_2|/|c_3|$, $|c_2|/|c_4|$ approach their maximum. For example, for the symmetric modes of a beam with clamped ends this condition is given by $\beta = \pi/2 + 2\pi m$, $m \geqslant 2$, i.e. the extensional terms approach their maximum when the eigenvalues approach the intermediate value between two successive eigenvalues of the symmetric modes of flexural vibrations of a straight beam.

As an illustration, the shape of the normal displacement v of the symmetric mode was calculated using (18) for the lowest membrane mode ($\lambda_0 = \bar{\kappa}^2$). The extensional terms of normal displacement are $\tilde{v}^{(1)} = 1$, $\tilde{v}^{(2)} = 0$. First, the value of curvature $\bar{\kappa}_0$ was taken so that λ_0 is between two consecutive eigenvalues of the symmetric modes of flexural vibrations

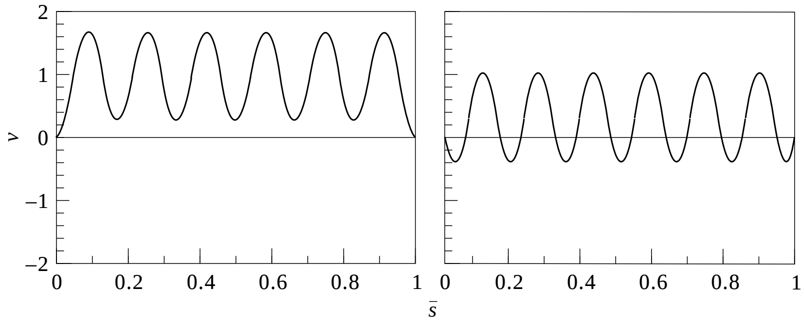

Figure 2. The shape of the normal displacement for two values of non-dimensional curvature $\bar{\kappa}$ for a beam of constant curvature and cross-section, $\varepsilon = 10^{-6}$. L.H. $\bar{\kappa} = 1{\cdot}5$; R.H. $\bar{\kappa} = 1{\cdot}75$.

of the corresponding straight beam. At this value of curvature the contribution of the extensional term is maximum (Figure 2a), the eigenfunction v is the sum of the oscillatory eigenfunction of flexural vibrations of a straight beam and a constant (eigenfunction of membrane). As the curvature departs from the value $\bar{\kappa}_0$, the eigenvalue λ_0 approaches one of the eigenvalues of the flexural vibrations of a straight beam, the amplitude of the extensional term decreases and the shape of the eigenmode becomes closer to the shape of the flexural vibration mode (Figure 2b). The number of half-waves in the oscillatory part is determined by the parameter $\beta = \lambda_0^{1/4}\varepsilon^{-1/4}$ and increases with increase in curvature ($\sim\bar{\kappa}^{1/2}$) and decrease in slenderness ($\sim\varepsilon^{-1/4}$).

3.2. BEAM OF VARYING CURVATURE AND CROSS-SECTION

Results similar to those of the preceding section can be obtained for beams with arbitrary curvature and arbitrary variation of cross-section along their length using a singular perturbation technique. Assume the following expansion of the solution to the problem (10).

$$u = \sum_{n=0}^{\infty} u^{(n)}\varepsilon^{n\mu}, \qquad v = \sum_{n=0}^{\infty} v^{(n)}\varepsilon^{n\mu}, \qquad \lambda = \sum_{n=0}^{\infty} \lambda_n\varepsilon^{n\mu}, \tag{33}$$

where $\varepsilon = h_0^2/(12l^2)$, h_0 is a characteristic thickness of the beam and μ is a positive real number. Following the results of the preceding section a leading approximation is sought as the sum of slowly and rapidly varying terms in the form

$$u^{(0)} = u_0(\bar{s}) + \varepsilon^{1/4}u_1(\xi), \qquad v^{(0)} = v_0(\bar{s}) + v_1(\xi), \tag{34}$$

where $\xi = \bar{s}/\varepsilon^{1/4}$. It is also assumed that

$$v_0 = O(1); \qquad v_1 = O(1); \qquad u_0 = O(1); \qquad u_1 = O(1); \tag{35}$$

$$\int_0^1 u_1(\bar{s})\,\mathrm{d}\bar{s}\to 0, \qquad \int_0^1 v_1(\bar{s})\,\mathrm{d}\bar{s}\to 0 \quad \text{as } \varepsilon\to 0. \tag{36}$$

and that the derivatives of v_0 and u_0 with respect to $\bar{s}$ are of the same order as the derivatives of v_1 and u_1 with respect to ξ. By substituting (34) in the first equation of (10) and equating to zero the coefficients of leading powers of ε the following equations are obtained for slowly varying and rapidly varying components

$$\varepsilon^{1/4}u_1' - \bar{\kappa}v_1 = \text{const}, \tag{37}$$

$$[\bar{A}(u_0' - \bar{\kappa}v_0)]' + \lambda\bar{A}u_0 = 0. \tag{38}$$

The derivatives here are with respect to $\bar{s}$. In view of the assumption about the oscillatory nature of u_1 and v_1 and the assumption (36) it can be seen that the constant in (37) is zero. The same procedure applied to the second equation of (10) yields

$$-\varepsilon(Iv_1'')'' + \bar{\kappa}\bar{A}(u_0' - \bar{\kappa}v_0) + \lambda\bar{A}v_0 + \lambda\bar{A}v_1 = 0.$$

By separating the oscillatory and slowly varying terms one obtains

$$\bar{\kappa}(u_0' - \bar{\kappa}v_0) + \lambda v_0 = 0, \qquad -\varepsilon(Iv_1'')'' + \lambda\bar{A}v_1 = 0.$$

Finally the equations for slowly varying components are

$$[\bar{A}(u_0' - \bar{\kappa}v_0)]' + \lambda\bar{A}u_0 = 0, \qquad \bar{\kappa}(u_0' - \bar{\kappa}v_0) + \lambda v_0 = 0 \tag{39}$$

and for oscillatory components

$$\varepsilon^{1/4}u_1' - \bar{\kappa}v_1 = 0, \qquad -\varepsilon(Iv_1'')'' + \lambda\bar{A}v_1 = 0. \tag{40, 41}$$

It can be seen that the system (39) is the equations of free vibrations of a membrane and represents the so-called "reduced problem" which can be obtained from (10) by setting $\varepsilon = 0$. Equation (41) describes the flexural vibrations of a straight beam with the cross-section varying in the same manner as for the curved beam.

It is convenient to introduce the function

$$\phi = -(1/\lambda)(u_0' - \bar{\kappa}v_0) \tag{42}$$

so that from equations (39)

$$(\bar{A}\phi)' = \bar{A}u_0, \qquad v_0 = \bar{\kappa}\phi. \tag{43}$$

These can be combined in a single equation for ϕ

$$((\bar{A}'/\bar{A})\phi)' + \phi'' + (\lambda - \bar{\kappa}^2)\phi = 0, \tag{44}$$

which represents the reduced problem.

The boundary conditions corresponding to a clamped end are

$$v_0 + v_1 = 0, \qquad v_0' + v_1' = 0, \qquad u_0 + \varepsilon^{1/4}u_1 = 0.$$

Using (43) the boundary conditions can be rewritten in terms of the variable ϕ as

$$\bar{\kappa}\phi + v_1 = 0, \qquad ((\bar{\kappa}'\bar{A} - \bar{\kappa}\bar{A}')/\bar{A})\phi + v_1' = 0, \qquad (\bar{A}\phi)'/\bar{A} + \varepsilon^{1/4}u_1 = 0. \tag{45–47}$$

With a precision up to $\varepsilon^{1/4}$ these boundary conditions can be split in the following way

$$(\bar{A}\phi)' = 0 \tag{48}$$

and

$$v_1 = -\bar{\kappa}\phi, \qquad v_1' = [(\bar{\kappa}'\bar{A} - \bar{\kappa}\bar{A}')/\bar{A}]\phi. \tag{49}$$

so that the reduced problem (44) is solved subject to boundary conditions (48) while equation (41) for oscillatory component is solved subject to boundary conditions (49). A discrepancy in boundary condition for u of order $\varepsilon^{1/4}$ that arises in the leading approximation can be removed in the next approximation if $\mu = 1/4$ in (33). This is also consistent with the equations derived above.

For a hinged end boundary conditions for equation (41) take the form

$$v_1 = -\bar{\kappa}\phi, \qquad v_1'' = \left[-\bar{\kappa}'' + 2\bar{\kappa}' \frac{\bar{A}'}{\bar{A}} - \bar{\kappa}\left(\frac{\bar{A}'}{\bar{A}}\right)^2 + \bar{\kappa}\left(\frac{\bar{A}'}{\bar{A}}\right)' + \bar{\kappa}(\lambda - \bar{\kappa}^2) \right]\phi. \tag{50}$$

The analysis of the properties of eigenvalues and eigenfunctions of Sections 3.1.2 and 3.1.3 is valid for a beam of varying curvature and cross-section, as it is based entirely on the form of the solution (34) and boundary conditions (12, 13). The asymptotic curves for eigenvalues in the general case, however, cannot be given by explicit formulae, but can be found by solving the eigenvalue problems (41), (44) and (48) with boundary conditions of a curved beam in terms of normal displacements. The eigenfunctions are a combination of the eigenfunctions of the vibrations of a curved membrane and the flexural vibrations of a straight beam. The extensional terms in the eigenfunctions disappear as the eigenvalues approach those of a straight beam. As the eigenvalues approach those of a membrane, the eigenfunctions are generally a combination of the eigenfunctions of a membrane and flexural vibrations of a straight beam with non-homogeneous boundary conditions in terms of the normal displacements of a membrane. The exception is the case when the boundary conditions of the problem (41) and (44) are completely uncoupled. This is possible when the right-hand sides of (49) or (50) do not depend on ϕ. It causes a situation in which oscillatory terms disappear as the eigenvalues approach the eigenvalues of a membrane. An example is given in Section 3.2.

4. NUMERICAL RESULTS

All the numerical results were obtained using a collocation software for boundary-value ODEs Colnew [12].

4.1. BEAM OF CONSTANT CURVATURE AND CROSS-SECTION

A plot of non-dimensional eigenvalues verses non-dimensional curvature for a beam with clamped ends is shown in Figure 3. The asymptotic curves are shown with dashed lines. An example of the transformation of the mode shape at the stage of the increase in eigenvalues is shown in Figure 4. Similar results have been obtained in [8] using finite element analysis. They agree well with the present results, although it seems that there is a tendency of the finite element method to overestimate the frequencies for larger values of curvatures (this agrees with the comparison of the finite-element results with the exact solution performed by the authors of [8]). Only the eigenvalues of the lowest membrane mode are shown in Figure 3 and, as predicted, only the symmetric bending modes are involved in transformation. The increase in eigenvalues is accompanied by the transformation of the symmetric bending mode with n half-waves to the bending mode with $n+2$ half-waves. During this transformation the mode shape takes the form corresponding to the lowest extensional mode shape shown in Figure 2a. The eigenvalues of the reduced model asymptotically approximate the eigenvalues of bending modes during the stage of their extensional transformation with the accuracy of approximation increased for smaller ε.

The convergence can be illustrated with the plot of the scaled eigenvalue $\beta = (\lambda/\varepsilon)^{1/4}$ versus scaled curvature $\eta = \sqrt{\bar{\kappa}}/\varepsilon^{1/4}$ (Figure 5). This plot is also convenient from a practical point of view. As the function $\beta(\eta)$ is invariant in ε in the region $\beta \geqslant \eta$, this plot allows a determination of the non-dimensional eigenvalue λ and the number of wiggles in the

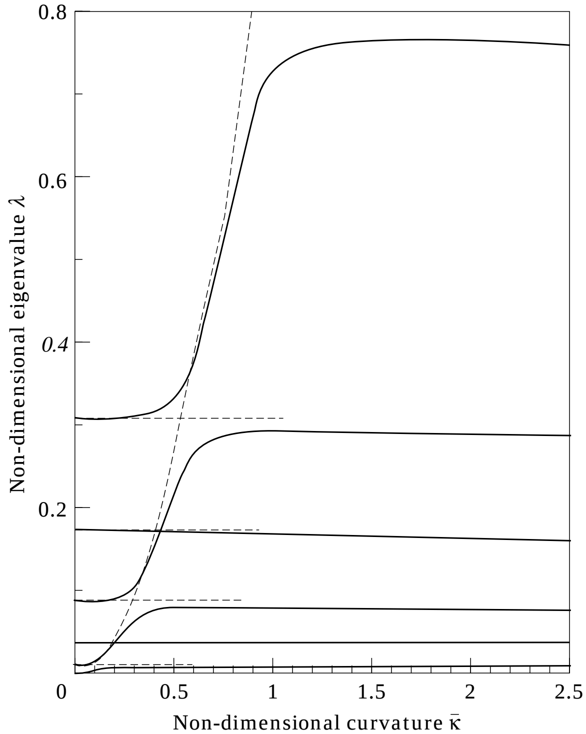

Figure 3. Non-dimensional eigenvalue as a function of non-dimensional curvature for a beam of constant curvature and cross-section, $\varepsilon = 10^{-6}$. Asymptotic curves are shown with dashed lines.

oscillatory component for any given values of curvature and slenderness ratio of the beam. It also shows the error of approximation and allows an estimation of the eigenvalues in the vicinity of the points of intersection of the asymptotic curves, where the asymptotic approximation is not valid.

Figure 6(a) shows the eigenvalues of the two lowest membrane modes (bold lines) and the eigenvalues of the first 14 bending modes (the symmetric modes are shown with full lines and the antisymmetric ones with dashed lines). It can be seen that the extensional transformation of the antisymmetric bending modes is connected with the second membrane mode (antisymmetric in v). The eigenfunction of beam vibrations at this stage

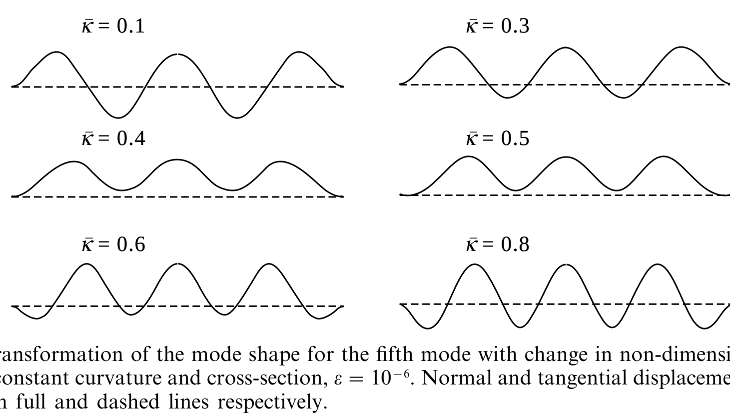

Figure 4. Transformation of the mode shape for the fifth mode with change in non-dimensional curvature $\bar{\kappa}$ for a beam of constant curvature and cross-section, $\varepsilon = 10^{-6}$. Normal and tangential displacements (almost zero) are shown with full and dashed lines respectively.

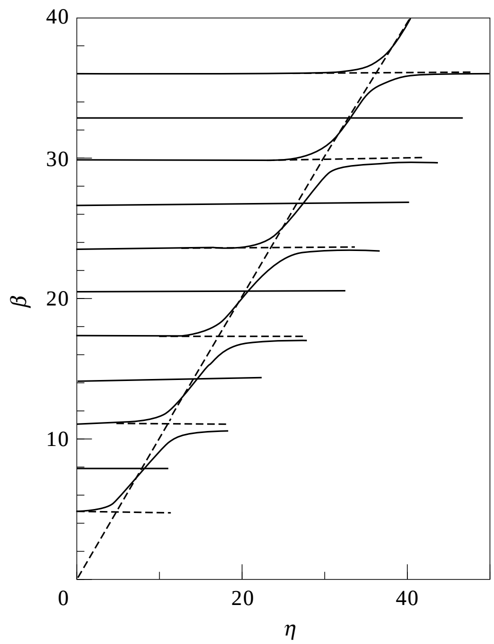

Figure 5. Scaled eigenvalue β versus scaled curvature parameter η for a beam of constant curvature and cross-section.

of extensional transformation is the sum of the eigenfunction of the second membrane mode and the eigenfunction of the flexural vibration of a straight beam (Figure 6(b)). The number of half-waves in oscillatory term of the extensional mode is connected with the

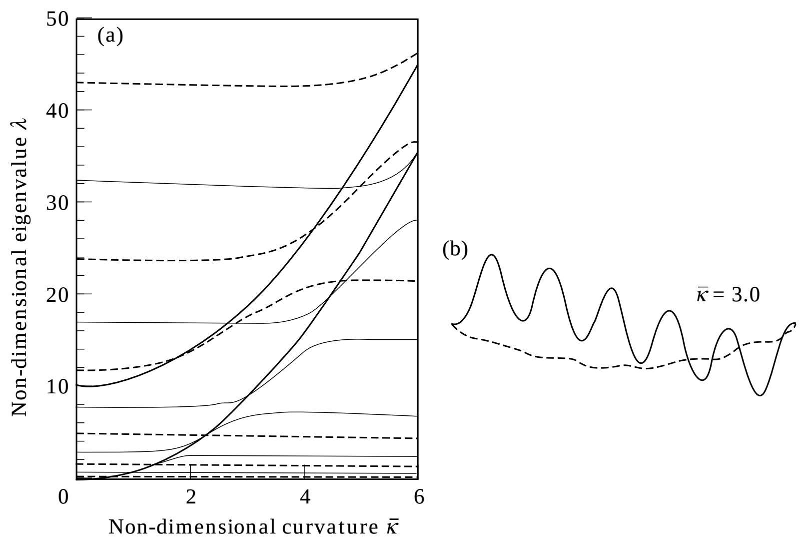

Figure 6. a) Non-dimensional eigenvalues of the first 14 bending modes (antisymmetric modes are shown with dashed lines) with two lowest membrane asymptotic curves (heavy lines) for a beam of constant curvature and cross-section. b) Typical shape of an extensional mode with eigenvalue close to the eigenvalue of the second membrane mode. (Normal and tangential displacements are shown with full and dashed lines respectively); $\varepsilon = 10^{-5}$.

number of half-waves in the bending mode that is transforming at the given value of curvature and, therefore, increases with increase in the curvature parameter and with decrease in ε.

4.2. UNIFORM BEAMS OF NON-CONSTANT CURVATURE

As an example of a beam of non-symmetric positive curvature a beam with curvature given by the asymmetric function is considered;

$$\bar{\kappa} = b \cos(\pi/2(\bar{s} - 0{\cdot}2)).$$

The eigenvalues versus shape parameter b are shown in Figure 7(a). As is expected, in this case all modes are involved in the transformation. As a typical example of the transformed mode shape the shape of the fourth bending mode at the stage of the transformation is shown in Figure 7(b). The shape is a combination of the lowest eigenmode of a membrane and the eigenmode of flexural vibrations of a straight beam.

Two examples of beams with the curvature changing sign are given in Figures 8 and 9. Figure 8 shows the eigenvalues of a beam of asymmetric curvature

$$\bar{\kappa} = b \cos(\pi(\bar{s} + 0{\cdot}3)),$$

while Figure 9 shows the eigenvalues of a beam of antisymmetric linear curvature

$$\bar{\kappa} = b(2\bar{s} - 1).$$

The behaviour of the eigenmodes during the transformation stage is similar to those in the previous example. For larger values of curvature, however, the decrease in eigenvalues is much more pronounced and is accompanied by further transformation of mode shape as shown in Figure 10. In the case of antisymmetric curvature it only affects the symmetric in v eigenmodes, while in the case of asymmetric curvature all modes are involved in transformation. It seems that this additional transformation phenomenon is a feature of the spectrum of beams whose curvature changes sign. It cannot be described by the asymptotic solution obtained in this paper, as it occurs outside the region of the spectrum

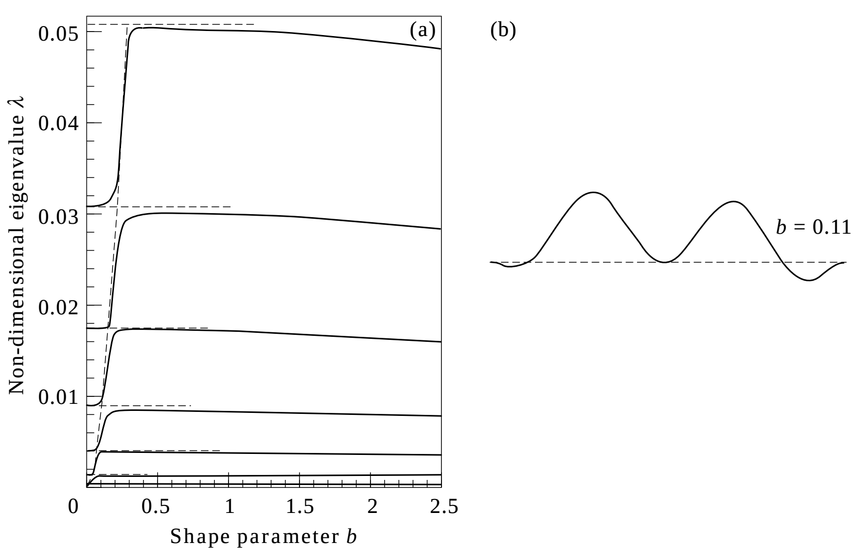

Figure 7. Beam of varying curvature $\bar{\kappa} = b \cos(\pi/2(\bar{s} - 0{\cdot}2))$; $\varepsilon = 10^{-7}$; *a*) Non-dimensional eigenvalue as a function of shape parameter b. b) Mode shape during the transformation for the fourth mode.

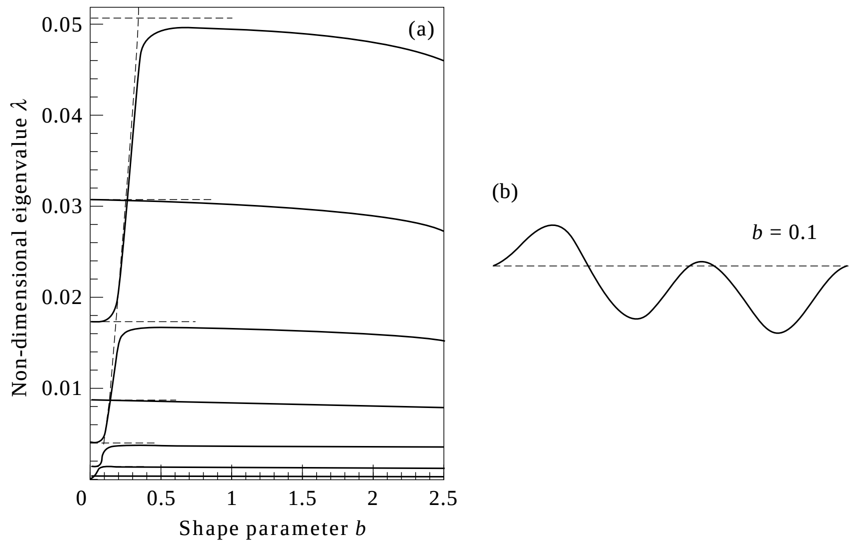

Figure 8. Beam of varying curvature $\bar{\kappa} = b\cos(\pi(\bar{s} + 0{\cdot}3))$; $\varepsilon = 10^{-7}$; *a*) Non-dimensional eigenvalue as a function of shape parameter b. b) Mode shape during the transformation for the fourth mode.

covered by the present asymptotic analysis. Although this phenomenon is pronounced for slender beams, it cannot be detected from a simplified model frequently used for the analysis of the vibration of a slender beam. In this simplified model the terms $(1 - \kappa z)$

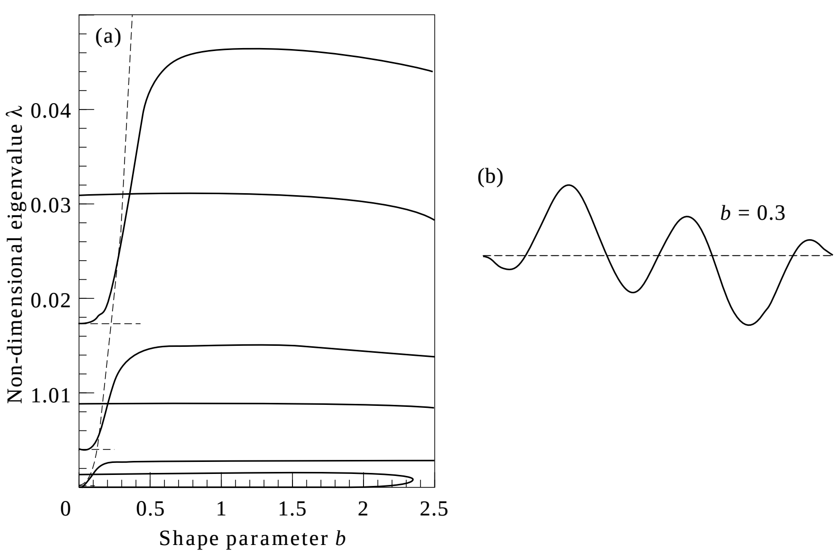

Figure 9. Beam of antisymmetric linear curvature, $\varepsilon = 10^{-7}$. *a*) Non-dimensional eigenvalue as a function of shape parameter b. b) Mode shape during the transformation for the fourth mode.

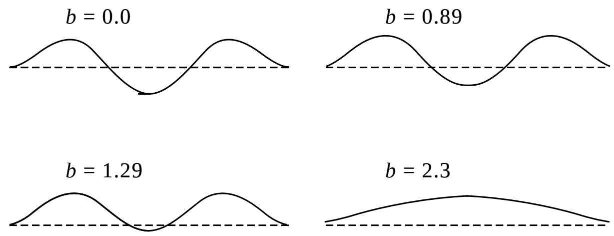

Figure 10. Transformation of the mode shape of third mode with change in shape parameter b for a beam of antisymmetric linear curvature; $\varepsilon = 10^{-7}$.

in the expression for the strain component (4) and potential energy (6) are replaced with unity, which can only be justified for reasonably small values of non-dimensional curvature. The present calculations according to the simplified model have shown that it gives accurate results for smaller values of the non-dimensional curvature ($\bar{\kappa} \leqslant 1$). For larger values of non-dimensional curvature the simplified model gives frequencies and mode shapes that are close to those of the flexural vibration of a straight beam, and fails to show the decrease in frequencies and the transformation of the mode shape that occur with increase in non-dimensional curvature. This limitation of the simplified slender beam model should be kept in mind.

An interesting case from a practical point of view is a beam whose curvature satisfies the conditions

$$\bar{\kappa}(0) = \bar{\kappa}(1) = \bar{\kappa}'(0) = \bar{\kappa}'(1) = 0.$$

In this case, as can be seen from (45, 46), there is complete uncoupling of boundary conditions for equations defining the rapidly varying and slowly varying terms of the solution. As an example of this one takes the curvature to be

$$\bar{\kappa} = b \sin^2(\pi \bar{s}).$$

The frequencies and transformation of mode shapes are shown in Figure 11. It can be seen that the oscillatory component of the eigenfunction vanishes at the extensional phase of the vibrations, the transformation from bending to extensional mode occurs rapidly and the frequencies of the extensional vibrations coincide almost exactly with those of the reduced model.

4.3. CURVED BEAM OF VARYING CROSS-SECTION

To examine the effect of varying cross-section on the frequency behaviour we consider beams of varying thickness and width separately.

As an example of a beam of varying thickness, a beam of constant curvature with non-dimensional thickness given by $H = 1 + \bar{s}$ is considered (so that the characteristic thickness h_0 is equal to the minimum thickness of the beam). The "membrane" asymptotic curves calculated as the lowest eigenvalue of (44) are almost indistinguishable from those of a beam of constant thickness and curvature (Figure 3). The "flexural" asymptotic curves for a beam of varying thickness are higher than the corresponding asymptotic curves of the beam with constant thickness. This determines the major difference in the spectrum of the beams of constant and varying thickness. Another difference is that in the case of varying thickness all modes are involved in transformation since the symmetry of modes is destroyed by the non-symmetric thickness variation.

As an example of a beam of varying width, a beam of constant curvature and thickness with non-dimensional width given by $B = 1 + \bar{s}$ is considered. In this case both families

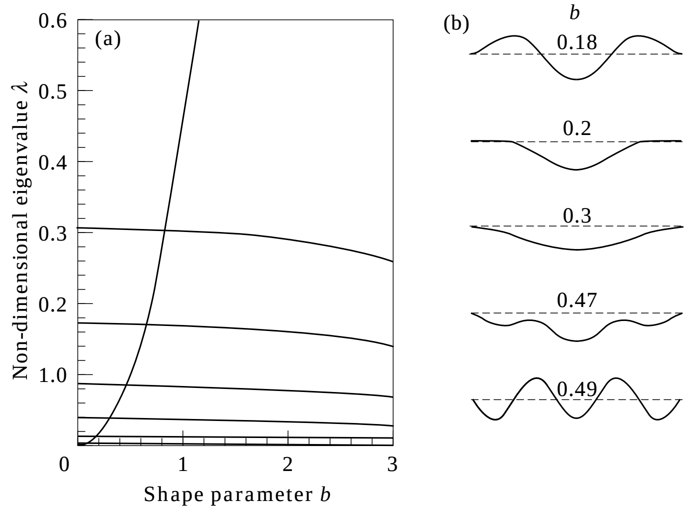

Figure 11. Beam of varying curvature $\bar{\kappa} = b\,\sin^2(\pi\bar{s})$; $\varepsilon = 10^{-6}$; *a*) Non-dimensional eigenvalue as a function of shape parameter *b*; b) Transformation of the mode shape for the third mode with change in shape parameter.

of asymptotic curves are almost indistinguishable from those of a beam with constant curvature and cross-section, which implies that the eigenvalues of the beam of linearly varying width are close to those of the beam with constant cross-section. The main difference in the behaviour of the eigenvalues and mode shapes with change in curvature in this case is the involvement of all modes in the transformation for a beam of varying width.

5. EXPERIMENTAL RESULTS

There are only a few reported experimental studies of curved bars in the literature, and these give only a few results. Petyt and Fleisher [13] report measurements of the variation with curvature of the frequencies of the first five modes of an aluminium sheet 50 cm long, 1·27 cm wide and 1·6 mm thick ($\varepsilon \approx 10^{-6}$) bent with constant curvature. The results agree with their calculations and with the behaviour shown in Figure 3. Scott and Woodhouse [9] similarly report a limited set of measurements on a narrow steel sheet about 1 m long, 150 mm wide and 0·8 mm thick ($\varepsilon \approx 5 \times 10^{-8}$) bent to an antisymmetric linearly varying S-shaped curvature. Again the present results agree with their calculations and with the behaviour shown in Figure 9a.

In the authors' experiments, the predictions of the theory were first examined using a beam consisting of a strip of steel, 800 mm long, 25 mm wide and 1 mm thick ($\varepsilon \approx 10^{-7}$) clamped between two heavy vices, the angle and position of which could be changed so as to fix the foil without introducing longitudinal tension. The bar was set into free vibration and its mode frequencies measured using the signal from a sub-minature accelerometer near one end. Two cases were examined: that of constant curvature, and that of linearly-varying antisymmetric curvature. In the first case, the general behaviour shown in Figure 3 was observed, while in the second case it was the antisymmetric modes that showed marked frequency dependence. Unfortunately, the experiment was not very satisfactory from a quantitative point of view because of the difficulties of achieving adequately rigid clamping. For this reason a much smaller and lighter foil beam was used in subsequent experiments.

In the second experimental approach the validity of the asymptotic approximation for high mode-number extensional vibrations was examined, using a narrow strip of PVDF foil, 25 μm in thickness, clamped between adjustable metallic jaws. Since the foil length varied from 10–55 mm in the experiments, ε lay in the range 2×10^{-8}–5×10^{-7}. The foil had electrodes evaporated on its two surfaces and could be excited into extensional vibrations by an applied oscillating potential through the piezoelectric effect. The motion of the foil was detected using a small microphone placed nearby. The ripples due to the transverse modes largely cancel, and the microphone responds to the average displacement of the foil. Since the foil was clearly not narrow enough, in the sense of being a beam, an experiment was carried out to check the validity of this approximation. In this experiment, after measuring the intact foil, it was slit longitudinally along its center line and measured again. The frequency of the beam-like mode being measured was unaltered. The experiment showed also a higher mode, associated with excitation of the foil in the transverse direction, the frequency of which approximately doubled as a result of the slitting operation, again confirming its identity.

In an initial set of experiments, three foils of length 20, 30 and 40 mm respectively and width 10 mm were flexed to the same uniform curvature. The resonance frequency was the same in each case, confirming the independence of this frequency upon length, as predicted by the theory.

In a second set of experiments, the angles and positions of the clamped jaws were chosen so as to achieve constant curvature of the strip. Figure 12(a) shows the measured dependence of the lowest extensional mode frequency on curvature for the case of a uniformly curved foil of length 45 mm and width 8 mm with the piezo-electric contraction along the foil length. It can be seen that the resonance frequency is proportional to the curvature as predicted by (11), (25). Absolute agreement between measured and calculated resonance could not be checked exactly because of uncertainty in the Young's modulus E and density ρ of the foil. Excellent agreement between theoretical and experimental results is obtained if the parameters of the foil are in the range $460\pi \leqslant \sqrt{E/\rho} \leqslant 480\pi$, which is consistent with the manufacturer's data.

In a third set of measurements, the two ends of the foil were clamped to opposite faces of a solid block, so as to be parallel and separated by a distance of 11 mm, and the length of the foil was varied. Under these circumstances the curvature was symmetrical about the center of the foil strip but not constant, and even changed in sign for the longer strips. Under these conditions it was expected all odd-numbered extensional modes to be clearly observed. Figure 12(b) shows the measured frequencies of these two resonances and, for

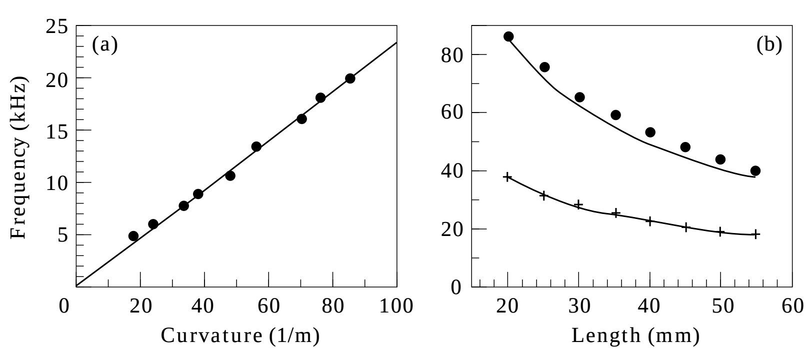

Figure 12. Comparison of theoretical and experimental results for piezoelectric foil; a) frequencies of the lowest membrane mode and experimental resonance frequencies for uniformly curved strip; b) frequencies of first and third membrane modes and experimental resonance frequencies for non-uniformly curved strip.

comparison, the calculated frequencies of the first and third extensional modes. In calculations the non-dimensional curvature of the strip was taken as

$$\bar{\kappa} = a_1 l \cos [a_2 l(\bar{s} - 0{\cdot}5)], \tag{51}$$

where the constants a_1 and a_2 were found from the conditions in which the ends of the strip are parallel and the distance b_1 between them is equal to 11 mm

$$a_1 = \frac{\pi a_2}{2 \sin (a_2 l/2)}, \qquad b_1 = l \int_0^1 \cos \left\{ \frac{\pi}{2 \sin (a_2 l/2)} \sin [a_2 l(\bar{s} - 0{\cdot}5)] \right\} \mathrm{d}\bar{s}.$$

For each value of length l in the experiment, the first and third eigenvalues of (44) were found for the curvature given by (51). The frequencies were calculated according to (11) with the ratio of Young's modulus to density taken from the experiments with the foil of constant curvature as $\sqrt{E/\rho} = 480\pi$. Once again the agreement between theory and experiment is good.

6. DISCUSSION AND CONCLUSIONS

The asymptotic analysis of the equations of free vibrations of beams having varying curvature and cross-section reveals that there are two types of asymptotic limits for eigenvalues in the region of spectrum satisfying the condition $\lambda_0 \geqslant \lambda_0^{(m)}(\bar{\kappa})$ ($\lambda_0^{(m)}$) is the lowest eigenvalue of the membrane vibrations problem; λ_0 is the leading approximation to the eigenvalue of free vibrations of a curved beam). The first type is given by the eigenvalues of a curved membrane with varying cross-section and approximates the eigenvalues of a curved beam at the stage of extensional vibrations. The second type is given by the eigenvalues of flexural vibrations of a straight beam with varying cross-section and approximates the eigenvalues of a curved beam at the stage of inextensional vibrations. Both limits approximate the eigenvalues of the same eigenmode of a curved beam. While the eigenfunctions of a straight beam serve as an asymptotic limit for eigenfunctions of inextensional vibrations of a curved beam, the eigenfunctions of a membrane differ from the eigenfunctions of extensional vibrations of a curved beam and contain additional oscillatory terms originating from the bending energy of deformation. The type of solution described exists for beams with boundary conditions corresponding to clamped and hinged ends and does not exist for beams with simply supported ends. This explains and provides a method of prediction of the transformation phenomenon noticed in the behaviour of eigenvalues and eigenfunctions of beams with clamped and hinged ends with change in curvature.

Only the leading approximations to the eigenvalues and eigenfunctions are obtained in the present paper. It is expected that the next approximation is of order $\varepsilon^{1/4}$. However, this matter awaits further study. Generally, the higher the mode number the more accurate the approximation. A number of numerical examples presented confirm the validity and satisfactory accuracy of the asymptotic approximation to the eigenvalues of the lower modes for beams of slenderness parameter ε up to 10^{-5} (that is with thickness to length ratio up to 0·01). Although the accuracy of the asymptotic approximation to the eigenvalues decreases with increase of slenderness parameter ε, the asymptotic solution is important for understanding the spectrum of relatively thick beams as it indicates the regions of the spectrum where the transformation phenomenon is present. In addition, scaled plots, such as presented in Figure 5, allow the estimation of the frequencies of curved beams for different, and rather larger values of ε.

Generally, the asymptotic approximation fails at the vicinities of the points of intersection of the asymptotic curves corresponding to flexural and membrane modes with the same symmetry of normal displacements. However, the analysis conducted reveals special types of curvature function for which the eigenvlaues of the vibrations of curved beam and their asymptotic approximations are almost indistinguishable. A complete decoupling of extensional and flexural vibrations occurs in this case.

The asymptotic solution obtained allows a simple approximation for high mode-number extensional vibrations of beams of arbitrarily varying curvature and cross-section. This is especially important for the analysis of vibrations caused by a high frequency external force. Experiments with piezo-electric foils confirm the validity of the asymptotic approximation for high mode-number extensional vibrations.

Yet another interesting phenomenon has been revealed numerically in the case of a beam whose curvature changes sign. An additional transformation of mode shape occurs for larger values of curvature in the region not covered by the present asymptotic analysis. Although this phenomenon has been noticed in the behaviour of the modes of beams of very small slenderness parameter $\varepsilon = 10^{-6}$–10^{-8}, it cannot be revealed using the simplified slender beam model. In connection with this it is worthwhile to notice that in all cases, except for the case of a beam of constant curvature, the error in frequencies and mode shapes determined from the simplified slender beam model becomes pronounced for values of the non-dimensional curvature parameter greater than unity. This is not surprising, since the assumption on which the simplified slender beam model is based is valid only for small values of non-dimensional curvature. This assumption is also used in classical thin shell theory which therefore must possess the same shortcoming.

The analysis can be extended by inclusion of shear deformation and rotary inertia effects, and it can also be applied to thin shells.

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