# Wave Propagation on Turbulent Jets: II. Growth

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Summary

The growth in amplitude of a sinuous disturbance propagating along a fully turbulent diverging planar air jet is studied experimentally for a range of slit widths, blowing pressures and acoustic excitation frequencies relevant to organ pipes. The wave amplitude is found to increase approximately exponentially with distance provided this amplitude remains small compared with the local jet width. For larger amplitudes the growth becomes linear. In the exponential growth regime the local behaviour is very similar to that predicted by simple theory for a laminar inviscid non-divergent jet with the same velocity profile. In particular, for a jet with local half-width $b$ and local exponential growth rate $\mu$, the quantity $\mu b$ rises to a peak value near 0.5 for values of the Strouhal number $kb$ near 0.6, then falls towards zero, or may even become negative, for $kb$ greater than about 2. The significance of these results for the sound generation mechanism in organ pipes is noted.

*Wellenausbreitung auf turbulenten Strahlströmen: II. Zunahme*

Zusammenfassung

Die Zunahme der Amplitude einer sinusförmigen Störung, welche sich längs eines vollständig turbulenten divergierenden ebenen Luftstrahls ausbreitet, wurde im für Orgelpfeifen relevanten Bereich von Schlitzweite, Anblasdruck und akustisch angeregten Frequenzen experimentell untersucht. Es wurde festgestellt, daß die Wellenamplitude näherungsweise exponentiell mit der Entfernung ansteigt, sofern die Amplitude klein bleibt gegenüber der örtlichen Strahlweite. Für größere Amplituden wird die Zunahme linear. Für den Fall der exponentiellen Zunahme ist das lokale Verhalten dem sehr ähnlich, welchesl durch eine einfache Theorie für einen laminaren, reibungsfreien nichtdivergierenden Strahl mit demselben Geschwindigkeitsprofil vorausgesagt wird. Das bedeutet im einzelnen, daß für einen Strahl mit der lokalen halben Weite $b$ und der lokalen exponentiellen Zunahme $\mu$ die Größe $\mu b$ bis zu einem Maximum bei 0,5 ansteigt für Werte der Strouhalzahl $kb$ bei 0,6, dann zu Null absinkt oder für $kb$ größer als ungefähr 2 sogar negativ wird. Die Bedeutung dieser Ergebnisse für den Schallerzeugungsmechanismus in Orgelpfeifen wird angemerkt.

*Propagations d'ondes sur jets turbulents: II. La croissance*

Sommaire

On étudie expérimentalement la croissance en amplitude d'une perturbation en sinus se propageant le long d'un jet d'air divergent complètement turbulent, dans une gamme de paramètres (largeur de fente, pression de soufflage, fréquences de l'excitation acoustique) qui correspondent à ceux qui s'observent pour les tuyaux d'orgue. L'amplitude de l'onde croît approximativement d'une manière exponentielle avec la distance, pourvu que cette amplitude reste petite par rapport à la largeur locale du jet. Pour des amplitudes plus grandes, la croissance devient linéaire. En régime de croissance exponentielle le comportement local est très semblable à celui que prédit une théorie simple établie pour un jet laminaire non visqueux et non divergent ayant le même profil de vitesse. En particulier, pour un jet de demi-largeur locale $b$ et de taux de croissance exponentielle $\mu$, le produit $\mu b$ s'élève jusqu'à un maximum de 0,5 pour des valeurs du nombre de Strouhal $kb$ avoisinant 0,6 puis il tombe à zéro ou devient même négatif pour $kb$ supérieur à 2. On note l'importance de ces résultats en ce qui concerne le mécanisme de génération du son dans les tuyaux d'orgue.

## 1. Introduction

Central to the elucidation of the driving mechanism of musical instruments such as the organ flue pipe, recorder and flute is the problem of wave propagation on planar air jets. This problem has, by its intrinsic interest, attracted the attention of many scientists and mathematicians starting with the classic work of Rayleigh [1]. In spite of the time and effort expended since then the extreme mathematical complexity of the problem has allowed only solutions to highly idealised situations to be obtained. Nevertheless these solutions, which in general bear a strong ressemblance to Rayleigh's original conclusions, still contain much that is of practical use, and experiments have shown some of them to be qualitatively applicable even outside their domain of formal validity.

It is as a consequence of this mathematical difficulty that attention has turned naturally to the experimental approach, a common occurrence in fluid mechanical problems. The vast majority of the experimental work done has investigated the problem in terms of the instability of the jet to the propagation of acoustically impressed disturbances [2, 3]. The results are usually expressed as contours on a graph of the Strouhal number versus the Reynolds number, separating regions of jet stability from regions of instability in which the wave grows.

The present authors have not been able to find any published experimental investigations of the actual form of the growth of the waves in the unstable region and the only prediction available concerning the form of this growth is essentially that of Rayleigh, which is that the growth is exponential. This result holds only for disturbances having infinitesimal amplitudes. The numerous photographs of realised jets in the literature all show disturbance amplitudes which are certainly at least of the order of the jet width and growth in this regime shows many complexities.

The preceding remarks have been concerned entirely with jets having small efflux velocities and carefully streamlined nozzle geometries. The flow is, as a consequence, laminar. In the context of the driving mechanisms of organ flue pipes, the efflux velocities are large enough and the nozzle configurations sufficiently unstreamlined as to ensure turbulent flow. This has been shown to be the case in a previous paper [4]. The region of interest on a jet in the organ flue pipe is that part of the jet at the lip which is, by virtue of its oscillation, blowing into and out of the pipe. Theoretical considerations by a number of workers have shown that the phase of this motion at the upper lip with respect to the perturbation at the flue, and thus the wave velocity, is important in controlling the frequency of the pipe oscillation [5 – 8]. The wave velocity on turbulent jets has been investigated experimentally by the authors in a previous paper [4] and the results will be briefly discussed later. The amplitude of the wave at the lip is also vital since it determines the amplitude of the fundamental mode and the harmonic development of the pipe tone [9]. The wave amplitude is obviously determined by the law of the growth of the waves on the jet and this growth is therefore the subject of the experiments described in this paper.

## 2. Experiment

The geometry of the jet nozzle used in the experiments was identical to the one used for the measurements of the wave velocity on the jet. It was constructed of perspex to function as the foot, languid and flue of an organ pipe when the appropriate resonating tube was attached. In this way the jet being studied was known to resemble those in real organ pipes.

For all the experiments described the coordinates were arranged in the usual way with $x$ designating distance from the slit along the plane jet, $y$ designating the distance through the jet perpendicular to its plane and the origin at the slit. The slit itself was 25 mm long and its width could be varied between 0 and 2 mm. The blowing pressures quoted were measured at the slit at $x=0$. All pressures were measured using a pitot tube, connected to a pressure transducer, mounted on a traversing mechanism with a multiturn potentiometer giving direct readout of distance.

Initial measurements on the unperturbed jet for blowing pressures, $P_0$, of 400, 800 and 1500 Pa showed that the variation of jet velocity, $V(x,y)$, with $y$, closely approximated the form

$$V(x,y) = V(x,0)\ \mathrm{sech}^2(y/b) \qquad (1)$$

for all values of $x$ up to $x \approx 20$ mm. The width of the jet was then defined in terms of the half-width $b$ where $V(x,b)/V(x,0) = 0.416$, this being $\mathrm{sech}^2\ 1$. The form (1) for the jet profile is identical to that calculated by Bickley [10] for a plane laminar jet using the boundary layer theory of Prandtl and arguments of self-preservation. For this jet he obtained $b \propto x^{2/3}$.

The variation of the jet velocity in the $x$ direction, $V(x,0)$, at $y=0$ was similar for the initial blowing pressures and also for the two slit widths $2l=$ 0.5 mm and 1 mm used, following the law

$$V(x) \propto (x - 6\,l)^{-1/2}. \qquad (2)$$

The jet width increased linearly with $x$ as would be expected from this result and showed no dependence on blowing pressure. Specifically

$$b(x) \approx 0.11\,x\,. \qquad (3)$$

The results (2) and (3) are consistent with the flow being fully turbulent and self-preserving. We note here that the momentum flux $\varrho \int V^2(x,y)\,\mathrm{d}y$, where $\varrho$ is the air density, is proportional to $b\,V^2(x,0)$ and is approximately conserved along the jet. Discrepancies in the spreading rate causing this slight deviation from true conservation have been attributed to induced flow in the surrounding air by Kotsovinos [11].

The propagation velocity of sinuous disturbances on this plane jet was measured for a number of blowing pressures, slit widths and frequencies and the results were described in the previous paper [4]. In brief the finding was that the wave velocity on the

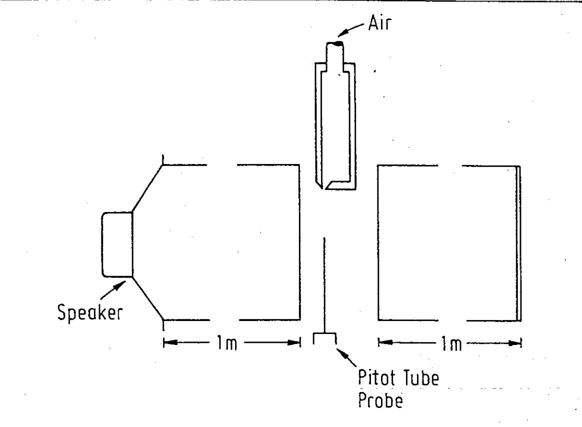


Fig. 1. Schematic diagram of the experimental arrangements. A loudspeaker feeds into a 2 m pipe which is divided in the middle and closed at the far end. The flue directs the jet across the gap and the pitot probe points into the air stream.

jet, $U$, varied like

$$U \propto (x - x_0)^{-1/2}, \tag{4}$$

where $x_0$ was of order of the slit width. For all the experiments

$$U \approx V(x, 0)/2 \tag{5}$$

within an accuracy of about 10 per cent.

The same apparatus was used in the present set of experiments to investigate the growth of this wave. It consisted of a 2 m tube having a diameter of 10 cm which was closed at one end and had a loudspeaker in the other end. The tube was split halfway along and the jet blew across this gap. This is shown in Fig. 1. The loudspeaker was driven sinusoidally at frequencies such that standing waves having velocity maxima at the gap were set up in the pipe. The air displacement amplitude $\xi$ at the position of the jet was calculated from the pressure gradient measured with two small condenser microphones mounted a fixed distance apart along the tube axis. Their outputs were normalised and then subtracted to give the pressure gradient.

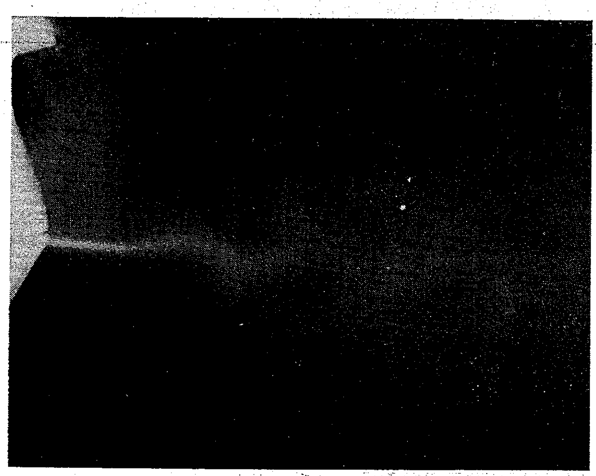

Fig. 2. Photograph in the $x, y$ plane of the perturbed jet made visible with suspended talc powder. The efflux pressure at the slit is 400 Pa, the frequency is 750 Hz and the slit width $2\,l$ is 0.5 mm.

The oscillating jet thus produced was realised and photographed, not to provide any numbers, but to show that the sort of jet behaviour envisaged in these experiments actually occurs. A number of methods for making jets visible have been described in the literature [2, 12]. For this jet a straightforward method which could cope with the high flow rates was sought. The method used was to suspend a large quantity of powder, in this case talc powder since it is cheap, readily available and of low density, in the air used to feed the jet. A reservoir situated upstream in the airline to the jet was used with the air being fed in at the bottom so that it sustained a tumbling motion and kept sufficient quantities of the powder suspended. The ensuing visible jet was lit with a strobe light triggered by the oscillator driving the loudspeaker, and photographs were taken with an ordinary camera. Although of only moderate quality the resulting picture shown in Fig. 2 clearly shows the form of the perturbed jet.

The amplitude $A$ of the sinuous disturbance as a function of $x$ was obtained in the following way. The pitot tube connected to the pressure transducer was used first to obtain the level of the mean dynamic pressure $P(x, 0)$ on the median plane of the unperturbed jet over the range from $x = 4$ mm to $x =$ 19 mm. Once a standing wave had been set up in the pipes the measurement was repeated. The pressure readings thus obtained represented the average dynamic pressure at $y = 0$ for given values of $x$ and $A$, $\overline{P}(x, A)$, of the jet as it oscillated back and forth in the $y$ direction. Assuming the oscillation of the jet is sinusoidal and using the measured jet dynamic pressure profile

$$P(0, y) \propto \operatorname{sech}^4(y/b) \tag{6}$$

the calculated form of the ratio $\overline{P}(x, A)/P(x, 0)$ is

$$\frac{\overline{P}(x, A)}{P(x, 0)} = \frac{\displaystyle\int_{-A}^{A} \frac{\operatorname{sech}^4(y/b)}{(1 - y^2/A^2)^{1/2}}\,\mathrm{d}y}{\displaystyle\int_{-A}^{A} \frac{\mathrm{d}y}{(1 - y^2/A^2)^{1/2}}}. \tag{7}$$

A graph of this function can then be used to determine $A/b$ from the measured value of $\overline{P}/P$.

Data was obtained for efflux pressures $P_0 = 400$ Pa, 800 Pa and 1500 Pa, slit widths $2\,l = 0.5$ mm and

1.0 mm, and frequencies, $f = 250$ Hz, 420 Hz, 920 Hz and 1590 Hz. The widest range of disturbance amplitudes possible with the given arrangement was measured. This was limited, for small amplitudes, by the obscuring effect of turbulence on the dynamic pressure readings and for large amplitudes by the limits of the loudspeaker and by practical considerations which will be discussed later.

An obvious limit to the applicability of this method derives from the assumption that the velocity profile of the jet maintains the form (1) and is simply displaced in an oscillatory fashion in the $y$ direction. This is certainly valid for small disturbance amplitudes but, as can be seen from Fig. 2, the jet tends to break up into vortices or other large-scale structures for $A/b \gtrsim 1$ so that the method is then no longer reliable.

## 3. Discussion

The initial analysis of the new data was performed assuming a growth law for the waves on the jet of an exponential form. This is reasonable in view of the fact that the simplest linear rule is that the growth in the amplitude is proportional to the amplitude itself. In fact Rayleigh [1] showed, in his treatment of the problem, that a sinuous disturbance on a plane laminar inviscid jet with a rectangular velocity profile grows in time as $\exp(\mu' t)$. This may be converted rather simplistically to growth in space by writing $\mu = \mu'/U$ and the value of $\mu$ is then

$$\mu = k(\coth k l)^{1/2} \tag{8}$$

for vanishingly small amplitudes, where $k$ is the wave number $2\pi/\lambda$. In this simple treatment, however, the jet is unstable at all frequencies and also $\mu$ increases without limit with increasing frequency which does not agree with experience. Using a simplified slab model to approximate the velocity profile, however, Rayleigh found that $\mu'$ increases with increasing frequency to $k b \approx 1$ and then decreases, becoming negative at $k b \gtrsim 2$.

Since then other researchers, all assuming an inviscid jet, have calculated the form of $\mu'$, using more realistic jet profiles. Lessen and Fox [13], using the Bickley jet profile, obtained the results from which Fig. 3 is derived, giving a spatial growth parameter $\mu$ which increases with $k b$ to a broad maximum at $k b \approx 0.6$ and then falls to zero at $k b \approx 2$. A more careful treatment of the spatial growth by Betchov and Criminale [15] gives the similar curve shown dashed in the figure.

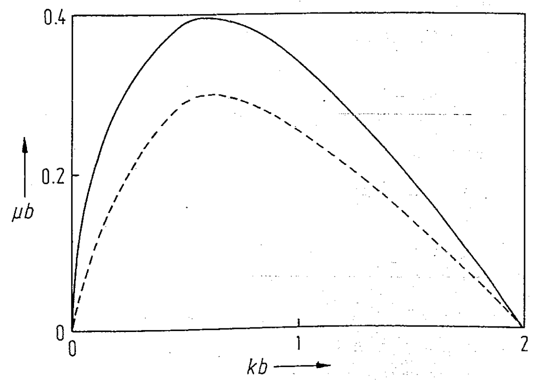


Fig. 3. The amplification factor, $\mu b$, as a function of $k b$, the frequency parameter, for the plane, laminar, inviscid jet. The solid line is the modified version of the calculation of Lessen and Fox [13]. The dashed line is the calculation of Betchov and Criminale [15].

The spreading of the real jet is caused by the viscosity of the fluid. When, in addition, turbulent flow is considered, as opposed to laminar flow, it is sufficient to a first approximation to allow for this in terms of an increase in the effective viscosity. This is because the main agent of momentum transport across the jet has shifted from viscous drag to eddy diffusion. The effective viscosity in this case is no longer constant but depends to the first order on the product $b V$. Since the viscosity acts as a diffusion coefficient for velocity in the Navier-Stokes equation, this form for the viscosity produces a jet having its half-width proportional to $x$. With the viscosity constant, as in laminar flow, this same reasoning predicts $b \propto x^{2/3}$ as calculated by Bickley [10]. Thus, by using local values on the jet for the calculation of $k b$, it is not unreasonable to hope that the predictions of Rayleigh and others for laminar flow will also generally hold for the turbulent jet in the limit of very small amplitudes.

Using our measured data, the value of $\mu = (1/A)(\mathrm{d}A/\mathrm{d}x)$ was calculated along the jet at 1 mm intervals for $x = 4$ mm to 19 mm, each data set having $P_0$, $l$, $f$ and $\xi$ constant. The aim was to categorise the data in terms of the three dimensionless pa-

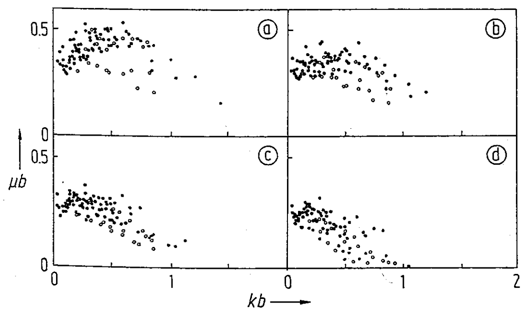


Fig. 4. The amplification factor, $\mu b$, as a function of $k b$ for all the experimental data obtained with the slit width $2l = 0.5$ mm. The open circles correspond to $f = 920$ Hz. The data contained in each graph is for $A/b$ constant, where in (a) $A/b = 0.1$, (b) $A/b = 0.4$, (c) $A/b = 0.7$ and (d) $A/b = 1.0$.

rameters $\mu b$, $k b$ and $A/b$. With this in view it was found to be most convenient initially to plot $\mu b$ against $A/b$, each graph having $P_0$, $l$, $f$ and $x$ constant. Note that $k b$ is a constant for each graph and increases through the set due to increasing $x$. Apart from the one or two points at small $A/b$, straight lines of reasonably good fit could be drawn through the data. When $A/b$ is small there is not only a large uncertainty associated with its measurement due to fluctuations caused by turbulence but also these amplitudes represent that region of pressure profile close to $y=0$ where the pressure varies least for some given displacement in the $y$ direction. Using the slopes of the lines drawn through these data, $\mu b$ was plotted against $k b$ for $A/b=0.1$, 0.4, 0.7 and 1.0 where data existed for these $A/b$ values. Fig. 4 shows the results, where the graph for each value of $A/b$ contains the data for all the four frequencies and three blowing pressures used, with $2l=0.5$ mm. The results for $2l=1.0$ mm followed very similar trends.

The sequence of anomalous points, plotted as open circles, which appears in each example corresponds to $f=920$ Hz. All the data for this frequency, although smooth and indeed repeatable, followed a different trend to the rest. Brown [12] found in his work similar well-defined frequencies where the response of the jet was either maximal or minimal independent of any other variables. Andrade [14] also observed a similar phenomenon but he attributed it to some characteristic of the jet having a natural frequency. Pursuing Andrade's hypothesis the equipment was checked with an accelerometer for any unusual resonances at or around $f=920$ Hz but none were found. Then, on the assumption that there may be some resonance or unusual turbulence in the air flow through the nozzle, the experiment was repeated using a nozzle arrangement of quite different dimensions. The results agreed remarkably well with the original set at $f=920$ Hz. Therefore these data are plotted as they stand but their anomalous behaviour is noted by the use of a different symbol on the graph.

The results in Fig. 4 a for $A/b=0.1$ show that $\mu b$ attains a broad maximum equal to approximately 0.5 for $k b$ between 0.5 and 1 and then decreases towards zero for $k b \approx 2$. This whole behaviour is in good general agreement with the trends shown in Fig. 3 predicted by theory for vanishingly small wave amplitudes on laminar non-diverging inviscid jets. However, the behaviour is already substantially modified in Fig. 4 b where $A/b=0.4$. This is hardly surprising since the peak-to-peak wave displacement is now almost half the jet width. It is of interest to note that $\mu b$ still shows an ill-defined maximum but for a smaller $k b$ value than in Fig. 4 a. These results strongly support wave growth of an exponential form on the jet, so long as the amplitude is much less than the half-width, with $\mu$ varying very much according to the simple theory.

The curves in Figs. 4 c and 4 d for $A/b=0.7$ and 1.0 differ progressively in shape from those for smaller amplitudes. There seems in fact to be no justification for assuming that the growth is any longer exponential. The theoretical calculations of the growth behaviour, in assuming $A$ and also the air velocity in the $y$ direction to be vanishingly small, attribute the growth of the wave to the Bernoulli force arising from the velocity in the $x$ direction of the wave crests relative to the surrounding air. When $A$ has become any substantial proportion of $b$ or of the wavelength there is an appreciable velocity in the $y$ direction. It is evident that both $k b$ and $A/b$ are thus important in specifying when the transition away from exponential growth occurs.

We surmise at this stage that the growth after transition becomes linear in the sense that $\mathrm{d}A/\mathrm{d}x$ becomes constant. The photograph of the jet in Fig. 2 lends some support to this hypothesis. If $\mathrm{d}A/\mathrm{d}x$ is assumed to become constant for $A/b$ greater than some transitional value then this would appear on the graphs of $\mu b$ plotted against $A/b$ as a change from straight lines to curves like $\mu b \sim A^{-1}$. The critical value of $A/b$ would be expected to decrease as $k b$ increased. Once there is a transition to linear growth at a point $x_0$ up-stream from the point $x$ where $A$ is being measured, the effect on $A$ of increasing acoustic excitation $\xi$ becomes progressively smaller as $\xi$ becomes larger. This arises from the fact that the amplitude $A$ at $x$ comes from the integration of all the growth up-stream from $x$, and this

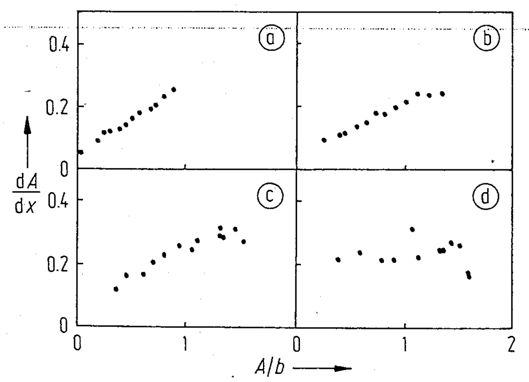


Fig. 5. A sample of the data plotted in the form $\mathrm{d}A/\mathrm{d}x$, where $A$ is the wave amplitude, against $A/b$. The slit width $2l=0.5$ mm, $f=225$ Hz, the efflux pressure $P_0=800$ Pa. For each graph $x$ is constant and therefore $k b$ is constant. (a) $k b=0.062$, (b) $k b=0.134$, (c) $k b=0.223$, (d) $k b=0.401$. Increasing $A/b$ corresponds to increasing air displacement amplitude, $\xi$, in the sound field.

is now a combination of exponential growth and linear growth. As $\xi$ is increased, the transition point moves towards the slit and a progressively greater part of the jet length is in the linear growth regime.

In order to test the hypothesis of linear growth the data was plotted in the form $\mathrm{d}A/\mathrm{d}x$ versus $A/b$ for each of the data sets. Thus $k\,b$ is constant for each of the resulting curves. The result is shown in Fig. 5. As would be expected the curves tend to consist of two straight-line segments. The initial sloping line were $\mathrm{d}A/\mathrm{d}x \propto A$ is the exponential growth region and the flat section where $\mathrm{d}A/\mathrm{d}x$ is constant represents linear growth. The transition depends on $k\,b$.

It is not unreasonable to hope that the behaviour displayed in these curves might be condensed into a single universal curve by plotting some relationship between the dimensionless quantities $\mu\,b$, $k\,b$ and $A/b$. Unfortunately we have not been able to find such a generalization.

## 4. Conclusions

Our measurements, as reported here and in our previous paper on the subject [4], show that the behaviour of sinuous waves propagating on a fully turbulent, divergent planar jet in a viscous fluid is locally very similar to the behaviour predicted for wave propagation on a laminar, non-divergent and inviscid jet having the same velocity profile. In particular the waves travel with a phase velocity closely equal to half the local central velocity of the jet and, provided their amplitude remains small in comparison with the jet width, grow exponentially in amplitude while their Strouhal number $k\,b$ remains less than about 2. If the wave amplitude becomes comparable with the jet width, then its growth tends to a linear law and the jet breaks up into vortices. If the Strouhal number exceeds about 2 then wave growth ceases and the wave may even be attenuated as it propagates further along the jet.

These conclusions are of interest not only because of their basic significance in fluid dynamics, but also because of the light they throw on the mechanism of sound generation in organ pipes and related musical instruments. When a turbulent divergent jet is excited by an acoustic field containing a fundamental frequency $f$ and its harmonics, as in an organ pipe, then waves having all these frequencies are excited and propagate along the jet. For each such wave, the Strouhal number $k\,b$ increases with distance along the jet and, if the distance to the upper lip of the pipe is large, essentially only the wave component associated with the fundamental will survive. This is the basis of the simplified harmonic-generation theory put forward by Fletcher and Douglas [9].

If, however, the cut-up of the pipe lip, and therefore the length of the jet, is small, or if the blowing pressure is considerably increased, waves associated with several harmonics may propagate to the lip, thus requiring a much more complicated treatment [8].

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