A Puzzle for Structuralism

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1 Introduction

Structuralism is the view that the subject-matter of a theory of pure mathematics is a *mathematical structure*. Different versions of structuralism tell different stories about what mathematical structures are.[1] But assuming that they agree about when a model[2] *exemplifies* a mathematical structure, they can agree about the conditions under which a sentence of pure mathematics is true:

> [*Mathematical Truth*]
>
> Let structure s be the subject-matter of a theory T of pure mathematics, and let $\mathcal{C}$ be the class of models that exemplify s. Then a sentence in the language of T is true if and only if it is true according to every model in $\mathcal{C}$.

This is a promising start, but it will not take us very far unless we supplement it with an account of how it is that mathematicians succeed in *specifying* the needed mathematical structure. We would like to know, for example, how it is that mathematicians are able to single out the structure of the natural numbers from every other mathematical structure as the subject-matter of arithmetic. It is tempting to think that mathematicians uniquely specify a mathematical structure—or, equivalently, the class $\mathcal{C}$ of models exemplifying that structure—by setting forth an appropriate set of axioms. For $\mathcal{C}$ might be identified as the class of all and only the models which make the axioms true.

This strategy will succeed on the assumption that three conditions are met. First, the axioms must be *satisfiable* (i.e. they must have at least one model), to ensure that at least one structure is exemplified by the models in $\mathcal{C}$. Second, the axioms must be *categorical* (i.e. they must have only pairwise isomorphic models), to ensure that at most one structure is exemplified by the models in

$\mathcal{C}$. Finally, the axioms must be *recursive*, to ensure that they can be set forth by finite beings like ourselves, at least in principle. (The set of true sentences of arithmetic, for example, is far too complex to be set forth by humans, even in principle.) When we combine the three conditions, we are in a position to offer the following thesis on behalf of the structuralist:

> [*Axiomatic Categoricity*]
> If a set of axioms $\mathcal{A}$ is (*i*) satisfiable, (*ii*) categorical and (*iii*) recursive, then mathematicians are in a position to use $\mathcal{A}$ to uniquely specify the mathematical structure exemplified by each of the members of the class $\mathcal{C}$ of models of $\mathcal{A}$.

In the special case of pure arithmetic, [*Mathematical Truth*] and [*Axiomatic Categoricity*] combine to produce a tidy account of the conditions under which a sentence of pure arithmetic is true. Since the set of second-order Dedekind-Peano Axioms is satisfiable, categorical and recursive, [*Axiomatic Categoricity*] implies that it can be used to specify the class $\mathcal{C}$ of models exemplifying a certain mathematical structure s—the structure of the natural numbers. Thus, when s is taken to be the subject-matter of pure arithmetic, it follows from [*Mathematical Truth*] that a sentence of the language of pure arithmetic is true just in case it is true according to each of the models of the second-order Dedekind-Peano axioms.

We will see, however, that (on certain reasonable assumptions) no recursive and satisfiable axiomatization of set theory is categorical. So friends of [*Mathematical Truth*] cannot rely on [*Axiomatic Categoricity*] to provide an account of the conditions under which the sentences of pure set theory are true. The aim of the paper is to assess the prospects of structuralism in light of this limitative result.

2 Axiomatic Categoricity

2.1 Arithmetic

We know from Gödel's Incompleteness Theorem that no recursive and satisfiable first-order axiom system sufficiently rich to interpret elementary arithmetic can ever be categorical. So no sufficiently rich first-order axiom system can ever satisfy the necessary conditions of [*Axiomatic Categoricity*].

Matters improve dramatically if we are allowed to help ourselves to *second-order* quantification. The set of second-order Dedekind-Peano Axioms, for example, is a satisfiable, categorical and recursive axiomatization of arithmetic. According to [*Axiomatic Categoricity*], mathematicians are therefore in a position to use the second-order Dedekind-Peano Axioms to specify a mathematical structure—the structure of the natural numbers.

There are, of course, important questions to be raised about second-order quantification in the present context. For instance, one might think that our understanding of second-order quantification is not sufficiently determinate to rule out non-standard Henkin-interpretations. And this would naturally lead to doubts about whether the second-order Dedekind-Peano Axioms can be any more successful than their first-order counterparts in the task of uniquely specifying the structure of the natural numbers. A defense of the claim that our understanding of second-order quantification is robust enough to exclude Henkin interpretations would take is too far afield,[3] but we shall nonetheless assume that structuralists can help themselves to second-order quantifiers. This is acceptable because our aim is to set forth a challenge for structuralism, and the legitimacy of second-order quantification can only strengthen the structuralist's case.

2.2 Set Theory

If there is an inaccessible cardinal, then the axioms of ZFC2 (second-order Zermelo-Fraenkel set theory with Choice) are not categorical.[4] [*Proof sketch:* Let In be a sentence of the language of ZFC2 to the effect that there is an inaccessible cardinal. Then, if there is an inaccessible, neither ZFC2 $\models In$ nor ZFC2 $\models \neg In$.] Thus, unlike the case of the Dedekind-Peano Axioms, [*Axiomatic Categoricity*] does not suffice to guarantee that axioms of ZFC2 succeed in uniquely specifying a mathematical structure.

Ernst Zermelo has shown, however, that ZFC2 has a remarkable feature: given any two of its models, (at least) one of them must be isomorphic to an initial segment of the other. In other words, the structure of a model of ZFC2 is completely determined by the height of its ordinal sequence. This means that all we need to obtain a categorical second-order axiomatization of pure set theory is to supplement the axioms of ZFC2 with an axiom fixing the height of the ordinals. For example, the result of supplementing the axioms of ZFC2 with an axiom to the effect that there are no inaccessibles, $\neg In$, is categorical.

$\neg In$ is not a sentence that many set theorists are inclined to accept. But the categoricity of ZFC2+$\neg In$ gives rise to the question of whether some other way of enriching ZFC2 might yield categoricity without betraying mathematical practice.

There is an important sense in which the answer is 'no'. For the following seems to be consistent with the practice of most set-theorists:

> *Principle of Second-order Global Reflection*
>
> There is a set S such that every true sentence of the language of second-order set theory is true when its quantifiers are restricted to elements of S;

and we have the following:

> **First Limitative Result**
>
> *If the Principle of Second-order Global Reflection obtains, then no recursive and satisfiable set of true sentences of the language of set theory is categorical.* (See appendix for proof.)

(Note that this is compatible with the categoricity of ZFC2 + $\neg In$, since $\neg In$ is inconsistent with the truth of Second-order Global Reflection.)

The First Limitative Result brings about a preliminary puzzle for Structuralism. If set theory has no satisfiable, categorical and recursive axiomatization, then [*Axiomatic Categoricity*] cannot be used to explain how it is that mathematicians are able to single-out the class of models exemplifying the structure of pure sets. How then, according to the structuralist, might the subject-matter of set-theory be specified?

Of course, the puzzle relies on the assumption that the Second-order Global Reflection Principle is true. But the Reflection Principle should presumably be accepted or rejected on the basis of strictly mathematical considerations, so it shouldn't be questioned merely on the grounds that it poses an obstacle for what seemed like an attractive account of how mathematicians manage to specify the subject-matter of set theory. The *mathematical* question of whether we should accept a thesis like the Reflection Principle ought not to depend on the *philosophical* question of how it is that mathematicians are in a position to specify the structure of the pure sets.

3 Impure Set Theory

It follows from Zermelo's result that any two models of ZFC2 of the same size are isomorphic. So if we enriched ZFC2 with an axiom specifying the size of the universe of sets, [*Axiomatic Categoricity*] would deliver the result that mathematicians can used the enriched system to specify the structure of pure sets. But the First Limitative Result shows that the enriched system would violate Global Reflection.

There is, however, a different way of taking advantage of Zermelo's result. The trick is to bring urelements into the picture. By building on Zermelo's result, [21] has shown that any two models of the same cardinality of ZFCU2 (second-order Zermelo-Fraenkel set theory with urelements and Choice) plus the Urelement Set Axiom (which says that there is a set whose members are all and only those individuals that are not sets) have isomorphic pure sets.[5] In particular, any two models of McGee's system with domains consisting of *absolutely everything* have isomorphic pure sets.[6] (Of course, when models are taken to be *sets*, there is no model with an absolutely unrestricted domain. For present purposes it is best not to think of models as sets.[7])

But now suppose one insists that the first-order quantifiers of McGee's axiom-system are to range over absolutely everything. Since any two models of the McGee's system with absolutely unrestricted domains have isomorphic pure sets, one might claim that the result is a unique specification of the structure of the pure sets. The problem posed by the First Limitative Result is avoided because one leaves it to the world to determine the size of the models involved in the specification, rather than setting forth an axiom to the effect that the universe of pure sets is of such-and-such a size.

The results in [39] show that McGee's result is not an isolated case. There are a number of axiomatizations of set theory with urelements whose models with unrestricted domains are pairwise isomorphic. This suggests a general strategy for improving on [*Axiomatic Categoricity*]. Consider first a slight simplification:

[*U-Axiomatic Categoricity*]

If a set of axioms $\mathcal{A}$ is (*ia*) satisfied by a model with an absolutely unrestricted domain, (*iia*) such that any two models of $\mathcal{A}$ with absolutely unrestricted domains are isomorphic, and (*iii*) recursive, then, by insisting that the first-order quantifiers take absolutely unrestricted range, mathematicians are in a position to use $\mathcal{A}$ to uniquely specify the mathematical structure exemplified by each of the members of the class $\mathcal{C}$ of models of $\mathcal{A}$ with absolutely unrestricted domains.

The reason this is not quite what we want is that McGee and Uzquiano's results don't show that, when the range of the quantifiers is absolutely unrestricted, any two models of ZFCU plus the Urelement Set Axiom are isomorphic. What they show is that the *pure sets* of one will be isomorphic to the *pure sets* of the other. The principle we need is therefore this:

[*U-Axiomatic P-Categoricity*]

If a set of axioms $\mathcal{A}$ is (*ia*) satisfied by a model with an absolutely unrestricted domain, (*iib*) such that any two models of $\mathcal{A}$ with absolutely unrestricted domains have isomorphic P-restrictions[8] for some predicate P in the language of $\mathcal{A}$, and (*iii*) recursive, then, by insisting that the first-order quantifiers take absolutely unrestricted range, mathematicians are in a position to use $\mathcal{A}$ to uniquely specify the mathematical structure exemplified by each of the P-restrictions of members of the class $\mathcal{C}$ of models of $\mathcal{A}$ with absolutely unrestricted domains.

Consider McGee's axiom-system. When we let P be the predicate 'x is a pure set', [*U-Axiomatic P-Categoricity*] delivers a guarantee that ZFCU2 plus the Urelement Set Axiom uniquely specifies a mathematical structure—provided, of course, that the axioms of ZFCU2 plus the Urelement Set Axiom are satisfied by some model with an absolutely unrestricted domain. Structuralists can therefore claim, in accordance with [*Mathematical Truth*], that a sentence of the language of pure set theory is true just in case it is true according to each member of the class $\mathcal{C}$ of models of ZFCU2 plus the Urelement Set Axiom with unrestricted domains.[9]

To illustrate the point, consider the status of In and $\neg In$, characterized above. Neither In nor

$\neg In$ is a (semantic) consequence of ZFCU2 plus the Urelement Set Axiom. The axioms themselves leave open the question of whether In or $\neg In$ is true, but by requiring our quantifiers to take absolutely unrestricted range, we let the world answer the question for us. For it follows from McGee's result that one of In and $\neg In$ is true in all models with absolutely unrestricted domains of ZFCU2 plus the Urelement Set Axiom. And, by [*Mathematical Truth*], this means that each of In and $\neg In$ must be either true or false. If the size of the universe is that of the first inaccessible, then $\neg In$ is true and In is false. But if the size of the universe is that of the seventeenth inaccessible, then In is true and $\neg In$ is false. (Here and throughout, claims about the size of the universe are to be understood in second-order terms. For instance, 'the size of the universe is that of the first inaccessible' is to be read as a sentence of pure second-order logic to the effect that there are precisely inaccessibly many objects and that any plurality consisting of precisely inaccessibly many objects is in one-one correspondence with the entire universe.)

4 Stability

On the assumption that conditions (ia), (iib), and (iii) are fulfilled, [*U-Axiomatic P-Categoricity*] delivers the result that mathematicians are in a position to use ZFCU2 plus the Urelement Set Axiom to uniquely specify the structure of the pure sets. But, of course, condition (ia) (i.e. satisfiability in a model with an absolutely unrestricted domain) is far from straightforward. If the size of the world is that of an inaccessible, then there will be many models of ZFCU2 plus the Urelement Set Axiom with absolutely unrestricted domains. But if the world happens to contain precisely power-of-an-inaccessible many objects, then there will be no such models. In general, the usefulness of [*U-Axiomatic P-Categoricity*] depends on whether or not the size of the world is such that the relevant axiom-system is satisfied in models with absolutely unrestricted domains.

Indeed, when we deal with unrestricted versions of mathematical theories, we run the risk that different theories might impose incompatible demands on the size of the universe. For instance, unrestricted versions of the class theory set forth in [19] and of second-order Morse-Kelley class theory with urelements (MKU2) plus the Urelement Set Axiom[10] both require the universe contain precisely power-of-an-inaccessible many objects. As a result, either of these unrestricted versions of class theory imposes constraints on the size of the universe that are incompatible with the constraints imposed by the unrestricted version of ZFCU2 plus the Urelement Set Axiom, which requires the size of the universe to be that of an inaccessible.

But such incompatibility makes it difficult to understand what reasons could be offered to accept one axiom-system over its rivals. Methods of justification internal to the mathematical community are unlikely to help, since current mathematical practice seems to allow set theory and class theory to coexist peacefully side by side. And it seems bizarre to suggest that one could use non-mathematical methods to determine whether the size of the universe satisfies the constraints imposed by one of the theories rather than the constraints imposed by the other.

It may be of interest to illustrate a further risk: that of incompatibility between a mathematical theory and a cluster of non-mathematical theses. It is a consequence of the axioms of classical mereology that any objects have a (unique) mereological fusion.[11] Hence, when the first-order quantifiers are taken to range over absolutely everything there is, classical mereology implies that any objects whatsoever—whether they be concrete or abstract—have a (unique) mereological fusion. This imposes an immediate constraint on the size of the universe:

> Let there be precisely κ mereological atoms. If there is no atomless gunk, then there must be precisely 2^κ objects (or $2^\kappa - 1$ objects if κ is finite).[12]

This means, in particular, that unless there is atomless gunk, the universe cannot have an inaccessible size (since 2^κ objects are never an inaccessible number of objects). But in order for ZFCU2 plus the Urelement Set Axiom to be true when the first-order variables take absolutely unrestricted range, the world *must* have the size of an inaccessible cardinal. So, when one's first-order quantifiers range over absolutely everything, one cannot accept both classical mereology and ZFCU2 plus the Urelement Set Axiom without being immediately committed to the conclusion that there is atomless gunk.

This is very troubling. For whether or not classical mereologists are happy with the conclusion that there is atomless gunk, it seems inappropriate for such a conclusion to be forced upon them merely as a result of accepting an unrestricted version of set theory rather than, say, an unrestricted version of class theory. Still worse is the suggestion that classical set theorists might make a decision about what system of mathematical axioms to accept on the basis of whether they believe in the existence of atomless gunk, or some other metaphysical thesis.

Something is amiss. But what? One tentative suggestion is that the unrestricted versions of set theory and class theory we have considered are too ambitious: *mathematical* theories should not be allowed to impose any specific constraints on the cardinality of the universe over and above the requirement that the universe be *at least* of a certain size. The suggestion is admittedly vague. But, at least as a first approximation, it might be partially expressed as the claim that mathematical theories should be *stable*, where stability is characterized as follows:[13]

> A theory T is stable if and only if there is some cardinal κ such that T is satisfied by a model of cardinality κ and, for all $\lambda > \kappa$, T is satisfied in a model of cardinality λ.

It is important to note that, because cardinals are sets, stability has been characterized on the basis of satisfiability in *set-sized* models.[14] Thus the stability of a theory T need not *guarantee* that T is satisfiable in models with unrestricted domains.[15] Nonetheless, the stability of T should be *prima facie* evidence that T imposes no structural constraints on the cardinality of the universe over and above the requirement that the universe be at least of a certain size.[16]

The versions of set theory and class theory we have considered are certainly not stable (provided, that is, that they are satisfiable in some set-sized domain). ZFCU2 plus the Urelement Set Axiom, for example, is never satisfied by models whose cardinality is a successor. Might one be able to improve upon the McGee-Uzquiano results by setting forth a categorical and *stable* axiomatization for set theory? In fact, the answer is 'no':

> **Second Limitative Result**
>
> *No theory extending ZFCU2 that is satisfiable in a set-sized domain is stable.* (See appendix for proof.)

5 A Puzzle for Structuralism

The limitative results of sections 2 and 4 suggest that no attempt to specify a mathematical structure as the subject-matter of set-theory can satisfy both of the following two conditions:

[*Success*]

We are in a position to justify the claim that our specification has succeeded in picking out some mathematical structure or other (whether or not some structure has been picked out uniquely).

[*Uniqueness*]

Our specification is capable of picking out a mathematical structure uniquely.

Here's why. The First Limitative Result suggests that, if [*Uniqueness*] is to be satisfied in the case of set theory, structuralists have little choice but to adopt the strategy of [*U-Axiomatic P-Categoricity*]. But, in light of the Second Limitative Result, it would seem that [*U-Axiomatic P-Categoricity*] can only deliver the right result if it is used in conjunction with an unstable axiom-system.

This undermines [*Success*]. Unstable axiom-systems are problematic because they impose constraints on the size of the universe other than the requirement that the universe be *at least* of a certain size. For instance, an unstable theory might impose a constraint to the effect that the universe contains power-of-an-inaccessible many objects. This means that different unstable theories might impose incompatible constraints on the size of the universe. And it is difficult to see how one could ever be justified in believing that the constraints imposed by a given theory obtain, rather than the incompatible constraints of its rivals. The methods of justification internal to the mathematical community are unlikely to help. It seems unreasonable to suppose, for example, that purely mathematical considerations would favor unstable axiomatizations of set theory over unstable axiomatizations of class theory, rather than allowing set theory and class theory to coexist peacefully side by side. And it is again difficult to imagine that one could use non-mathematical methods to determine whether, e.g. there are precisely power-of-an-inaccessible many objects. But, unless we are justified in believing that the relevant constraints on the size of the universe obtain, we cannot be justified in thinking that our specification has succeeded in picking out some mathematical structure or other. So [*Success*] fails.

If this is right, then structuralists face a dilemma. They must either abandon [*Success*] or abandon [*Uniqueness*]. Neither option is very comfortable. If they give up [*Success*], then they are unable to guarantee that their specification of the structure of the pure sets is, in fact, compatible with structure of the pure sets (since they are unable to rule-out a scenario in which it doesn't pick out any mathematical structure at all). If, on the other hand, they give up [*Uniqueness*], then they are left with a choice. They might decide to hold on to the claim that there is a unique mathematical structure which constitutes the subject-matter of set theory in spite of the failure of [*Uniqueness*], and endeavor to explain how it is that set theoretic terms come to be associated with just one of a large family of rival structures even though mathematicians lack the resources to specify which. But it is not easy to see how such a story would go. Alternatively, structuralists might take the failure of [*Uniqueness*] to show that set theory does not have a *unique* structure as its subject-matter. Instead, its truth conditions are associated with several different mathematical structures.[17] But this position comes at a cost. For one might have thought that the difference between *interpreted* mathematical theories—such as arithmetic and set theory—and *uninterpreted* mathematical theories—such as group theory—is that, whereas the former have a unique structure as their subject matter, the latter do not.[18] Structuralists must therefore tell a novel story about the distinction between interpreted and uninterpreted mathematical theories. But, again, it is not easy to see how such a story would go.

Appendix

- **First Limitative Result**

  Proof Sketch: Assume, for *reductio*, that A is a recursive, satisfiable and categorical set of true sentences of the language of set theory. Within the language of second-order set theory, define the predicates Model(x) and Sat(x, y). Model(x) is satisfied by a set M just in case M is a (set-sized) model for second-order set theory, and Sat(x, y) is satisfied by a pair of a set M and (the Gödel number of) a formula ϕ just in case M is a (set-sized) model that satisfies ϕ. Since A is a recursive set of sentences, "x is a (Gödel number of a) member of A" is represented by some formula of second-order set theory. Let $\ulcorner \overline{\mathrm{A}} \models \overline{\phi} \urcorner$ be a formula of the language of second-order set theory that states that (the Gödel number of) ϕ is satisfied in every (set-sized) model in which (the Gödel numbers of) all members of A are satisfied.

In the presence of Second-order Global Reflection, $\ulcorner \overline{\mathrm{A}} \models \overline{\phi} \urcorner$ is a truth predicate for the language of second-order set theory. For suppose that ϕ is a true formula of set theory. Since every sentence in A is true, all members of $\mathrm{A} \cup \{\phi\}$ are true. Second-order Global Reflection therefore guarantees that there is a model M that satisfies all members of $\mathrm{A} \cup \{\phi\}$. Since A is categorical, it follows that $\ulcorner \overline{\mathrm{A}} \models \overline{\phi} \urcorner$ is true. If, on the other hand, ϕ is false, then all members of $\mathrm{A} \cup \{\neg\phi\}$ are true. Second-order Global Reflection therefore guarantees that there is a model M that satisfies all members of $\mathrm{A} \cup \{\neg\phi\}$, from which it follows that $\ulcorner \overline{\mathrm{A}} \models \overline{\phi} \urcorner$ is false. Accordingly, $\ulcorner \overline{\mathrm{A}} \models \overline{\phi} \urcorner$ iff ϕ is true. This contradicts Tarski's Theorem, according to which no language expresses its own truth predicate.□

- **Second Limitative Result**

Proof Sketch: We proceed in two steps. First we note that the cardinality of any set-sized model of ZFCU2 (or extensions thereof) must fulfill a certain condition.[19] Then we observe that, given any cardinal that satisfies the condition, there is a higher cardinal that fails to satisfy the condition.

First step. Let κ be an infinite cardinal and suppose that $\langle M, E, S\rangle$ is a model of ZFCU2 of cardinality κ (where M is the domain of the model, E is the interpretation of '$\in$' and S is the interpretation of 'Set'). We show that there is an inaccessible $\lambda \leq \kappa$ such that $\kappa^{<\lambda} \leq \kappa$ (where $\kappa^{<\lambda}$ is the cardinality of the set of subsets of κ of cardinality strictly less than λ).

For each $a \in S$, let $(a)_E = \{x \in M : xEa\}$. Notice that for no $a \in S$, $|(a)_E| = \kappa$. For otherwise, by second-order replacement in $\langle M, E, S\rangle$, there would be $b \in S$ such that $(b)_E = M$, which is impossible. Let λ be the least cardinal $\leq \kappa$ such that for no $a \in S$, $|(a)_E| = \lambda$. Then $(a)_E \in [M]^{<\lambda}$ for any $a \in S$ (where $[M]^{<\lambda}$ is the set of subsets of M of cardinality strictly less than λ).

Let $X \in [M]^{<\lambda}$. By minimality of λ there is an $a \in M$ such that $|X| \leq |(a)_E|$. Let F be a function from $(a)_E$ onto X. By second-order replacement in $\langle M, E, S\rangle$, there is some $b \in S$ such that $(b)_E = X$. Since extensionality holds in $\langle M, E, S\rangle$, this yields the result that, for any $X \in [M]^{<\lambda}$, there is a unique $b \in M$ such that $(b)_E = X$. So the members of $[M]^{<\lambda}$ are precisely the $(a)_E$ for $a \in S$. Since $\langle M, E, S \rangle$ is a model of ZFCU2, λ must be an inaccessible. But $|M| = \kappa$. So λ is an inaccessible $< \kappa$ such that $\kappa^{<\lambda} \leq \kappa$.

Second step. It can be shown that $\beth_{\alpha+\omega}$ has as many *countable* subsets as subsets altogether.[20] So, when $\kappa = \beth_{\alpha+\omega}$, there can be no inaccessible $\lambda < \kappa$ such that $\kappa^{<\lambda} \leq \kappa$. By the first step, it follows that the cardinals that fail to satisfy ZFCU2 are unbounded. □

Notes

[1]Compare, for instance, [14], [25], [21] and [36].

[2]For present purposes, it is best not to think of models as sets. A second-order conception of model is set forth in [30] and further developed in [31]. In what follows we shall use 'model' in the second-order sense. (This means that it makes no literal sense to speak of a *class* of models. Claims to the effect that every member of a class $\mathcal{C}$ of models satisfies a certain condition Φ are therefore to be understood as shorthand for claims to the effect that every model satisfying a *predicate* '$C(X)$' satisfies Φ.)

[3]See, however, [2], [3], [4], [21], [22], [32], [29] and [43]. Boolos's work is criticized in [34], [25] and [20]. Relevant texts also include [35], [17], [18] and [26].

[4]Assuming, of course, that they are satisfiable.

[5]It may be of interest to note that when logically unrestricted quantifiers '$\forall^U$'are allowed—quantifiers taking unrestricted range in every model—McGee's partial categoricity result translates into a full categoricity result for ZFCU2 plus the Urelement Set Axiom: two models of ZFCU2^U plus the Urelement Set Axiom^U have isomorphic pure sets. For more on logically unrestricted quantifiers, see [42], [31] and [43].

[6]For discussion on quantifying over everything see [23], [9] chapters 14-16, [7], [6], [42], [22], the postscript to [10] in [11], [27], [28], [31], [12], and [43].

[7]See footnote 2.

[8]The P-restriction of a model M of $\mathcal{A}$ to P is the elementary submodel M whose domain is the extension of P according to M.

[9] Consider the theory of cardinal numbers embodied in HP (Hume's Prinpicle):

$$\forall X \forall Y [Num(X) = Num(Y) \leftrightarrow X \approx Y],$$

where'$X \approx Y$' abbreviates a second-order formula expressing one-one correspondence between the objects falling under 'X' and the objects falling under 'Y'.

HP need not be categorical: on the assumption that there are uncountable models of HP, an uncountable model of HP will not be isomorphic to a countable one. So, [*Axiomatic Categoricity*] does not guarantee that HP can be used on its own to uniquely specify a mathematical structure. But HP has the same feature as McGee's system: any two models of HP of the same cardinality have isomorphic numbers. So [*U-Axiomatic P-Categoricity*] delivers the result that, when the range of the quantifiers is taken to be absolutely unrestricted, mathematicians can use HP to to specify the structure of the numbers.

[10]In the case of Morse-Kelley class theory, this presupposes that the class variables are taken to be genuinely individual variables. For an alternative treatment, see [40].

[11]See, for instance, [**?**]. The mereological fusion of the Fs is the unique x such that (i) x has each of the Fs as parts, and (ii) every part of x has a part in common with an F.

[12]A mereological atom is an object with no parts other than itself. There is atomless gunk if there is some object which is not the fusion of mereological atoms.

[13]Cf. [13] n. 5.

[14]What if we had characterized stability in terms of satisfiability in arbitrary domains (set-sized or not)? The problem with this suggestion is that satisfiability in models with unrestricted domains becomes a necessary and sufficient condition for stability. So the new notion does not deliver a useful test for determining whether a mathematical theory is satisfiable by the universal domain. (It is also worth noting that the second limitative result below does not hold for the revised notion of stability.)

[15]For example, if—*pace* Second-order Global Reflection—there is a finite axiomatization of T that is satisfied in models with an unrestricted domain but false in all set-sized models, then the negation of the Ramsey sentence for T will be a stable theory that is not satisfiable in models with

unrestricted domains.

[16]One informal way of motivating this claim makes use of the thought underlining the standard motivation for the principle of Second-order Reflection—the thought that any structural feature of the universe of sets ought to be mirrored by some set-sized model of set-theory. Suppose a set theory T imposes some structural constraint $\mathcal{F}$ on the cardinality of the universe which happens not to be met. On reasonable assumptions, the universe of all sets is just as large as the universe of all objects. So the universe of all sets will also fail to meet $\mathcal{F}$. But if any structural feature of the universe of sets is mirrored by some set-sized model of set-theory, then there must be models of arbitrarily high cardinality which satisfy $\mathcal{F}$ (since, given some structural feature and some cardinal κ, we can think of a further structural feature satisfied by the universe that combines the preceding one with the condition that the universe be of cardinality greater than κ). T is therefore not stable. Vague as it is, this sort of argument has proven extremely valuable in the development of set theory (see, for example, [33]).

[17]This means that [*Mathematical Truth*] must be given up. The most natural substitute is presumably a version of supervaluationism:

> [*Supervaluational Mathematical Truth*]
>
> Let $\mathcal{S}$ be a non-empty class of structures picked out by mathematicians as candidates for constituting the subject-matter of a theory T of pure mathematics, and, for $s \in \mathcal{S}$, let $\mathcal{C}_s$ be the class of models that exemplify s. Then a sentence in the language of T is true if, for every $s \in \mathcal{S}$, it is true according to every model in $\mathcal{C}_s$; the sentence is false if its negation is true; and otherwise the sentence is neither true nor false.

[18]See, for instance, [41].

[19]The proof of this fact is essentially taken from [15].

[20]More generally, if γ is a singular limit cardinal of cofinality ζ, there are as many subsets of γ of cardinality ζ as there are subsets of γ altogether. See Theorem 12 in [37].

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