Quantum Field Theory and Representation Theory

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Outline of the talk

- Quantum Mechanics and Representation Theory: Some History

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- Quantum Field Theories in 1+1 dimensions
- Twisted K-theory and the Freed-Hopkins-Teleman theorem

Some History

Quantum Mechanics

- Summer 1925: Observables are operators (Heisenberg)
- Fall 1925: Poisson Bracket $\longrightarrow$ Commutator (Dirac)
- Christmas 1925: Representation of operators on wave-functions (Schrödinger)

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Representation Theory

- Winter-Spring 1925: Representation Theory of Compact Lie Groups (Weyl)
- Spring 1926: Peter-Weyl Theorem (Peter, Weyl)

Schrödinger and Weyl

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Weyl’s Book

1928: Weyl’s "Theory of Groups and Quantum Mechanics", with alternate chapters of group theory and quantum mechanics.

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Weyl.

The Gruppenpest

Wolfgang Pauli: the "Gruppenpest", the plague of group theory.

For a long time physicists mostly only really needed representations of:

- $\mathbf{R}^n, \;\; U(1)$: Translations, phase transformations. (Fourier analysis)
- $SO(3)$: Spatial rotations.
- $SU(2)$: Spin double cover of $SO(3)$, isospin.

Widespread skepticism about use of representation theory until Gell-Mann and Neeman use $SU(3)$ representations to classify strongly interacting particles in the early 60s.

Representation Theory: Lie Groups

Definition. *A representation of a Lie group G on a vector space V is a homomorphism*

$$\rho : g \in G \to \rho(g) \in GL(V)$$

We're interested in representations on complex vector spaces, perhaps infinite dimensional (Hilbert space). In addition we'll specialize to unitary representations, where $\rho(g) \in U(V)$, transformations preserving a positive definite Hermitian form on V.

For $V = \mathbf{C}^n$, $\rho(g)$ is just a unitary n by n matrix.

Representation Theory: Lie Algebras

Taking differentials, from ρ we get a representation of the Lie algebra $\mathfrak{g}$ of G:

$$\rho' : \mathfrak{g} \rightarrow End(V)$$

For a unitary ρ, this will be a representation in terms of self-adjoint operators.

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- Schrödinger Equation: H generates time evolution of states

$$i\frac{d}{dt}|\Psi >= H|\Psi >$$

Symmetry in Quantum Mechanics

Schrodinger's equation: H is the generator of a unitary representation of the group R of time translations.

Physical system has a Lie group G of symmetries $\rightarrow$ the Hilbert space of states $\mathcal{H}$ carries a unitary representation ρ of G.

This representation may only be projective (up to complex phase), since a transformation of $\mathcal{H}$ by an overall phase is unobservable.

Elements of the Lie algebra $\mathfrak{g}$ give self-adjoint operators on $\mathcal{H}$, these are observables in the quantum theory.

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- Phase transformations: Charge Q, $G = U(1)$.

Quantization

Expect to recover classical mechanical system from quantum mechanical one as $\hbar \to 0$

Surprisingly, can often “quantize" a classical mechanical system in a unique way to get a quantum one.

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- Observables: functions on M
- Hamiltonian: distinguished observable H corresponding to the energy.
- Hamilton's equations: time evolution is generated by a vector field X_H on M determined by

$$i_{X_H}\omega = -dH$$

where ω is the symplectic form on M.

Quantization + Group Representations

Would like quantization to be a functor

(Symplectic manifolds M, symplectomorphisms)

$$\downarrow$$

(Vector spaces, unitary transformations)

This only works for some subgroups of all symplectomorphisms. Also, get projective unitary transformations in general.

Mathematics $\longrightarrow$ Physics

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Example: In Grand Unified Theories, particles form representations of groups like $SU(5)$, $SO(10)$, E_6, E_8.

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What can mathematicians learn from quantum mechanics?

- Constructions of representations starting from symplectic geometry (geometric quantization).
- Interesting representations of infinite dimensional groups (quantum field theory).

Canonical Example: R^{2n}

Standard flat phase space, coordinates $(p_i, q_i),\ i = 1 \ldots n$:

$$M = \mathbf{R}^{2n}, \omega = \sum_{i=1}^{n} dp_i \wedge dq_i$$

Quantization:

$$[\hat{p}_i, \hat{p}_j] = [\hat{q}_i, \hat{q}_j] = 0, \quad [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$$

(This makes $\mathbf{R}^{2n+1}$, a Lie algebra, the Heisenberg algebra)

Schrödinger representation on $\mathcal{H} = L^2(\mathbf{R}^n)$:

$$\hat{q}_i = \text{mult. by } q_i, \quad \hat{p}_i = -i\hbar\frac{\partial}{\partial q_i}$$

Metaplectic representation

Pick a complex structure on $\mathbf{R}^{2n}$, e.g. identify $\mathbf{C}^n = \mathbf{R}^{2n}$ by

$$z_j = q_j + ip_j$$

Then can choose $\mathcal{H} = \{\text{polynomials in } z_j\}$.

The group $Sp(2n, \mathbf{R})$ acts on $\mathbf{R}^{2n}$ preserving ω, $\mathcal{H}$ is a projective representation, or a true representation of $Mp(2n, \mathbf{R})$ a double cover.

Segal-Shale-Weil = Metaplectic = Oscillator Representation

Exponentiating the Heisenberg Lie algebra get H^{2n+1}, Heisenberg group (physicists call this the Weyl group), $\mathcal{H}$ is a representation of the semi-direct product of H^{2n+1} and $Mp(2n, \mathbf{R})$.

Quantum Field Theory

A quantum field theory is a quantum mechanical system whose configuration space ($\mathbf{R}^n$, space of q_i in previous example) is infinite dimensional, e.g. some sort of function space associated to the physical system at a fixed time.

- Scalar fields: $Maps(\mathbf{R}^3 \to \mathbf{R})$
- Charged fields: sections of some vector bundle
- Electromagnetic fields: connections on a $U(1)$ bundle

These are linear spaces, can try to proceed as in finite-dim case, taking $n \longrightarrow \infty$.

A Different Example: S^2

Want to consider a different class of example, much closer to what Weyl was studying in 1925.

Consider an infinitely massive particle. It can be a non-trivial projective representation of the spatial rotation group $SO(3)$, equivalently a true representation of the spin double-cover $Spin(3) = SU(2)$.

$\mathcal{H} = \mathbf{C}^{n+1}$, particle has spin $\frac{n}{2}$.

Corresponding classical mechanical system:

$$M = S^2 = SU(2)/U(1), \quad \omega = n \times \text{ Area 2-form}$$

This is a symplectic manifold with $SU(2)$ action (left multiplication).

Geometric Quantization of S^2

What is geometric construction of $\mathcal{H}$ analogous to Fock representation in linear case?

Construct a line bundle L over $M = SU(2)/U(1)$ using the standard action of $U(1)$ on $\mathbf{C}$.

$$L = SU(2) \times_{U(1)} \mathbf{C}$$

$$\downarrow$$

$$M = SU(2)/U(1)$$

M is a Kähler manifold, L is a holomorphic line bundle, and $\mathcal{H} = \Gamma_{hol}(L^n)$, the holomorphic sections of the n'th power of L.

Borel-Weil Theorem (1954) I

This construction generalizes to a geometric construction of all the representations studied by Weyl in 1925.

Let G be a compact, connected Lie group, T a maximal torus (largest subgroup of form $U(1) \times \cdots \times U(1)$). Representations of T are “weights", letting T act on $\mathbf{C}$ with weight λ, can construct a line bundle

$$L_\lambda = G \times_T \mathbf{C}$$

$$\downarrow$$

$$G/T$$

G/T is a Kähler manifold, L_λ is a holomorphic line bundle, and G acts on $\mathcal{H} = \Gamma_{hol}(L_\lambda)$.

Borel-Weil Theorem II

Theorem (Borel-Weil). *Taking λ in the dominant Weyl chamber, one gets all elements of $\hat{G}$ (the set of irreducible representations of G) by this construction.*

In sheaf-theory language

$$\mathcal{H} = H^0(G/T, \mathcal{O}(L_\lambda))$$

Note: the Weyl group $W(G,T)$ is a finite group that permutes the choices of dominant Weyl chamber, equivalently, permutes the choices of invariant complex structure on G/T.

Relation to Peter-Weyl Theorem

The Peter-Weyl theorem says that, under the action of $G \times G$ by left and right translation,

$$L^2(G) = \sum_{i \in \hat{G}} End(V_i) = \sum_{i \in \hat{G}} V_i \times V_i^*$$

where the left G action acts on the first factor, the right on the second.

To extract an irreducible representation j, need something that acts on all the V_i^*, picking out a one-dimensional subspace exactly when $i = j$.

Borel-Weil does this by picking out the subspace that

- transforms with weight λ under T
- is invariant under $\mathfrak{n}_+$, where $\mathfrak{g}/\mathfrak{t} \otimes C = \mathfrak{n}_+ \oplus n_-$

Borel-Weil-Bott Theorem (1957)

What happens for λ a non-dominant weight?

One gets an irreducible representation not in $H^0(G/T, \mathcal{O}(L_\lambda))$ but in higher cohomology $H^j(G/T, \mathcal{O}(L_\lambda))$.

Equivalently, using Lie algebra cohomology, what picks out the representation is not $H^0(\mathfrak{n}_+, V_i^*) \neq 0$ (the $\mathfrak{n}_+$ invariants of V_i^*), but $H^j(\mathfrak{n}_+, V_i^*) \neq 0$. This is non-zero when λ is related to a λ' in the dominant Weyl chanber by action of an element of the Weyl group.

In some sense irreducible representations should be labeled not by a single weight, but by set of weights given by acting on one by all elements of the Weyl group.

The Dirac Operator

In dimension n, the spinor representation S is a projective representation of $SO(n)$, a true representation of the spin double cover $Spin(n)$. If M has a "spin-structure", there is spinor bundle $S(M)$, spinor fields are its sections $\Gamma(S(M))$.

The Dirac operator $\not{D}$ was discovered by Dirac in 1928, who was looking for a “square root" of the Laplacian. In local coordinates

$$\not{D} = \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}$$

$\not{D}$ acts on sections of the spinor bundle. Given an auxiliary bundle E with connection, one can form a “twisted" Dirac operator

$$\not{D}_E : \Gamma(S(M) \times E) \to \Gamma(S(M) \times E)$$

Borel-Weil-Bott vs. Dirac

- Instead of using the Dolbeault operator $\overline{\partial}$ acting on complex forms $\Omega^{0,*}(G/T)$ to compute $H^*(G/T, \mathcal{O}(L_\lambda))$, consider $\not{D}$ acting on sections of the spinor bundle.
- Equivalently, instead of using $\mathfrak{n}_+$ cohomology, use an algebraic version of the Dirac operator as differential (Kostant Dirac operator).

Basic relation between spinors and the complex exterior algebra:

$$S(\mathfrak{g}/\mathfrak{t}) = \Lambda^*(\mathfrak{n}_+) \otimes \sqrt{\Lambda^n(\mathfrak{n}_-)}$$

Instead of computing cohomology, compute the index of $\not{D}$ This is independent of choice of complex structure on G/T.

Equivariant K-theory

K-theory is a generalized cohomology theory, and classes in K^0 are represented by formal differences of vector bundles. If E is a vector bundle over a manifold M,

$$[E] \in K^0(M)$$

If G acts on M, one can define an equivariant K-theory, $K^0_G(M)$, whose representatives are equivariant vector bundles.

Example: $[L_\lambda] \in K^0_G(G/T)$

Note: equivariant vector bundles over a point are just representations, so

$$K^0_G(pt.) = R(G) = \text{ representation (or character) ring of } G$$

Fundamental Class in K-theory

In standard cohomology, a manifold M of dimension d carries a “fundamental class" in homology in degree d, and there is an integration map

$$\int_M : H^d(M, \mathbf{R}) \to H^0(pt., \mathbf{R}) = \mathbf{R}$$

In K-theory, the Dirac operator provides a representative of the fundamental class in “K-homology" and

$$\int_M : [E] \in K_G(M) \to \text{index}\not{D}_E = \text{ker}\not{D}_E - \text{coker}\not{D}_E \in K_G(pt.) = R(G)$$

Quantization = Integration in K-theory

In the case of $(G/T, \omega = curv(L_\lambda))$ this symplectic manifold can be "quantized" by taking $\mathcal{H}$ to be the representation of G given by the index of $\not{D}_{L_\lambda}$.

This construction is independent of a choice of complex structure on G/T, works for any λ, not just dominant ones.

In some general sense, one can imagine "quantizing" manifolds M with vector bundle E, with $\mathcal{H}$ given by the index of $\not{D}_E$.

Would like to apply this general idea to quantum field theory.

History of Gauge Theory

- 1918: Weyl's unsuccessful proposal to unify gravity and electromagnetism using symmetry of local rescaling of the metric
- 1922: Schrödinger reformulates Weyl's proposal in terms of phase transformations instead of rescalings.
- 1927; London identifies the phase transformations as transformations of the Schrödinger wave-function.
- 1954: Yang and Mills generalize from local $U(1)$ to local $SU(2)$ transformations.

Gauge Symmetry

Given a principal G bundle over M, there is an infinite-dimensional group of automorphisms of the bundle that commute with projection. This is the gauge group $\mathcal{G}$. Locally it is a group of maps from the base space M to G.

$\mathcal{G}$ acts on $\mathcal{A}$, the space of connections on P.

Example: $M = S^1$, $P = S^1 \times G$, $\mathcal{G} = Maps(S^1, G) = LG$

$\mathcal{A}/\mathcal{G} =$ conjugacy classes in G, identification given by taking the holonomy of the connection.

Loop Group Representations

See Pressley and Segal, Loop Groups.

LG/G is an infinite-dimensional Kähler manifold, and there is an analog of geometric quantization theory for it, with two caveats:

- One must consider “positive energy" representations, ones where rotations of the circle act with positive eigenvalues.
- Interesting representations are projective, equivalently representations of an extension of LG by S^1. The integer classifying the action of S^1 is called the “level".

For fixed level, one gets a finite number of irreducible representations.

QFT in 1+1 dimensions

Consider quantum field theory on a space-time $S^1 \times \mathbf{R}$. The Hilbert space $\mathcal{H}$ of the theory is associated to a fixed time S^1.

The level 1 representation for $LU(N)$ is $\mathcal{H}$ for a theory of a chiral fermion with N “colors".

At least in 1+1 dimensions, representation theory and quantum field theory are closely related.

Freed-Hopkins-Teleman

For loop group representations, instead of the representation ring $R(LG)$, one can consider the Verlinde algebra V_k. As a vector space this has a basis of the level k positive energy irreducible representations.

Freed-Hopkins-Teleman show:

$$V_k = K^{k+\tau}_{dim\ G,G}(G)$$

The right-hand side is equivariant K-homology of G (under the conjugation action), in dimension $dim\ G$, but “twisted" by the level k (shifted by τ). This relates representation theory of the loop group to a purely topological construction.

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- Apply physicist's “BRST" formalism.
- Get explicit representative of a K-homology class in $K_{\mathcal{G}}(\mathcal{A})$.
- Idenitfy $K_{\mathcal{G}}(\mathcal{A})$ with FHT's $K_G(G)$, since for a free H action on M, $K_H(M) = K(M/H)$.

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- Work in Progress: 1+1 dim QFT and twisted K-theory. Relate path integrals and BRST formalism to representation theory and K-theory.
- QFT in higher dimensions?