APOLOGY FOR THE PROOF OF THE RIEMANN HYPOTHESIS

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The Riemann hypothesis is the product of a renaissance in mathematics which began in the seventeenth century after more than a thousand years in which Greek mathematics lay dormant in monasteries. When mathematics is viewed as the science of numbers, two aspects of that science are found to be strikingly present in antiquity. The mysterious nature of prime numbers was explored. The Euclidean algorithm was introduced to show that a positive integer is an essentially unique product of primes. The existence of an infinite number of primes was known. In another direction the infinite nature of the number system was illustrated in paradoxical examples. The problem of determining the quantity of wine in a round tub was proposed by Archimedes. Infinite processes, which are uncomputable, are implicitly used by every wine merchant. An axiomatic treatment is required for mathematical analysis even at the most fundamental level. The elusive nature of infinite processes and the fascination of the integers remain as permanent attractions of mathematics.

The rebirth of mathematics is exemplified by the lives of Renée Descartes (1596–1652), Pierre de Fermat (1601–1665), and Blaise Pascal (1623–1662). Mathematical discoveries are stimulated by the Cartesian philosophy that the universe of mathematics, and even the physical universe, is subject to analysis. Mathematics acquired unity when viewed as the science of numbers from which are derived more geometrical mathematical quantities. Remarkable properties of prime numbers were discovered by Fermat. Every prime which is congruent to one modulo four admits an essentially unique representation $a^2 + b^2$ for integers a and b. Every prime which is congruent to one modulo six admits an essentially unique representation $a^2 - ab + b^2$ for integers a and b. The education of Blaise Pascal illustrates the fertile nature of mathematical activity in his day.

Pascal scarcely knew his mother since she died when he was two years old. His father took him to Paris when he was a child and assumed personal responsibility for his education. The child was taught to express himself in writing, to read the religious, ethical, and political texts which presented the foundations of the society in which he lived, and to know something of its history. But the existence of disciplines of less immediate import was not concealed. One of these was Euclidean geometry. Pascal received sufficient information to reconstruct the subject without destroying its interest. The learned societies of the day welcomed a fresh view of results which were thought to be completely understood. They received with enthusiasm the theorems which Pascal continued to discover throughout his life.

The mathematical awakening which took place in the seventeenth century is inseparable from a religious awakening permitting advances in education. The revolutionary doctrine that every child deserves an education, regardless of social rank, was practiced by Jansenist nuns inspired by the writings of Saint Augustine. A regrettable conflict arose with the recently consolidated authority of the French state. In 1709 Louis XIV ordered the destruction of the Jansenist monastery in Port-Royal. The museum which now commemorates the site contains a portrait of Jacqueline Pascal painted by the court artist Philippe de Champagne. In the preface to Pascal's *Pensées*, she describes with meticulous care the education of her younger brother.

A massive exodus of talent from France was caused by persecution of Huguenots in the seventeenth century. Although mathematics continued to flourish in Paris, Basel and Petersburg emerged as mathematical centers. The Cartesian tradition was continued outside of France by Isaac Newton (1640–1727) and Leonard Euler (1707–1783). The image of Newton as inventor of the infinitesimal calculus reveals only one facet of his mathematical genius. Newton was unlike the majority of English mathematicians who define mathematics by its applications. Although Newton gave applications to astronomy and optics, he also contributed to the concept of mathematics as the science of numbers. The Newton polynomials

$$\frac{s(s-1)\ldots(s+1-n)}{1.2.\ldots n}$$

serve as a link between the two aspects of number already evident in antiquity. The integers are present in an analytic structure which is an expression of the infinite nature of mathematics.

The publication of Newton's *Principia Mathematica* in 1687 was an exciting event in the lives of the mathematical members of the Bernoulli family in Basel, where Euler later came for graduate work. Although the greater part of his career was devoted to applications of mathematics, he also made fundamental contributions to mathematics defined as the science of numbers. The discovery of the gamma function, which is the first contribution, seems to have been made accidentally. The result is an application of techniques essentially due to Newton. The gamma function $\Gamma(s)$ is a solution of the recurrence relation

$$s\Gamma(s) = \Gamma(s+1)$$

which has the value one when s is equal to one. The function has the value

$$1.2\ldots n$$

when s is equal to $n+1$ for a nonnegative integer n. Asked the value of $\Gamma(s)$ when s is not a nonnegative integer, Euler gave an answer which is determined by the product

$$\Gamma(s)^{-1} = e^{\gamma s} s \prod (1+s/n)e^{-s/n}$$

taken over the positive integer n. The positive constant γ which is needed to validate the functional identity is known as Euler's constant. Although other answers can be given, they are easily derived from the answer he gave, which is generally accepted as the simplest answer.

The discovery of the gamma function in 1730 would be less significant without his discovery of the zeta function in 1737. The classical zeta function

$$\zeta(s) = \sum n^{-s}$$

is defined, when the real part of s is greater than one, as a sum over the positive integers n. The function can also be defined by the Euler product

$$\zeta(s)^{-1} = \prod (1-p^{-s})$$

taken over the primes p. The advantage of the product is that it exhibits the absence of zeros in the half–plane. The classical zeta function is another link between the two aspects of numbers discovered in antiquity. The primes are present in a structure, now called analysis, which is an expression of the infinite nature of mathematics. The infinitessimal calculus, Newton polynomials, the gamma function, and the zeta function all belong to analysis.

The functional identity for the zeta function was discovered by Euler only in 1761. The identity states that the function

$$\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s)$$

remains unchanged when s is replaced by $1 - s$. The meaning of the identity is not immediate since the functions are defined in disjoint half–planes separated by the critical strip. The strip contains the complex numbers whose real part lies strictly between zero and one. What is meant is that the two functions have natural extensions across the boundaries of the strip and that the extensions are equal.

Euler made his career in a century of scientific advances which changed the economies of Europe. He received research funds, initially from Catherine the Great and later from Frederic the Great, for results of benefit to society. The bulk of his publications, which comprise more than one hundred volumes, contains applications of mathematics which seem of no permanent value. But the five volumes which treat mathematics as the science of numbers have lost none of their interest. Debates were held at the Prussian court between Leonard Euler and François Voltaire (1694–1778) concerning the meaning of Cartesian philosophy for scientific research. Euler was unable to give a convincing defence of principles which guided a lifetime of work. He was replaced as mathematician in residence by Louis de Lagrange (1736–1813).

The significance of the classical zeta function for the distribution of prime numbers did not excape reader's of Euler's work such as Lagrange and Carl Friederich Gauss (1777–1855). The logarithmic integral

$$\int_e^x \frac{dt}{\log(t)}$$

is a good approximation to the number of primes less than a given positive number x. A consolidation of the theory of the classical zeta function was however needed to know how good is the approximation. A first step was made in 1811 by Gauss in a systematic treatment of the hypergeometric series

$$F(a, b; c; z) = 1 + \frac{ab}{1c}\, z + \frac{a(a+1)b(b+1)}{1 \cdot 2c(c+1)}\, z^2 + \dots$$

which clarifies the relationship of the gamma function to the Newton polynomials.

The definition of the classical zeta function in the critical strip is clarified by the methods of complex analysis introduced by Augustin Cauchy (1789–1857). The extension of the zeta function across the boundaries of the critical strip is an example of analytic continuation. The Cauchy formula represents a function in a region of analyticity when the function is known on the boundary. The classical zeta function $\zeta(s)$ is an analytic function of s in the complex plane with the exception of a singularity at s equal to one. The product $(s-1)\zeta(s)$ is an analytic function of s in the complex plane when it is given the value one at s equal to one. The concept of analysis acquires a special meaning as a result of these applications. Complex analysis is that part of analysis which makes effective use of the properties of complex numbers such as the Cauchy formula.

The nature of the classical zeta function was clarified by Lejeune Dirichlet (1805–1859) when it was identified as a member of a large class of related functions. The classical zeta function is an exceptional member of the class because of its singularity. A typical zeta function is analytic in the complex plane. This observation gives interest to a theory of entire functions. These functions are analytic in the complex plane. The theory of entire functions is implicitly a theory of functions which resemble zeta functions.

Dirichlet pursues an iniative of Gauss which leads to a functional identity for Dirichlet zeta functions. Bernhard Riemann (1826–1866) returns to the fundamental issue raised by the classical zeta function. How well does the logarithmic integral estimate the number of primes less than a positive number x? Riemann observes that this issue is determined by the position of the zeros of the classical zeta function in the critical strip. The critical line, which runs down the middle of the critical strip, is the set of complex numbers whose real part is equal to one–half. The zeros of the classical zeta function are symmetric about the critical line because of the functional identity. The Riemann hypothesis is the conjecture that these zeros lie on the critical line. The statement of the conjecture in 1859 completes the consolidation of Euler's work.

Since the subsequent history of the Riemann hypothesis is complicated, I will approach it from the proof of the view of my own involvement. The solution of a celebrated problem creates a disturbance in the otherwise quiet flow of mathematical events. The solution escapes the planning of committees. Colleagues are unprepared because the possibility of a solution has not been included in their research proposals. Students have avoided related thesis topics because of the risk that the work will not be welcome to a prospective employer. Friends are discouraged from research activity by the demands of the situation created by the solution. The manuscript, which is necessarily written at the highest research level, is readable only to a limited audience. An introduction is therefore needed which makes available the opportunities created by the solution. This is done by supplying motivation for the argument in a chronological order which also gives an account of how the solution was obtained.

The first aim is to account for the choice of the Riemann hypothesis as a research objective. Mathematicians ordinarily choose a research career under the guidance of a professor who supervises a doctoral thesis. But I came to mathematics at an earlier age under the influence of someone not associated with a university. I would not have had such a teacher had it not been for unusual circumstances resulting from the Second World War. I was born in Paris in 1932 and attended school there until 1941 when the German occupation compelled the departure of my mother and her three children to the United States. Since my father remained in France, my maternal grandparents assumed a responsibility that would normally fall to parents.

When the United States entered the war six months later, we were safe in a seashore cottage at Rehoboth Beach, Delaware. My adaptation to English as a primary language was eased by the summer visits made to my grandparents in my childhood. My earliest mathematical experience was obtained solving cryptograms in the Philadelphia Inquirer. The mystery of wartime secrecy stimulated logical thought. An effort to decipher coded information was seen as essential to survival.

My progress in school was sufficient for me to omit seventh grade. When I was twelve, I entered Saint Andrew's School, Middletown, Delaware. Since the cottage at Rehoboth was then sold, the house of my grandparents in Wilmington, Delaware, was home during vacations. My grandfather, Ellice Mc Donald, was a former surgeon and university lecturer who had turned to research. Research careers had recently become possible at the Rockefeller Institute. My grandfather found an alternative at the Franklin Institute by creating the Biochemical Research Foundation, of which he became director. The foundation moved from its original quarters in Philadelphia to new quarters off the campus of the University of Delaware as the United States entered the war.

I took my studies more seriously than other students did because my grandfather encouraged me to do so. I was stimulated by elementary algebra, which I studied in the third form. In the following summer vacation I solved the problems of an exercise book of intermediate algebra and was advanced to plane geometry for the fourth form. When I was home on vacations, I accompanied my grandfather to the Biochemical Research Foundation. And I caddied for him on Sunday mornings when he played golf with Irénée du Pont, the former president of the du Pont Company who supplied the funds for the operation of the foundation.

Mr. du Pont always drank a glass of rhum and orange juice in the clubhouse after playing golf. One morning he showed an unexpected interest in my mathematical education by posing a problem: Find positive integers a, b, and c such that

$$a^3 + b^3 = 22c^3.$$

Since the problem was more interesting than plane geometry, I spent the fourth form year solving it. For this purpose I had access to the libraries of Saint Andrew's School, the Biochemical Research Foundation, and the University of Delaware. With the help of these sources I was able to acquire the representation theory of positive integers in the form $a^2 - ab + b^2$ for integers a and b. This information is a prerequisite to a solution of the problem, which I have unfortunately lost. It was an achievement comparable to my doctoral thesis written ten years later. The result was difficult to check because the numbers obtained had five and six digits. Cubing them was beyond the capacity of the Marchant calculators available at the Biochemical Research Foundation. Mr. du Pont conceded the correctness of the solution but never revealed the source of the problem. This variant of the Fermat problem originates with Louis de Lagrange, who states it however with 10 in place of 22. The similarity between his name and mine confirms my impression that Mr. du Pont was amusing himself at my expense. Teasing was a common expression of his Mephistophelian temperament.

Another significant mathematical experience occurred in my fifth form year. I learned from a graduate text of the existence of a generalization of the factorial called the gamma function. In the course of the year I discovered the Euler product for the function without any formal training in complex analysis. It was sufficient to have a good understanding of the calculus. There remained little to learn of the calculus when it was offered as a course in the sixth form. Experience in calculation made instruction superfluous. The gamma function remained as an interest decisive in the proof of the Riemann hypothesis.

Since my grandfather was pleased with my mathematical progress, he decided that I should have a college education. In the fall semester 1949 I entered the Massachusetts Institute of Technology, the university at which Mr. du Pont had been an undergraduate. I took my first mathematics course with George Thomas as he was writing his calculus text. I used his manuscript to prepare the remaining three semesters of the calculus, which I then disposed of in proficiency examinations. I was able to take a graduate course in linear algebra from Witold Hurewicz in the second semester. The text on *Modern Algebra* by Garret Birkhoff and Saunders Mac Lane, Macmillan, 1941, was familiar as it had been in the library of the Biochemical Research Foundation. During the summer I worked through the recently published *Lectures on Classical Differential Geometry* by Dirk Struik, Addison-Wesley, 1950. Since he made an error in the proficiency examination which he administered at the end of the summer, I received a score of 105 for correcting the statement of a problem before solving it. Walter Rudin taught me a course in my sophomore year on the *Principles of Mathematical Analysis* as his book was called when it was published by McGraw-Hill in 1953. I took many courses outside of mathematics under the influence of my grandfather. What I learned in theoretical physics is now called scattering theory. What I learned in electrical engineering is now called the theory of linear systems.

The concentration of my interests in mathematics was disappointing to my grandfather, who wanted an heir to his pioneering efforts in biochemistry. Although I obtained an undergraduate degree in chemistry as well as mathematics, my talents lay in theory rather than in applications. The direction of my education was consistent with the increased sophistication of research in the postwar era. The price of this evolution is the difficulty scientific contributions have in spanning the gap between fundamental research and its eventual applications.

An important decision was then made since I ceased to prepare for a career in a privately endowed research institute and began one supported by public funds linked to education. When I decided on graduate school in mathematics, I was discouraged by my mathematical advisors from applying to the prestigious universities Harvard and Princeton because of an insufficient concentration on courses in mathematics. With the backing of George Thomas I received a teaching assistantship at Cornell University for the fall semester 1953. The terrain lost as an undergraduate was already recovered in the first graduate year. My grandfather died in the middle of the second year as I was taking qualifying examinations for the doctoral program.

I returned to number theory on passing the examinations. Preparation for them included lecture notes of Emil Artin and Emma Nöther on *Moderne Algebra* taken by Bartel van der Waerden, Springer-Verlag, 1940. Three treatises by Edward Titchmarsh then determined the direction of my efforts: *Introduction to the Theory of Fourier Integrals, Eigenfunction Expansions Associated with Second-Order Differential Equations,* and *The Theory of the Riemann Zeta-Function,* Clarendon Press, Oxford, 1937, 1946, and 1951. The Riemann hypothesis is a unifying theme of these volumes which became the ultimate goal of my research.

The attack on the Riemann hypothesis begins in my doctoral thesis, which concerns a problem in Fourier analysis, due to Arne Beurling, which was posed by Harry Pollard to Wolfgang Fuchs. An axiomatization in Hilbert space was made in postdoctoral work. Assume that a nonnegative measure μ is given on the Borel subsets of the real line with respect to which all polynomials are square integrable. Determine the closure of the polynomials in $L^2(\mu)$ when the closure is not the whole space. A Hilbert space of entire functions is obtained which has these properties:

- (H1) Whenever $F(z)$ is in the space and has a nonreal zero w, the function $F(z)(z - w^-)/(z - w)$ belongs to the space and has the same norm as $F(z)$.
- (H2) A continuous linear functional is defined on the space by taking $F(z)$ into $F(w)$ for every nonreal number w.
- (H3) The function $F^*(z) = F(z^-)^-$ belongs to the space whenever $F(z)$ belongs to the space, and it always has the same norm as $F(z)$.

Examples of spaces with these properties appear in the Colloquium Publication of Raymond Paley and Norbert Wiener, *Fourier Transforms in the Complex Domain,* American Mathematical Society, 1934. If a positive number a is given, the Paley-Wiener space of index a is the set of entire functions $F(z)$ of the form

$$2\pi F(z) = \int_{-a}^{a} f(t) \exp(itz) dt$$

with a finite integral

$$2\pi \|F\|^2 = \int_{-a}^{a} |f(t)|^2 dt.$$

The elements of the space are the entire functions of exponential type at most a which are square integrable on the real axis. The norm of the space is computable in several ways in terms of function values on the real axis since the identity

$$\int_{-\infty}^{+\infty} |F(t)|^2 dt = (\pi/a) \sum_{-\infty}^{+\infty} |F(n\pi/a)|^2$$

holds for every element $F(z)$ of the space. The identity was observed in a related context in 1814 by Carl Friederich Gauss, “Methodus nova integralium valores per approximationen inveniendi,” *Werke,* Königliche Gesellschaft der Wissenschaften, Göttingen, 1886, volume 3, pp. 163–196.

A generalization of Gaussian quadrature applies in Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3). The structure theory for such a space is related to the theory of entire functions $E(z)$ which satisfy the inequality

$$|E(x - iy)| < |E(x + iy)|$$

for $y > 0$. Write

$$E(z) = A(z) - iB(z)$$

where $A(z)$ and $B(z)$ are entire functions which are real for real z and

$$K(w, z) = \frac{B(z)A(w)^- - A(z)B(w)^-}{\pi(z - w^-)}.$$

Then the set of entire functions $F(z)$ such that the integral

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt$$

is finite and such that the inequality

$$|F(z)|^2 \leq \|F\|^2 K(z, z)$$

holds for all complex numbers z, is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3). The function $K(w, z)$ of z acts as a reproducing kernel function for function values at w since it is the unique element of the space which satisfies the identity

$$F(w) = \langle F(t), K(w, t)\rangle$$

for every element $F(z)$ of the space. A Hilbert space, whose elements are entire functions, which satisfies the axioms (H1), (H2), and (H3), and which contains a nonzero element, is isometrically equal to a space $\mathcal{H}(E)$.

The Paley-Wiener space of index a is obtained when

$$E(z) = \exp(-iaz)$$

in which case

$$A(z) = \cos(az)$$

and

$$B(z) = \sin(az).$$

The definition of the norm of the space $\mathcal{H}(E)$ simplifies since $E(z)$ has modulus one on the real axis.

Multiplication by z in a space $\mathcal{H}(E)$ is the transformation which takes $F(z)$ into $G(z)$ whenever $F(z)$ and $G(z)$ are elements of the space such that

$$G(z) = z\,F(z).$$

Multiplication by z need not be densely defined in a space $\mathcal{H}(E)$, but its domain is nearly dense. The orthogonal complement of the domain consists of those elements of the space which are of the form

$$A(z)u + B(z)v$$

for complex numbers u and v. Since such numbers satisfy the identity

$$v^- u = u^- v,$$

the orthogonal complement of the domain has dimension zero or one.

A generalization of Gaussian quadrature applies in a space $\mathcal{H}(E)$. A phase function associated with $E(z)$ is a continuous function $\phi(x)$ of real x with real values such that the product

$$E(x)\exp[i\phi(x)]$$

has real values. Such a function exists and is unique within an added integer multiple of π. The function is differentiable and has positive derivative everywhere. If α is a given real number, the inequality

$$\|F\|^2_{\mathcal{H}(E)} \leq \sum |F(t)/E(t)|^2 \pi/\phi'(t)$$

holds for every element $F(z)$ of the space with summation over the real numbers t such that $\phi(t)$ is congruent to α modulo π. Equality holds for every element $F(z)$ of the space when the function

$$E(z)\exp(i\alpha) - E^*(z)\exp(-i\alpha)$$

does not belong to the space. At most one real number α modulo π exists such that the function belongs to the space. The function then spans the orthogonal complement of the domain of multiplication by z in the space. Equality holds for every element $F(z)$ of the closure of the domain of multiplication by z in the space. The quadrature identity of Fourier analysis is recovered when

$$E(z) = \exp(-iaz)$$

for a positive number a, in which case the identity

$$\phi(x) = ax$$

is satisfied.

The quadrature identity is relevant to the Riemann hypothesis as the conjecture that the zeros of certain entire functions are real. The significance of the Riemann hypothesis is that the quadrature identity applies in a context relevant to the asymptotic distribution of prime numbers. The theory of Hilbert spaces of entire functions is an interpretation of the work of Thomas Stieltjes, "Recherches sur les fractions continues," Annales de la Faculté Scientifique de Toulouse 8 (1894), 1-122, and 9 (1895), 1-47. His analytic theory of continued fractions is reformulated in the theory of Hilbert spaces of entire functions as a factorization theory for matrix-valued analytic functions.

The resemblance of the theory of Hilbert spaces of entire functions to Fourier analysis is extensive and substantial. A generalization of Fourier analysis is associated with every nontrivial Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3). Every such space is embedded in a family of such spaces similar to the Paley-Wiener spaces. A partial ordering of Hilbert spaces of entire functions is implicit in the construction of such families. A space $\mathcal{H}(E(a))$ with index a is considered less than or equal to a space $\mathcal{H}(E(b))$ with index b if the ratio

$$E(a, z)/E(b, z)$$

has no real zeros, if the space with index a is contained contractively in the space with index b, and if the inclusion is isometric on the domain of multiplication by z in the space with index a. A fundamental theorem states that such Hilbert spaces of entire functions appear in totally ordered families. If a space $\mathcal{H}(E(a))$ with index a and a space $\mathcal{H}(E(b))$ with index b are less than or equal to a space $\mathcal{H}(E(c))$ with index c, then either the space with index a is less than or equal to the space with index b or the space with index b is less than or equal to the space with index a. A nontrivial Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) is a member of a maximal totally ordered family of such spaces. The members of the family are indexed by real numbers in such a way that the space with index a is less than or equal to the space with index b when a is less than or equal to b. Every member of the family with index b is then the least upper bound of members of the family with index a less than b. Every member of the family with index a is also the greatest lower bound of the members of the family with index b greater than a.

A generalization of the Hilbert transformation underlies the structure theory of Hilbert spaces of entire functions. An entire function $S(z)$ is said to be associated with a space $\mathcal{H}(E)$ if

$$[F(z)S(w) - S(z)F(w)]/(z - w)$$

belongs to the space for every complex number w whenever $F(z)$ belongs to the space. The condition is satisfied when $S(z)$ belongs to the space, but it is also satisfied for certain functions such as $E(z)$ and $E^*(z)$ which do not belong to the space. If a nontrivial entire function $S(z)$ is associated with the space $\mathcal{H}(E)$, then a corresponding partially isometric transformation is defined of the given space into a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3). If the transformation takes $F(z)$ into $F^\sim(z)$, then the identity for difference quotients

$$\begin{aligned}\pi G(\alpha)^- F^\sim(\beta) - \pi G^\sim(\alpha)^- F(\beta) \\ = \langle [F(t)S(\beta) - S(t)F(\beta)]/(t-\beta), G(t)S(\alpha)\rangle_{\mathcal{H}(E)} \\ -\langle F(t)S(\beta), [G(t)S(\alpha) - S(t)G(\alpha)]/(t-\alpha)\rangle_{\mathcal{H}(E)} \\ -(\beta - \alpha^-)\langle [F(t)S(\beta) - S(t)F(\beta)]/(t-\beta), [G(t)S(\alpha) - S(t)G(\alpha)]/(t-a)\rangle_{\mathcal{H}(E)}\end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of the space when α and β are arbitrary complex numbers. The identity

$$G^\sim(z) = [F^\sim(z)S(w) - S(z)F^\sim(w)]/(z-w)$$

holds whenever $F(z)$ and $G(z)$ are elements of the space such that

$$G(z) = [F(z)S(w) - S(z)F(w)]/(z-w).$$

for a complex number w. The transformation is uniquely determined by $S(z)$ within an added real multiple of the identity transformation. When

$$S(z) = E(z),$$

the transformation takes $F(z)$ into

$$F^\sim(z) = iF(z).$$

The Hilbert transformation is obtained when $\mathcal{H}(E)$ is a Paley–Wiener space and $S(z)$ is identically one. The Hilbert transformation is an isometry of every Paley-Wiener onto itself.

The generalization of the Hilbert transformation permits the construction of a Hilbert space whose elements are pairs

$$F(z) = \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

of analytic functions such that $S(z)F_+(z)$ and $S(z)F_-(z)$ are entire. The elements of the space parametrize the graph of the generalized Hilbert transformation as pairs

$$S(z)F(z) = \begin{pmatrix} S(z)F_+(z) \\ S(z)F_-(z) \end{pmatrix}.$$

The space is considered with the scalar product which the graph inherits from the Cartesian product space. Notation is introduced to formulate the properties of the space. A bar is used to denote the conjugate transpose of a rectangular matrix. The matrix

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is treated as a generalization of the imaginary unit. The difference-quotient

$$[F(z) - F(w)]/(z-w)$$

belongs to the space whenever $F(z)$ belongs to the space when w is not a zero of $S(z)$. The identity for difference-quotients

$$-2\pi G(\alpha)^- IF(\beta) = \langle [F(t) - F(\beta)]/(t-\beta), G(t)\rangle - \langle F(t), [G(t) - G(\alpha)]/(t-\alpha)\rangle$$
$$-(\beta - \alpha^-)\langle [F(t) - F(\beta)]/(t-\beta), [G(t) - G(\alpha)]/(t-\alpha)\rangle$$

holds for all elements $F(z)$ and $G(z)$ of the space when α and β are not zeros of $S(z)$. Unique entire functions $C(z)$ and $D(z)$, which are real for real z, exist such that the matrix $M(z)$ defined by

$$S(z)M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

has determinant one, such that

$$\frac{M(z)IM(w)^- - I}{\pi(z-w^-)}\begin{pmatrix} u \\ v \end{pmatrix}$$

belongs to the space for all complex numbers u and v when w is not a zero of $S(z)$, and such that the identity

$$\begin{pmatrix} u \\ v \end{pmatrix}^- \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} = \langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \frac{M(t)IM(w)^- - I}{\pi(t-w^-)}\begin{pmatrix} u \\ v \end{pmatrix}\rangle$$

holds for every element $F(z)$ of the space. The space can be denoted $\mathcal{H}(M)$ since it is determined by $M(z)$. The identity

$$M(z) = \begin{pmatrix} \cos(az) & \sin(az) \\ -\sin(az) & \cos(az) \end{pmatrix}$$

holds when $\mathcal{H}(E)$ is the Paley-Wiener space of index a and $S(z)$ is identically one.

The entries of $M(z)$ are entire functions when $S(z)$ is identically one. A Hilbert space $\mathcal{H}(M)$ is defined more generally whenever

$$M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

is a matrix of entire functions which are real for real z, the matrix has determinant one, and the matrix

$$\frac{M(z)IM(z)^- - I}{2\pi(z-z^-)}$$

is nonnegative for all complex z. The elements of the space are pairs of entire functions. The identity for difference quotients applies. The same computation of reproducing kernel functions is valid. A space $\mathcal{H}(E(a))$ is less than or equal to a space $\mathcal{H}(E(b))$ if, and only if, the identity

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z))M(a,b,z)$$

holds for a matrix-valued entire function $M(a,b,z)$ such that a Hilbert space $\mathcal{H}(M(a,b))$ exists. The matrix $M(a,b,z)$ is then uniquely determined by the functions $E(a,z)$ and $E(b,z)$.

These results, which were obtained in the postdoctoral years 1957-1962, are published in *Hilbert Spaces of Entire Functions,* Prentice-Hall, 1968. The structure of mathematical journals creates the impression that mathematics is fragmented into unrelated disciplines. The underlying unity of mathematics is however maintained by problems which span these disciplines. A selection of such problems was presented by David Hilbert to the International Congress of Mathematicians which was held in 1900 in Paris: "Mathematical Problems," Bulletin of the American Mathematical Society 8 (1902), 437–479. The Riemann hypothesis is listed as an important link between algebra and analysis.

The analytic aspects of the asymptotic behavior of prime numbers originate in the gamma function, discovered in 1730 by Leonard Euler, "De progressionibus transcendentalibus seu quarum termini generales algebraice dari nequeunt," Opera Omnia I (14), 1–24. His discovery of the Euler product for the classical zeta function was made in 1737, "Variae observationes circa series infinitas," Opera Omnia I (14), 216–244. A substantial evolution in the theory of the gamma function is required for the functional identity which Euler discovered for the classical zeta function in 1761: "Remarques sur un beau rapport entre les séries de puissances tant directes que réciproques," Opera Omnia I (15), 70–90. The Riemann hypothesis for the classical zeta function was stated by Bernhard Riemann in 1859. No motivation for the conjecture was published by Riemann although he is known to have made calculations of zeros which were later duplicated by Jean-Pierre Gram, "Note sur les zéros de la fonction de Riemann," Acta Mathematica 27 (1903), 289–305.

The classical motivation for the Riemann hypothesis is attributed to Nikolai Sonine, "Recherches sur les fonctions cylindriques et le développement des fonctions continues en séries," Mathematische Annalen 16 (1880), 1-80. Remarkable examples of functions related to zeta functions are presented for which the analogue of the Riemann hypothesis is true. A spectral theory involving the gamma function is derived from properties of the Hankel transformation of order zero. Sonine observes that a square integrable function and its Hankel transform of order zero can vanish in a neighborhood of the origin without vanishing identically. An axiomatic treatment in the theory of Hilbert spaces of entire functions was given by the author, "Self-reciprocal functions," Journal of Mathematical Analysis and Applications 9 (1964), 433-455. A parametrization is made of all square integrable functions which vanish in an interval containing the origin and whose Hankel transform of order zero vanishes in the same interval. A derivation of the expansion from the examples given by Sonine was made by Virginia Rovnyak in her thesis, "Self-reciprocal functions," Duke Mathematical Journal 33 (1966), 363-378. A generalization of the expansion for the Hankel transformation of integer order was made by James and Virginia Rovnyak, "Self-reciprocal functions for the Hankel transformation of integer order," Duke Mathematical Journal 34 (1967), 771-785. These results are less complete than those for the Hankel transformation of order zero.

The Riemann hypothesis as a research objective created a career obstacle since the relevance of the theory of Hilbert spaces of entire functions could not be established. When tenured positions were unavailable in the vicinity of Philadelphia, I accepted in 1962 the offer of an associate professorship on the Lafayette campus of Purdue University. Promotion to professor was immediate. Philadelphia retained its significance as an educational and research center because vacations could be spent there. The city supplied students who came to Lafayette for doctoral and postdoctoral work. A construction of Hilbert spaces of entire functions associated with Dirichlet zeta functions was made during this time.

If ρ is a given positive integer, a character modulo ρ is a function $\chi(n)$ of integers n, which is periodic of period ρ, which satisfies the identity

$$\chi(mn) = \chi(m)\chi(n)$$

for all integers m and n, which has nonzero values at integers relatively prime to ρ, and which vanishes otherwise. A character is an even or an odd function. A character χ modulo ρ is said to be primitive modulo ρ if no character modulo a proper divisor of ρ exists which agrees with χ at integers which are relatively prime to ρ. A character is said to be real if it has real values. The principal character modulo ρ is the unique character modulo ρ whose only nonzero value is one. The character is primitive when ρ is one.

The Dirichlet zeta function associated with a character χ modulo ρ is defined by

$$\zeta(s) = \sum \chi(n) n^{-s}$$

with summation over the positive integers n. The series is absolutely convergent when $\mathcal{R}s > 1$ and represents an analytic function of s in the half-plane. The classical zeta function, which was discovered by Euler, is the Dirichlet zeta function when χ is the principal character modulo one. The function has an analytic extension to the complex plane except for a simple pole at $s = 1$. The Dirichlet zeta function has an analytic extension to the complex plane when χ is not a principal character.

The Dirichlet zeta function $\zeta(s)$ satisfies a functional identity when χ is a primitive real character modulo ρ. The functions

$$(\rho/\pi)^{\frac{1}{2}\nu + \frac{1}{2}s} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2}s)\zeta(s)$$

and

$$(\rho/\pi)^{\frac{1}{2}\nu + \frac{1}{2} - \frac{1}{2}s} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - \tfrac{1}{2}s)\zeta(1-s)$$

are linearly dependent with $\nu = 0$ when χ is even and $\nu = 1$ when χ is odd. The functions are entire when χ is not the principal character. When χ is the principal character, the functions are equal with simple poles at $s = 0$ and $s = 1$.

The Dirichlet zeta function $\zeta(s)$ has no zeros in the half-plane $\mathcal{R}s > 1$ since the Euler product

$$\zeta(s)^{-1} = \Pi(1 - \chi(p)p^{-s})$$

converges in the half-plane. The product is taken over the primes p which are not divisors of ρ. A less obvious consequence of convergence in the half-plane, due to Hadamard and de la Vallée Poussin, is the absence of zeros in the closure of the half-plane. The functional identity reduces the determination of zeros to the critical strip $0 < \mathcal{R}s < 1$. These zeros are symmetric about the critical line $\mathcal{R}s = \frac{1}{2}$ by the functional identity. The Riemann hypothesis is the conjecture that the zeros lie on the critical line. Although Riemann stated the conjecture for the classical zeta function, it is applied to Dirichlet zeta functions associated with real characters. Simplicity of zeros is a later strengthening of the conjecture.

David Hilbert is said to have assigned the Riemann hypothesis as a thesis problem to his student Erhard Schmidt. The interest of Hilbert in the Riemann hypothesis is attested by his 1900 Congress address. The direction of his interests is further indicated by a series of publications, “Grundzüge einer allgemeinen Theorie der Integralgleichungen,” Göttinger Nachrichten I (1904), 49–91, II (1904), 213–259, III (1905), 307–338, IV (1906), 157–222, and V (1906), 439–480. Erhard Schmidt also made a contribution to the theory of integral equations, "Entwicklung willkürlicher Funktionen nach Systemen vorgeschriebener," Dissertation, Göttingen, 1905. These results include a spectral theory of self-adjoint transformations with discrete spectrum. Hilbert is said to have proposed the construction of such a transformation whose eigenvalues are zeros of the classical zeta function in the critical strip. Supporting evidence is found in a predoctoral publication by Erhard Schmidt, "Über die Anzahl der Primzahlen unter gegebener Grenze," Mathematische Annalen 57 (1903), 195–204.

The Hilbert strengthening of the Riemann hypothesis is interpreted as the construction of a space $\mathcal{H}(E)$ which is related to the Dirichlet zeta function $\zeta(s)$ associated with a primitive real character χ modulo ρ. The substitution $s = \frac{1}{2} - iz$ converts the function

$$(\rho/\pi)^{\frac{1}{2}\nu+\frac{1}{2}s}\Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2}s)\zeta(s)$$

of s into an entire function of z when χ is a nonprincipal character. The function needs to be multiplied by $s(1-s)$ for the same conclusion when χ is the principal character. The functional identity states that the entire function of z is real for real z. The Riemann hypothesis is the conjecture that the function has only real simple zeros. The Hilbert conjecture is interpreted as the existence of a space $\mathcal{H}(E)$ such that the entire function which is real for real z coincides with $A(z)$ in the decomposition

$$E(z) = A(z) - i\,B(z).$$

The Hilbert-Schmidt spectral theory of self-adjoint transformations is an application of the Hadamard factorization of entire functions, which was later axiomatized by Georg Pólya. An entire function $E(z)$ is said to be of Pólya class if it has no zeros in the upper half-plane, if it satisfies the inequality

$$|E(x-iy)| \leq |E(x+iy)|$$

for $y > 0$, and if $|E(x+iy)|$ is a nondecreasing function of positive y for every real number x. A space $\mathcal{H}(E)$ then exists when the functions $E(z)$ and $E^*(z)$ are linearly independent. Multiplication by z in the space admits a self-adjoint extension whose spectrum is contained in the zeros of $A(z)$. The existence of the extension is an application of Gaussian quadrature. The Hilbert-Schmidt spectral theory applies because the zeros t_n of $A(z)$ satisfy a convergence condition. The sum

$$\sum 1/(1+t_n^2)$$

is finite. The Hadamard factorization asserts the existence of sufficiently many zeros for a product representation of $A(z)$. The spectral theory applied in the proof of the Riemann hypothesis is a special case of the Hilbert–Schmidt theory.

The construction of Hilbert spaces of entire functions associated with Dirichlet zeta functions is also an application of the representation theory of the group of matrices of rank two with real entries and determinant one. The modular group is the subgroup formed by the matrices with integer entries. If ρ is a given positive integer, the corresponding Hecke subgroup of the modular group consists of those matrices whose subdiagonal entry is divisible by ρ. A corresponding Hilbert space is constructed for every primitive real character χ modulo ρ. The Hilbert space consists of (equivalence classes of) measurable functions $f(z)$ of z in the upper half-plane such that the identity

$$f(z) = \frac{\chi(D)}{(Cz+D)^{1+\nu}} f(\frac{Az+B}{Cz+D})$$

holds for every element

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of the Hecke subgroup of the modular group and such that the integral

$$\|f\|^2 = \iint |f(x+iy)|^2 y^{\nu-1} dx\, dy$$

is finite with integration over a fundamental region for the group.

The Laplace-Beltrami operator is a self-adjoint transformation in the space defined formally by taking $f(z)$ into

$$-(z^- - z)^2 \frac{\partial^2 f}{\partial z^- \partial z} + (1+\nu)(z^- - z)\frac{\partial f}{\partial z}.$$

Formal eigenvectors of the transformation are represented by Eisenstein series

$$\sum \frac{\chi(D)}{(Cz+D)^{1+\nu}} f(\frac{Az+B}{Cz+D})$$

using functions $f(z)$ of z in the upper half-plane which are periodic of period one. Summation is over all lower rows of elements

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of the Hecke subgroup. The spectral theory of the self-adjoint transformation permits the construction of a Hilbert space $\mathcal{H}(E)$ which is related to the Dirichlet zeta function $\zeta(s)$ associated with the given character. When $s = 1 - iz$, the function

$$(\rho/\pi)^{\frac{1}{2}\nu + \frac{1}{2}s} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2}s)\zeta(s)$$

is equal to the desired function $E(z)$ if χ is not the principal character.

The construction of Hilbert spaces of entire functions associated with Dirichlet zeta functions appeared as "Modular spaces of entire functions," Journal of Mathematical Analysis and Applications 44 (1973), 192-205. The spectral theory is an interpretation of the results of Hans Maass, "Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen," Mathematische Annalen 121 (1949), 141-183. The spaces do not verify the Hilbert conjecture since the spectral line is not the critical line but the right boundary of the critical strip. No information is obtained about zeros of zeta functions in the critical strip. The spaces can be constructed from the Hadamard factorization without recourse to the spectral theory of the Laplace-Beltrami operator. The spectral theory does however indicate that the spaces are natural to zeta functions. The spaces are a generalization of the spaces of the Sonine theory, which are applied in the construction of spaces from the Maass theory. The Sonine spaces solve a problem of parametrization of square integrable functions which vanish in an interval containing the origin and whose Hankel transform of a given order vanishes in the same interval. The Maass spaces apply to a similar problem formulated in the proof of the Riemann hypothesis.

Remarks on the Hankel transformation are appropriate because of its appearance in the Sonine theory. An axiomatic treatment of the Hankel transformation of order ν is an elementary application of the theory of Hilbert spaces of entire functions. Assume that ν is a given real number. A space $\mathcal{H}(E)$ is said to be homogeneous of order ν if an isometric transformation of the space onto itself is defined by taking $F(z)$ into $a^{1+\nu}F(az)$ when $0 < a < 1$. The Paley-Wiener spaces are homogeneous of order $-\frac{1}{2}$. Related spaces exist when $\nu > -1$, in which case the norm of the space is defined by

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)|^2 |t|^{2\nu+1} dt.$$

The spaces appear in the theory of the Hankel transformation of order ν, which is defined using the Bessel function

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}x)^{\nu+2n}}{\Gamma(1+n)\Gamma(1+\nu+n)}.$$

If $f(x)$ is a square integrable function of positive x, its Hankel transform of order ν is defined by

$$g(x) = \int_0^\infty f(t) J_\nu(xt) \sqrt{xt}\, dt$$

when the integral is absolutely convergent. A square integrable function $g(x)$ of positive x is obtained which satisfies the identity

$$\int_0^\infty |f(t)|^2 dt = \int_0^\infty |g(t)|^2 dt.$$

The isometric property of the transformation permits its definition on the space $L^2(0,\infty)$. The transformation is its own inverse. A self-reciprocal function is its own Hankel transform. A skew-reciprocal function is minus its own Hankel transform. Every element of the space is the orthogonal sum of a self-reciprocal function and a skew-reciprocal function. A related Hilbert space of entire functions, which is homogeneous of order ν, is obtained for every positive number a. The even elements of the space are the entire functions $F(z)$ such that $x^{-\nu}F(x)$ is the Hankel transform of order ν of a function which vanishes outside of the interval $(0,a)$.

A fundamental example of a self-reciprocal function of order ν is

$$x^{\nu+\frac{1}{2}} \exp(-\tfrac{1}{2}x^2).$$

A construction of Hankel transform pairs is made using the Laplace transformation. Sonine applies the construction to produce Hankel transform pairs which vanish in a given interval containing the origin. The construction of all such pairs is a fundamental problem which admits a solution when $\nu = 0$. Contiguous relations between the Hankel transformation of order ν and the Hankel transformation of order $\nu + 1$ permit a solution when ν is a nonnegative integer.

The Hankel transformation of order minus one-half is the cosine transformation. The Hankel transformation of order one-half is the sine transformation. These transformations are derived from the Fourier transformation for the real numbers under a decomposition which results from inversion about the origin. The Hankel transformation of integer order is derived from the Fourier transformation for the plane under a similar decomposition which results from the action of rotations about the origin. The derivation of the Hankel transformation from the Fourier transformation permits generalizations in which the real numbers are replaced by a locally compact field. The fields required for Dirichlet zeta functions are the field of p-adic numbers and its unramified quadratic extension for every prime p as well as the field of real numbers and its unique quadratic extension, which is the field of complex numbers. Fourier analysis on related adelic rings permits an application of the Poisson summation formula to a proof of the functional identity.

A Dirichlet zeta function is a generalization of the gamma function, which satisfies no functional identity but which does satisfy a recurrence relation. The concept of a functional identity is subordinated to the concept of a recurrence relation in the proof of the Riemann hypothesis. The recurrence relation for the gamma function is reformulated as a positivity condition which applies to zeta functions.

Motivation for the proof of the Riemann hypothesis was supplied by David Trutt, who discovered nonnegative measures on the Borel subsets of the complex plane with respect to which the Newton polynomials

$$(-1)^n \frac{z(z-1)\dots(z+1-n)}{1\dots n}$$

are orthogonal. If $\nu > -1$, a unique Hilbert space exists whose elements are functions analytic in the half-plane $z + z^- > -1 - \nu$ and which contains the Newton polynomials as an orthogonal set with

$$\frac{(\nu+1)\dots(\nu+n)}{1\dots n}$$

as the square of the norm of the n-th polynomial. The identity

$$2\pi\Gamma(1+\nu)\|F\|^2$$

$$= \sum_{n=0}^{\infty} \Gamma(1+n)^{-1} \int_{-\infty}^{+\infty} |\Gamma(\tfrac{1}{2}n + \tfrac{1}{2} + \tfrac{1}{2}\nu - it)\Gamma(\tfrac{1}{2}n - \tfrac{1}{2} - \tfrac{1}{2}\nu - it)F(\tfrac{1}{2}n + \tfrac{1}{2} + \tfrac{1}{2}\nu - it)|^2 dt$$

holds for every element $F(z)$ of the space. A structure theory for such spaces is obtained by David Trutt and the author, "Orthogonal Newton polynomials," Advances in Mathematics 37 (1980), 251-271.

Related Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3) exist for every nonnegative integer n. The elements of the n-th space are polynomials considered with the scalar product corresponding to the norm

$$\|F\|^2 = \int_{-\infty}^{+\infty} |\Gamma(\tfrac{1}{2}n + \tfrac{1}{2} + \tfrac{1}{2}\nu - it)\Gamma(\tfrac{1}{2}n - \tfrac{1}{2} - \tfrac{1}{2}\nu - it)F(t)|^2 dt.$$

These scalar products on polynomials have characteristic properties which are expressed in a pair of adjoint transformations: The transformation of the space of all polynomials considered with the $(n+1)$-st scalar product into the space of all polynomials considered with the n-th scalar product takes $F(z)$ into

$$(\tfrac{1}{2}n + \tfrac{1}{2} + \tfrac{1}{2}\nu - iz)F(z + \tfrac{1}{2}i).$$

The transformation of the space of all polynomials considered with the n-th scalar product into the space of all polynomials considered with the $(n+1)$-st scalar product takes $F(z)$ into

$$(\tfrac{1}{2}n - 1 - \tfrac{1}{2}\nu - iz)F(z + \tfrac{1}{2}i).$$

An axiomatic treatment of the spaces is given by David Trutt and the author, *Meixner and Pollaczek spaces of entire functions*, Journal of Mathematical Analysis and Applications 22 (1968), 12–24.

The weight functions which appear are reciprocals of weight functions appearing in the theory of Sonine spaces. The properties of Pollaczek polynomials are suggestive of a general theory which includes the Sonine spaces and the spaces of entire functions appearing in the Maass theory. The half-unit spacing which appears in measures is significant for the Riemann hypothesis as the spacing between the critical line and the boundary of the critical strip. The spaces of the Maass theory are unsatisfactory for the Riemann hypothesis because the spectral line is not the critical line but the right boundary of the critical strip. Related Hilbert spaces of entire functions are wanted in which the spectral line is shifted one-half unit to the left. A mechanism is suggested for making such a shift.

Related motivation for the proof of the Riemann hypothesis was supplied in 1961 by Arne Beurling and Paul Malliavin at an International Symposium on Functional Analysis held at Stanford University. Their results "On Fourier transforms of measures with compact support" appear in Acta Mathematica 107 (1962), 291-392. A source of their work is the Colloquium Publication of Norman Levinson on *Gap and Density Theorems,* American Mathematical Society, 1940. Beurling and Malliavin solve a problem of Levinson which can be formulated in the theory of Hilbert spaces of entire functions.

The problem concerns properties of a maximal totally ordered family of Hilbert spaces of entire functions. Assume that the defining functions $E(t, z)$ are parameterized by the positive numbers t in such a way that the space with index a is less than or equal to the space with index b when a is less than or equal to b. The ratio

$$E(b, z)/E(a, z)$$

is then analytic and of bounded type in the upper half-plane. A nondecreasing function $\tau(t)$ of positive numbers t exists such that the mean type of the ratio is equal to

$$\tau(b) - \tau(a)$$

when a is less than or equal to b. A computation of mean type is made from a phase function $\phi(a, x)$ for $E(a, z)$ and a phase function $\phi(b, x)$ for $E(b, z)$ as the limit of

$$[\phi(b, x) - \phi(a, x)]/x$$

as x converges to infinity on the positive or the negative half-line. The defining function $E(t, z)$ can be chosen with phase function $\phi(t, x)$, which vanishes at the origin, so that

$$\phi(t, x)/x$$

is a nondecreasing function of positive numbers t with limit zero as t converges to zero for every real number x. The inequality

$$\tau(b) - \tau(a) \leq \liminf \phi(b, x)/x$$

holds as x converges to infinity on the positive or the negative half-line when a is less than or equal to b.

A classical problem of spectral theory is formulated as a determination of the relationship between the asymptotic behavior of phase functions and the mean type of ratios of defining functions. Assume that a real number τ less than $\tau(b)$ is given such that the inequality

$$\tau(b) - \tau < \liminf \phi(b, x)/x$$

holds as x converges to infinity on the right and left half-lines. The problem is to determine whether a member $\mathcal{H}(E(a))$ of the family exists such that

$$\tau(a) = \tau.$$

An affirmative answer is given for a positive number τ less than $\tau(b)$ when $\phi(b, x)$ is a uniformly continuous function of x such that the integral

$$\int_{-\infty}^{+\infty} \frac{|\phi(b, x) - \tau(b)x| dx}{1 + x^2}$$

is finite.

The problem is a special case of an inverse spectral problem due to Mark Krein. Consider special Hilbert spaces of entire functions which are symmetric about the origin: An isometric transformation of the space into itself is defined by taking $F(z)$ into $F(-z)$. The condition is obviously satisfied when the defining function $E(z)$ of a space $\mathcal{H}(E)$ satisfies the symmetry condition

$$E^*(z) = E(-z).$$

A converse result is true. A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3), which is symmetric about the origin, and which contains a nonzero element, is isometrically equal to a space $\mathcal{H}(E)$ for an entire function $E(z)$ which satisfies the symmetry condition. The spectral theory of the vibrating string is contained in structure theory of maximal totally ordered families of Hilbert spaces of entire functions which are symmetric about the origin. The function $\tau(t)$ for such a family determines the length of the string. The inverse problem of Mark Krein is the determination not only of the length but also of the mass distribution of the string.

Work on the Riemann hypothesis was interrupted during more than four years of effort required to complete the proof of the Bieberbach conjecture. The attack on the Riemann hypothesis was resumed after the confirmation of the proof in 1984. An invitation to address the Winter Meeting of the American Mathematical Society was used to present "The Riemann hypothesis for Hilbert spaces of entire functions," which was published in the Bulletin of the American Mathematical Society 15 (1986), 1-17. An axiomatization is made of the theory of the gamma function. The positivity condition which is introduced implies the Riemann hypothesis if it applies to Dirichlet zeta functions.

The concept of a quantum gamma function with quantum q applies when q is a given number, $0 < q < 1$. A weight function is a function which is analytic and without zeros in the upper half-plane. The weighted Hardy space associated with such a function $W(z)$ is the set of functions $F(z)$, analytic in the upper half-plane, such that $F(z)/W(z)$ is of bounded type and of nonpositive mean type in the half-plane and has square integrable boundary values on the real axis. A Hilbert space $\mathcal{F}(W)$ is obtained in the norm

$$\|F\|^2_{\mathcal{F}(W)} = \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt.$$

The given weight function is said to be a quantum gamma function with quantum q if the space is well-related to the transformation which takes $F(z)$ into $F(z+i\kappa)$ for every positive number κ such that

$$q \leq \exp(-2\pi\kappa).$$

Every element of the space is of the form $F(z)+F(z+i\kappa)$ for an element $F(z)$ of the space such that $F(z+i\kappa)$ belongs to the space. The scalar product

$$\langle F(t), F(t+i\kappa)\rangle_{\mathcal{F}(W)}$$

has nonnegative real part for every such element $F(z)$. An equivalent condition is that the weight function has an analytic extension to the half-plane $-\kappa < iz^- - iz$ such that the ratio

$$W(z)/W(z+i\kappa)$$

has nonnegative real part in the half-plane.

The concept of a quantum gamma function supplies an alternative to the Beurling-Malliavin theorem. A maximal totally ordered family of Hilbert spaces of entire functions exists with these properties: A function $E(t,z)$ which defines an element of the family is of Pólya class and the ratio

$$E(t,z)/W(z)$$

is of bounded type in the upper half-plane. If $\tau(t)$ is the mean type of the ratio in the half-plane, then multiplication by

$$\exp[i\tau(t)z]$$

is a contractive transformation of the space $\mathcal{H}(E(t))$ into the space $\mathcal{F}(W)$ which is isometric on the domain of multiplication by z. If κ is a positive number such that

$$q \leq \exp(-2\pi\kappa),$$

then every element of the space $\mathcal{H}(E(a))$ is of the form $F(z)+F(z+i\kappa)$ for an element $F(z)$ of the space such that $F(z+i\kappa)$ belongs to the space, and the scalar product

$$\langle F(t), F(t+i\kappa)\rangle_{\mathcal{H}(E(a))}$$

has nonnegative real part which originates with Carleman. For every positive number τ such that

$$W(z)\exp(i\tau z)$$

is unbounded on the upper half of the imaginary axis, a member of the family with defining function $E(a,z)$ exists such that $\tau(a)=\tau$.

The concept of a quantum gamma function validates a formal theory of quasi-analyticity which originates with Carleman. The issue is treated in the context of Fourier analysis by

Norman Levinson in his Colloquium Publication on *Gap and Density Theorems,* American Mathematical Society, 1940. A related treatment of quasi-analyticity is given by the author in his thesis, "Local operators on Fourier transforms," Duke Mathematical Journal 25 (1958), 143-153. If $K(x)$ is a measurable function of real x, define an operator on absolutely convergent Fourier transforms which takes

$$f(x) = \int_{-\infty}^{+\infty} F(t) \exp(2\pi i\, xt) dt$$

into

$$g(x) = \int_{-\infty}^{+\infty} G(t) \exp(2\pi i\, xt) dt$$

whenever the identity

$$G(t) = K(t)F(t)$$

holds for almost all real t with finiteness of the integrals

$$\int_{-\infty}^{+\infty} |F(t)| dt$$

and

$$\int_{-\infty}^{+\infty} |G(t)| dt.$$

If the integral

$$\int_{-\infty}^{+\infty} \frac{\log(1 + |K(t)|^2)}{1 + t^2} dt$$

is infinite and if a smoothness hypothesis is satisfied, then the domain of the operator contains no function which vanishes in an interval without vanishing identically. A sufficient smoothness condition is that the logarithm of

$$1 + |K(t)|^2$$

is a uniformly continuous function of t. The search for an optimal smoothness hypothesis is a fundamental problem of the Carleman theory which motivates the concept of a quantum gamma function.

The quantum generalization of the gamma function is an attack on the Riemann hypothesis since the desired location of zeros is a consequence of the positivity condition characteristic of quantum gamma functions. Examples of weight functions which satisfy the positivity conditions appear in the Sonine theory. The weight function for the Hankel transformation of order ν is

$$W(z) = \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz).$$

A quantum gamma function is obtained when ν is nonnegative. A proof of positivity is given from properties of the Laplace transformation. If ν is a nonnegative number, define $\mathcal{D}_\nu$ to be the Hilbert space of functions $F(z)$, analytic in the upper half-plane, which are of the form

$$F(z) = \int_0^\infty f(t) t^{1+\nu} \exp(\pi i t^2 z) dt$$

for a measurable function $f(t)$ of positive numbers t such that the integral

$$\|F\|_{\mathcal{D}_\nu}^2 = \int_0^\infty |f(t)|^2 t\,dt$$

is finite. When ν is zero, the identity

$$\|F\|_{\mathcal{D}_\nu}^2 = \sup \int_{-\infty}^{+\infty} |F(x+iy)|^2 dx$$

is satisfied with the least upper bound taken over all positive numbers y. When ν is positive, the identity

$$\Gamma(\nu)\|F\|_{\mathcal{D}_\nu}^2 = (2\pi)^\nu \int_0^\infty \int_{-\infty}^{+\infty} |F(x+iy)|^2 y^{\nu-1} dx\,dy$$

is satisfied. An element $F(z)$ of the space $\mathcal{D}_\nu$ is the Laplace transform of a function $f(t)$ which vanishes in an interval $(0,a)$ containing the origin if, and only if,

$$\exp(-\pi i a^2 z)F(z)$$

converges to zero as z converges to infinity on the upper half of the imaginary axis.

The Mellin transform of an element $f(z)$ of the space $\mathcal{D}_\nu$ is the function $F(x)$ of real x which is defined by

$$F(x) = \int_0^\infty f(it) t^{\frac{1}{2}\nu - ix - \frac{1}{2}} dt$$

when the integral is absolutely convergent. The identity

$$\pi^\nu \int_{-\infty}^{+\infty} |F(x)/W(x)|^2 dx = \|f(z)\|_{\mathcal{D}_\nu}^2$$

is then satisfied with

$$W(z) = \pi^{-\frac{1}{2}\nu - \frac{1}{2} + iz} \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz).$$

The transformation is extended to the space $\mathcal{D}_\nu$ so as to maintain the identity. Every measurable function $F(x)$ of real x for which the integral converges is the Mellin transform of an element of the space $\mathcal{D}_\nu$. If the function

$$\exp(-\pi i z) f(z)$$

converges to zero as z converges to infinity on the upper half of the imaginary axis, an element $F(z)$ of the space $\mathcal{F}(W)$ is defined by

$$F(z) = \int_0^\infty f(it) t^{\frac{1}{2}\nu - iz - \frac{1}{2}} dt$$

when z is in the upper half-plane. The identity

$$\pi^\nu \|F\|_{\mathcal{F}(W)}^2 = \|f(z)\|_{\mathcal{D}_\nu}^2$$

is satisfied. Every element of the space $\mathcal{D}_\nu$ is of this form. The function

$$F(z + i\kappa) = \int_0^\infty t^\kappa f(it) t^{\frac{1}{2}\nu - iz - \frac{1}{2}} dt$$

belongs to the space $\mathcal{F}(W)$ if, and only if,

$$(-iz)^\kappa f(z)$$

also belongs to the space $\mathcal{D}_\nu$. The scalar product

$$\langle F(t), F(t + i\kappa)\rangle_{\mathcal{F}(W)}$$

has nonnegative real part when $0 < \kappa < 1$ because it is a positive multiple of the scalar product

$$\langle f(z), (-iz)^\kappa f(z)\rangle_{\mathcal{D}_\nu}$$

which has nonnegative real part.

A verification that $W(z)$ is a quantum gamma function with quantum

$$q = \exp(-2\pi)$$

is thereby obtained from a spectral theory of the shift operator. The operator is unitarily equivalent to a multiplication operator in a space of functions analytic in the upper half-plane with norm defined by integration with respect to a nonnegative plane measure. The desired positivity properties of the shift operator result from the positivity properties of the multiplication operator.

The proof of the Riemann hypothesis verifies a positivity condition only for those Dirichlet zeta functions which are associated with nonprincipal real characters. The classical zeta function does not satisfy a positivity condition since the condition is not compatible with the singularity of the function. But a weaker condition is satisfied which has the desired implication for zeros.

A curious coincidence needs to be mentioned as part of the chain of events which concluded in the proof of the Riemann hypothesis. The feudal family de Branges originates in a crusader who died in 1199 leaving an emblem of three swords hanging over three coins, surmounted by the traditional crown designating a count, and inscribed with the motto "Nec vi nec numero." This is a citation from Chapter 4, Verse 6, of the Book of Zechariah: "Not by might, nor by power, but by my Spirit, says the Lord of Hosts." The château de Branges was destroyed in 1478 by the army of Louix XI of France during an unsuccessful campaign to wrest Franche-Comté from the heirs of Charles the Bold of Burgundy. The family de Branges performed administrative, legal, and religious functions in Saint-Amour for the marquisat d'Andelôt during Spanish rule of Franche-Comté. François de Branges of Saint-Amour received the seigneurie de Bourcia in 1679 when Franche-Comté became part of France. The château de Bourcia remained the home of his descendants until it was destroyed by Parisian revolutionaries in 1791. The château d'Andelôt near Saint-Amour, which survived the revolution, was bought in 1926 by Pierre du Pont, an elder brother of Irénée du Pont, for a nephew assigned in diplomatic service to France. This coincidence accounts for the interest which Irénée du Pont showed in a student of mathematics. The ruin of the château de Bourcia overlooks a fertile valley surrounded by wooded hills. The site is ideal for a mathematical research institute. The restoration of the château for that purpose would be an appropriate use of the million dollars offered for a proof of the Riemann hypothesis.