The Price of Universality

Gabriel Uzquiano

May 30, 2004

1. I would like to present a puzzle not in ontology but in the meta-theory of ontology. I would like to argue that when we engage in perfectly general inquiry we risk facing unexpected conflicts between different comprehensive packages. The problem is exacerbated by the fact that while one package may be independently motivated by metaphysical considerations, another may be primarily a mathematical package.

The puzzle will require the assumption that there is no impediment for the development of perfectly comprehensive hypotheses in which our quantifiers are wide open and range over absolutely all there is. That is, I will assume for the purposes of this paper that genuinely unrestricted quantification is available for the purposes of ontology.[1] This is not unreasonable since, on the face of it, ontology seems to require genuinely unrestricted quantification. Maybe ontology is not even possible; that remains to be shown. But for now I will proceed on the assumption that we are in a position to engage in perfectly general inquiry.

I will likewise assume the ontological innocence of plural quantification and make extensive use of plurals for the formulation of different perfectly comprehensive ontological hypotheses.[2]

In what follows, I will look at two different packages: one concerns the part-whole relation whereas the other is concerned with membership. Each of these relations has been characterized as topic-neutral and has been taken to apply across ontological categories. Thus it is very tempting to make use of genuinely unrestricted quantification in the formulation of their respective theories. Unfortunately, I will argue that the two packages are in tension in the presence of what would otherwise seem two independently plausible hypotheses. I am interested in the question of how to

[1]This assumption has been recently defended in [9] and [17].

[2]The ontological innocence of plural quantification has been defended in [1] and [6].

revolve the conflict.

But let me first introduce each of the packages under consideration. Our first package concerns the part-whole relation and consists of the following hypotheses:

(1) The part-whole relation is a two-place relation.

The part-whole relation is the subject matter of classical mereology. Our next hypothesis is the statement that, when stated with the help of plural quantification, classical mereology provides us with an adequate characterization of the structure of the part-whole relation:

(2) When stated with the help of plural quantification, the axioms of classical mereology are true of the part-whole relation.

David Lewis is a philosopher who has made use of plural quantification is the formulation of the following basic axioms of classical mereology in [6]:

Reflexivity: x is part of itself.

Transitivity: If x is part of some part of y, then x is part of y.

Unrestricted Composition: No matter what some objects are, there is a sum of them.[3]

Uniqueness of Composition: No matter what some objects are, there is at most one sum of them.

These axioms, in conjunction with standard definitions, guarantee that the structure of the part-whole relation is a complete Boolean algebra without a zero element.[4] And it is precisely this fact that will play a prominent role in our discussion.

The third and last hypothesis of the package is motivated by the assumption that the part-whole relation is topic-neutral and thus applies across ontological categories. Thus there are no restrictions on the field of the part-whole relation, which sometimes relates material objects, sometimes time intervals, sometimes regions of space, sometimes abstract objects, sometimes objects of a different ontological kind entirely. Because there are no restrictions on the field of the part-whole relation,

[3] Notice the use of plural quantification in the formulation of the axiom.

[4] I will take for granted the following definitions of 'overlap', 'distinct', and 'sum':
Definition: x *overlaps* y iff they have some common part z. x and y are *distinct* iff they have no part in common.
Definition: x is a *sum* of some objects iff x has all of them as parts and has no part distinct from each of them.

every object is part of itself. Maybe some objects have no parts other than themselves. Most objects, however, are sums of some proper parts. All there are, it seems, are atoms, if any, and sums of some proper parts.

What is important for present purposes, however, is that we should take our quantifiers in the formulation of the basic axioms of mereology to be wide open:

(3) The axioms of classical mereology take all objects of whatever ontological category into their domain.

This concludes the presentation of our first ontological package.

The preceding package is sometimes accompanied with the thesis that there is no mode of composition other than mereological composition. On this view, whenever a complex object is made out of some given objects, the complex is a mereological sum of them. Thus mereology is more than a theory of composition; mereology is *the* general theory of composition:

Compositional Monism: There is exactly one mode of composition.[5]

David Lewis is one philosopher who has famously defended this view in several places.[6] If Compositional Monism is true, then all cases in which an object is made out of some objects are instances of mereological composition. Since membership is different from the part-whole relation, it seems that *if* sets are made out of their members, they supply a counterexample to Compositional Monism. But the assumption that sets are made out of its members is not beyond doubt. David Lewis has recently challenge this assumption in [6], where he proposed to conceive of sets as made out of singletons instead: sets are mereological sums of singletons.

Whatever its merits, Compositional Monism is not forced upon one by one's allegiance to (1) – (3). For it is open to one to insist that mereology applies to all there is, individuals or sets, and nevertheless remain neutral with respect to the question of whether, for example, the composition of sets out of their members reduces to mereological compositon.[7]

[5] I borrow the term 'compositional monism' from [7].

[6] For example in [6].

[7] In what follows, I will use 'individual' to refer exclusively to non-sets that are members. My usage here differs from David Lewis's usage in [6] where he classifies the null set as an individual on the grounds that it has no members.

The time has come to lay down the ontological hypotheses comprising the package concerned with membership. The first hypothesis parallels (1) above:

(4) Membership is a two-place relation.

Membership is the subject matter of an applied mathematical theory: Zermelo-Fraenkel set theory (plus the axiom of choice) with urelements (ZFCU).[8] Our second hypothesis states that standard set theory provides us with an adequate characterization of the structure of membership:

(5) When stated with the help of plural quantification, the axioms of ZFCU are true of membership.

So I will assume some of the axioms of ZFCU are formulated with the help of plural quantification. In particular, I will take the axiom of replacement to read:

> *Replacement*: Given some ordered pairs whereby each member of a set x is paired with exactly one object, and if for each such object there is a member of x that is paired with it, the objects in question form a set.[9]

The third and last hypothesis is motivated by the assumption that there are no restrictions on the domain of the membership relation and all objects stand in the membership relation to some set or another. So we should take our quantifiers in the axioms of applied set theory to be wide open and range over absolutely all there is:

(6) The axioms of ZFCU take all objects of whatever ontological category into their domain.

This thesis is perfectly parallel to (3) above and summarizes the view that ZFCU is a perfectly comprehensive theory whose quantifiers again range over absolutely all there is. To be sure, not every object is a set. But because there are no restrictions on the ontological category of objects

I will use the term 'atom' to refer to objects that are not mereological sums. My use of the terms, then, leaves entirely open whether all individuals are mereological atoms or, alternatively, whether some individuals are mereological sums of some proper parts.

[8]Urelements are individuals or, on our present usage, non-sets.

[9]The other axiom that is sometimes formulated with the help of plural quantification is the axiom of separation, but, in the presence of replacement, the latter is redundant.

that are members, it seems that, in the presence of (2), all objects are members. Some will, in addition, have members and so they will be sets. Others will not have members and so they will be individuals. We will distinguish them within the language of set theory by the fact that the former are all and only those objects satisfying the predicate $Set(x)$. So we have that every object is either a set or an individual.

This concludes the presentation of our second package.

The question that immediately arises now is whether the two packages are compatible. There is no denying that each package should exert considerable attraction on many philosophers and mathematicians, respectively. So it would be very unfortunate if we had to surrender one or the other. The remainder of this paper argues that the two packages are incompatible with two independently plausible assumptions concerning the structure of the part-whole relation and membership, respectively.

2. The prospects of combining (1) – (3) with (4) – (6) depend to a large extent on what view one takes of the interaction between the part-whole relation and membership. One attractive hypothesis at this point is that sets have parts: their subsets. Perhaps she will want to allow for sets to have other parts as well. But for now we have:

> **First Thesis**: One set is part of another if and only if the first is a non-empty subset of the second.[10]

The First Thesis seems to conform well with ordinary speech. For as David Lewis reminds us in [6] we often speak of the set of even integers, for example, as part of the set of integers. Or the set of women as part of the set of human beings. And in fact it was not uncommon for the forefathers of set theory to refer to talk of sets as composed of parts and refer to the subsets of a set as its parts.[11]

[10] This is parallel to the First Thesis stated in [6], but, given (6), we make no room for classes that are not sets.

[11] Thus Georg Cantor writes in [4]:

> We will call by the name 'part' or 'partial set' of an set M any other set M_1 whose elements are also elements of M.

And, perhaps more interestingly, Ernst Zermelo writes in [18]:

> A subset of M that differs from 0 and M is called a *part* of M. The sets 0 and $\{a\}$ do not have parts.

Admittedly, this consideration has only limited force: one could object that we speak metaphorically when we speak of subsets of a set as parts of the set. Perhaps we are guided by a formal analogy between the part-whole relation and the subset relation. But, the objection continues, we should not read much into a mere analogy between the structure of the part-whole relation and the structure of the subset relation.

But notice that the First Thesis is much more difficult to resist in the presence of (1) – (3). Since there are no restrictions on the field of the part-whole relation, sets themselves enter into the part-whole relation. So the existence of structural analogies between the part-whole relation and the subset relation is indeed evidence for the First Thesis.[12]

And David Lewis has persuasively argued in [6] that the compositional monist has an incentive for combining the First Thesis with the thesis that the parts of a set are exhausted by its subsets:

> **Second Thesis**: No set has any part that is not a set.

For the combination of the two enable a compositional monist to identify a set with the mereological sum of some singletons, which would, according to the Second Thesis, qualify as mereological atoms. But because sets are mereological sums of singletons, the compositional monist will argue, we no longer need to postulate some unmereological way to make a set out of their members. The First Thesis and the Second Thesis jointly give compositional monists what they want:

> **Main Thesis**: The parts of a set are all and only its non-empty subsets.[13]

Unfortunately, we are in a position to derive a contradiction from the following ingredients:

- The framework of (1) – (3) according to which the axioms of classical mereology govern the part-whole relation and there is no restriction on the field of the part-whole relation.
- The framework of (4) – (6) according to which the axioms of applied set theory govern the membership relation and all there is is either an individual or a set.

[12] Notice that, on the present proposal, not every mereological sum of sets is a set. Take the sum, for example, of all non-self-membered sets. This sum exists, by Unrestricted Composition, but fails to form a set on pain of contradiction. This need not be a problem by itself. For we could treat some mereological sums as individuals.

[13] When the null set is treated as an individual, this tells us that the parts of a set are all and only its non-empty subsets.

- The Main Thesis.

We will make use, in particular, of the following consequence of the framework of (4) – (6):

> *Singletons*: Every object has a singleton.

One route to a contradiction would begin with suitable definitions of membership and set and derive a principle of naive comprehension for sets. Russell's paradox would then ensue.[14]

Without the detour through membership, the problem arises as follows. Some singletons overlap their members. Take the mereological sum of all there is, U. By Singletons, U has a singleton, $\{U\}$. But this singleton overlaps U because $\{U\}$ is itself part of U. Most singletons, however, do not overlap their members. My singleton is, on the Main Thesis, presumably not part of me, so we do not overlap. Thus some singletons, which we will call the Rs, fail to to overlap their members:

1. For every o, o is one of the Rs iff there is some x such that $o = \{x\}$ and o does not overlap x,

Unrestricted composition tells us that some object, r, is the mereological sum of the Rs:

2. For every o, o overlaps r iff o overlaps one of the Rs.

A paradox arises when we ask whether $\{r\}$, which exists according to Singletons, overlaps r.

3. $\{r\}$ overlaps r iff $\{r\}$ overlaps one of the Rs.

But since each R is a singleton and, by the Main Thesis, singletons have no proper parts, we have:

4. $\{r\}$ overlaps r iff $\{r\}$ is one of the Rs.

5. $\{r\}$ overlaps r iff there is some x such that $\{r\} = \{x\}$ and $\{r\}$ does not overlap x,

By extensionality, we infer:

6. $\{r\}$ overlaps r iff $\{r\}$ does not overlap r.

[14] This is the route David Lewis takes in [6], and, in his case, the primary reason to deny that all classes are sets. In the presence of proper classes that are not sets, the contradiction is avoided by denying Singletons.

No reason to be surprised. The following Cantorian argument provides us with an independent route to the incompatibility stated above. In the presence of Singletons and the Main Thesis, the operation that takes each object to its singleton provides us, in effect, with a 1-1 map of all objects in general into mereological atoms. But a generalization of Cantor's theorem shows that no such map exists on pain of contradiction. This is because classical mereology entails that there are strictly more sums than atoms. (Unrestricted Composition and Uniqueness of Composition jointly guarantee that no matter what some atoms may be, there is a unique sum of them. So there is a 1-1 correspondence between the power domain of all atoms and the domain of sums thereof.)

Two parenthetical remarks are in order. First, I have spoken of the domain of all objects and in what follows I will often speak of other domains that fail to form a set. Such talk should be understood as eliminable in favor of plural locutions that refer in the plural to whatever objects are supposed to lie in the domain. This is acceptable because I have assumed the availability of plural quantification at the outset. Because talk of domains that fail to form a set is highly convenient, I shall indulge in it. But the official translation into plural talk should be kept in mind.

Cantor's theorem states that the power set of a set is strictly larger than the set. Similarly, one might attempt to explain that the argument just now given relies on a certain generalization of Cantor's result: the power domain of a domain is strictly larger than the domain. Unfortunately, talk of a power domain of a domain that fails to form a set is not obviously eliminable in favor of plural quantification. So we have to be devious. First we must have resources to simulate relations that do not form a set.[15] If R is a relation taking all objects of whatever kind in its domain and x bears R to some objects, think of x as a *code* for them. The generalization of Cantor's theorem we are after states that no matter what relation R may be, there are some objects not coded by any object in the domain of R.[16]

At the end of the day, we have a generalization that is not specific to set theory. As Gideon Rosen has noted in [12], the following are incompatible:

- The framework of (1) – (3) according to which the axioms of mereology govern the part-whole

[15] In terms of plural quantification over ordered pairs, or perhaps by means of the machinery developed in the Appendix of [6].

[16] [13] contains a second-order statement and proof of the present generalization of Cantor's theorem.

relation and there is no restriction on the field of the part-whole relation.

- The existence of an operation ϕ satisfying the following constraints:

  Totality: $\forall x \exists y \; y = \phi x$

  Uniqueness: $\forall x \forall y \; (\phi x = \phi y \rightarrow x = y)$

  Atomicity: $\forall x \forall y (y$ is a part of $\phi x \rightarrow y = \phi x)$.

The combination of (4) – (6) and the Main Thesis guarantees the existence of such an operation: the singleton operation. (4) and (5) entail that the singleton operation satisfy Uniqueness and Totality, respectively, and the Main Thesis guarantees that singleton operation satisfies Atomicity.[17]

It is tempting to view the incompatibility between the Main Thesis and (1) – (3) and (4) – (6) as a problem specific to Compositional Monism. Since the Main Thesis is highly attractive for a compositional monist, it seems she had better reject one of the packages given at the outset. But perhaps it is open to the rest of us to just reject the Main Thesis as an account of how sets enter into the part-whole relation and still be able to reconcile our antecedent commitment to each of the original packages. One promising strategy at this point would be to give up the Second Thesis and thus allow for singletons to have proper parts themselves. This move could be made more plausible by noting that some singletons at least seem to occupy the extended spatiotemporal regions occupied by their members. But if spatiotemporal overlap is a sufficient condition for mereological overlap, then we should presumably admit that we overlap our singletons. So maybe there is independent reason to deny the Second Thesis. Once we do this, we are no longer compelled to endorse the Main Thesis and the difficulty appears to vanish.

Or does it? I will now argue that we should not try to take refuge in the denial of the Main Thesis. For whatever our account of how sets enter into the part-whole relation, we all face an uncomfortable predicament.

[17] It may be of interest to note that the conditions on the operation ϕ can be weakened further. Call some objects *relative atoms* iff they do not overlap. The moral of the Cantorian argument used above is that there are strictly fewer relative atoms than there are mereological sums thereof.

3. Before I introduce the predicament, however, let me quickly address the question of whether, absent specific constraints on the interaction between the part-whole relation and membership, our two original packages are in fact compatible.

Notice that (1) – (3) require the existence of a model of classical mereology whose domain comprises all there is. And (4) – (6) likewise require the existence of a model of ZFCU whose domain encompasses all objects there are. Because we have made use of plural quantification in the formulation of the axioms of each set of axioms, they impose far from trivial constraints on the size of their models. There are, in particular, no denumerable models of ZFCU when its axioms are formulated with the help of plural quantification.[18] Because I have insisted on the use of plural quantification, I will take ZFCU, for example, to require models of size greater or equal to some strong inaccessible.[19] But are there unrestricted models of ZFCU and classical mereology?

If we follow orthodoxy in identifying a model with a structured set of a certain sort, then the answer to our question is trivial and uninformative. Because there is no set of all objects, there is no model whose domain comprises all there is. Fortunately, there is an alternative definition of 'model' which makes use of plurals and differs from currently standard usage in permitting models larger than any set; indeed, models that take in the universe of all objects as a domain.[20] Given our global concerns, the latter usage is preferable for our purposes.

In what follows, when I speak of the cardinality of a domain that fails to form a set, that should not be understood in terms of the existence of a set-theoretic cardinal that measures its size. Instead, we should understand such talk in second-order terms as outlined in [13]. When I speak, for example, of the domain of all objects as strongly inaccessible in size, I mean to assert the official (conveniently pluralized) second-order translation of the following: (i) the domain is

[18] When couched in first-order terms, ZFCU is, if satisfiable, satisfiable in denumerable models on account of the Löwenheim-Skolem theorem. Unfortunately, as George Boolos has persuasively argued in [1], a first-order formulation of Zermelo-Fraenkel set theory with individuals give no more than partial expression to the principles set theorists do believe. Plural quantification allows us to do justice to their commitments by expressing some of the very general principles underlying the first-order axiomatization of the theory.

[19] A cardinal κ is strongly inaccessible just in case:
(i) κ is not denumerable.
(ii) For any $\lambda < \kappa$, $2^\lambda < \kappa$, and
(iii) κ cannot be represented as the supremum of fewer than κ smaller ordinals.
In sum, κ cannot be reached from lower cardinalities through applications of the operations used in (ii) and (iii). For present purposes, however, we should understand (i) – (iii) in second-order terms along the lines indicated in [13].

[20] This model theory is set forth in [11] and is structurally similar to a second-order account offered in [13].

not denumerable, (ii) the size of the domain cannot be reached from below by taking unions of fewer domains of smaller size, and (iii) the size of the domain cannot be reached from below by taking powers: the power domain of any domain of smaller size is still smaller than the domain of all objects. To be sure, the official translation of some of the clauses will have to make use the techniques employed in the formulation of the generalization of Cantor's theorem above in order to eliminate apparent talk of domains of domains but the details have been worked out in [13].

To return to our question, there is no formal obstacle to the existence of unrestricted models of each ZFCU and classical mereology. If the size of the universe of all objects is inaccessible, then there is an unrestricted model of ZFCU. More delicate is the question of whether there are models of classical mereology of inaccessible size. The crucial observation for present purposes is that there are complete Boolean algebras of any inaccessible size.[21] Since a model of classical mereology is basically a complete Boolean algebra, we can rest assured that if the universe of all objects is inaccessible, then there is an unrestricted model of classical mereology.

The situation changes drastically when we allow two seemingly plausible and independently motivated principles into the picture. And this will be the source of our predicament.

First there is the principle that there are as many pure sets as there are objects altogether.[22]

Maximality Principle: There is a 1-1 map from the universe into the pure sets.

The Maximality Principle is not a deductive consequence of ZFCU.[23] There are, however, independent reasons to think it is true. The axioms of standard set theory are motivated by a mixture of two different conceptions of set. While the iterative conception motivates the axioms of set theory with the exception of replacement, the doctrine of limitation of size motivates a different fragment of set theory but omits much of importance.[24]

The iterative conception is the view that sets are generated in a cumulative hierarchy of stages. We begin with individuals. At stage 0 all sets of individuals are formed. So stage 0 consists of all individuals and sets thereof. The sets formed at stage 1 are all sets of individuals and sets formed

[21] More precisely, there are complete Boolean algebras of *infinite* cardinality κ iff $\kappa^{\aleph_0} = \kappa$ as shown in [5].

[22] A set is pure if its elements, the elements of its elements, the elements of those, etc are all sets.

[23] A model of second-order ZFCU falsifying the Maximality Principle is discussed in [15].

[24] George Boolos persuasively argues for this claim in [2].

at stage 0, i.e., sets of individuals, sets of sets of individuals and sets of individuals and sets of individuals. After stage 1 comes stage 2 at which all sets of individuals and sets formed at stage 1 are formed.

Immediately after all stages 0, 1, 2, ..., there is a stage, stage ω. The sets formed at stage ω are all sets formed at finite stages earlier than ω. After stage ω comes stage $\omega + 1$ where all sets of sets formed at stage ω are formed, etc. In general, at stage α all sets of individuals and sets formed at stages earlier than α are formed.[25]

On the iterative conception, then, a set of individuals is formed at stage 0. But in the presence of the axioms of ZFCU, the existence of a set of all individuals entails the Maximality principle.[26] So *modulo* the axioms of ZFCU, the Maximality Principle seems to fall out of the picture of sets as generated in a transfinite sequence of stages that is implicit in the iterative conception of set.

The doctrine of limitation of size provides a different view of sets on which some objects form a set if and only if they are *few* in that there is no 1-1 map of the universe of all objects into them:

> *Limitation of Size*: Some objects form a set if and only if there is no 1-1 map of the universe of all objects into them.

The limitation of size doctrine motivates all of the axioms of ZFCU with the exception of infinity and power set. But the set-theoretic paradoxes show that the pure sets do not form a set on pain of contradiction. By Limitation of Size, it follows that there is a 1-1 map of the universe of all objects into the pure sets. Thus the Maximality Principle is an immediate consequence of the limitation of size conception of set.

So the Maximality Principle is in line with each of the views standardly cited in support of the axioms of ZFCU. This by itself is ample evidence in support of the Maximality Principle.

The last ingredient for our predicament is the mereological thesis that every object has a decomposition into atoms, or, to use David Lewis's phrase, that there is no atomless gunk:

[25] The result is a transfinite sequence of stages, which provides us with a map of the set-theoretic universe. If U is the set of individuals:
$U_0 = \mathcal{P}(U)$;
$U_{\alpha+1} = U \cup \mathcal{P}(U_\alpha)$;
$U_\lambda = \bigcup_{\alpha<\lambda} U_\alpha$, for λ a limit.

[26] This fact is proved, for example, in [8].

No Gunk Thesis: There are no objects whose parts all have further proper parts.

Classical mereology does not pronounce on the question of whether all objects are ultimately composed of mereological atoms. So No Gunk is independent from the axioms of classical mereology, but it is nonetheless a hypothesis with a distinguished history. Philosophers have generally found the No Gunk Thesis congenial, though not beyond doubt.[27] Hence one would expect to be able to combine this thesis with the Maximality Principle and the two packages given at the outset.

We are now in a position to state the predicament. The following are incompatible:

- The framework of (1) – (3) according to which the axioms of classical mereology govern the part-whole relation and there is no restriction on the field of the part-whole relation.
- The framework of (4) – (6) according to which the axioms of ZFCU govern the membership relation and all there is is either an individual or a set.
- The Maximality Principle.
- The No Gunk Thesis.

The problem is this. The combination of (1) – (3) with the No Gunk Thesis ensure that the universe of all objects has the structure of a complete atomic Boolean algebra without a zero element. But the universe of a complete atomic Boolean algebra is in 1-1 correspondence with the power domain of the domain of mereological atoms. Therefore the size of the universe of all objects can indeed be reached from below by taking powers and is thus not inaccessible.[28]

Unfortunately, the combination of (4) – (6) and the Maximality Thesis requires the universe to be inaccessible. Because the axioms of ZFCU guarantee that the domain of pure sets is inaccessible and, by the Maximality Thesis, that is in in 1-1 correspondence with the entire universe, it follows that the universe of all objects should be inaccessible.

So either the universe of all objects fails to model ZFCU or it fails to model classical mereology.

[27] As argued, for example, in [19].

[28] The conveniently pluralized official translation of these claims should be kept in mind.

We face a predicament because we took each of the four ingredients involved in the preceding argument to be antecedently plausible. While the combination of (1) – (3) and the No Gunk Thesis is supported by familiar metaphysical considerations, the combination of (4) – (6) and the Maximality Principle seems supported by primarily mathematical considerations.

4. It seems we must choose at least one of the following options:

(i) Reject the framework of (1) – (3); in particular, reject (2), i.e., the thesis that classical mereology governs the part-whole relation, or (3) according to which there no restriction on the field of the part-whole relation.

(ii) Reject the framework of (4) – (6); in particular, reject (5), i.e., the thesis that the axioms of ZFCU govern membership, or (6) according to which all objects are members.

(iii) Reject the Maximality Principle.

(iv) Reject the No Gunk Thesis.

But mathematical practice in combination with two background assumptions would seem to provide support for the combination of (4) – (6) and the Maximality Principle. Yet, some philosophers often provide metaphysical considerations in favor of the combination of (1) – (3) with the No Gunk Thesis. My primary concern has to do with the methodological question of how to proceed from here. In what follows, I will explore each of the four options in detail.

First in our list is (i) or the rejection of the first package: (1) – (3). I will assume (1) is not on the table for present purposes.[29] This leaves (2) or the view that classical mereology governs the part-whole relation and (3) or the view that there is no restriction on the field of the part-whole relation. Let me start with (2). One might circumvent our predicament by relaxing Unrestricted Composition or, alternatively, by denying Uniqueness of Composition. One weakening

[29] Some endurantists have, to be sure, powerful reasons to question (1); they may prefer to conceive of the part-whole relation as a three-place relation. The endurantist reasons, however, are tangential to the present debate and there is no reason to think that the denial of (1) would, by itself, do much to alleviate the conflict.

I regret not to have space to discuss another alternative according to which the part-whole relation is indeed not a two-place relation but rather a relation taking two plural arguments: *are among*. Suffice it to say that such a drastic move would hardly be motivated by the present conflict alone.

of Unrestricted Composition that would suit our purposes requires the assumption that there is no atomless gunk. In the presence of the No Gunk Thesis, one might introduce the following restriction on mereological composition:[30]

> *Limitation of Composition*: Whenever there are some objects whose atomic parts are not in 1-1 correspondence with all the objects there are, there is a mereological sum of them.

One attractive feature of this restriction on composition is that it need not require for there to be strictly more mereological fusions than atoms. Limitation of Composition and Uniqueness of Composition now guarantee that no matter what some atoms are, if they are not in 1-1 correspondence with the universe, there is a unique mereological sum of them. For set-sized domains, this amounts to the existence of a 1-1 map from the number of mereological atoms to the set of subsets of the domain of cardinality less than κ. But this set has cardinality κ when κ is inaccessible. When we generalize to cases in which the domain need not form a set, we have that if the universe is inaccessible in size, then there can be a relation R which codes any objects in the domain that are not in 1-1 correspondence with the domain by some object in the domain. So there is no obstacle to have a model of the amended axioms with unrestricted domain of inaccessible size.

There is more. Limitation of Composition fits well with the Main Thesis. For according to this combination of theses, some singletons will have a sum iff they are not in 1-1 correspondence with the universe. But in the presence of Limitation of Size, any such singletons will themselves form a set. This latter set will be their sum.

The obvious problem with this response is that it lacks any motivation besides its ability to bail us out from our predicament. In the absence of any independent support, it seems reasonable to dismiss the restriction on composition as an ad hoc move that would be largely unacceptable in the presence of a better alternative.

The other move in this vein is to deny Uniqueness of Composition. In the presence of the No Gunk Thesis, Unrestricted Composition simply places a lower bound on the size of the domain of a model of classical mereology. For set-sized models, the existence of κ mereological atoms guarantees

[30] David Lewis and Gideon Rosen consider similar suggestions in [6] and [12], respectively.

at least 2^κ mereological sums. But in the absence Uniqueness, nothing in the remaining axioms of mereology requires this to be an upper bound on the size of the domain. There are set-sized models of classical mereology minus Uniqueness in which (a) the No Gunk Thesis is true and (b) the domain is inaccessible due to a massive failure of Uniqueness. Unfortunately, the rampant coincidence that would be required for this move is likely to offend even the most enthusiast opponent of Uniqueness of Composition. For in such a set-sized model, at least some objects would need to have as many sums as there are objects in the domain.[31] But there is no reason to think the situation is different when we make allowance for models with unrestricted domain. For the denial of Uniqueness to be helpful, its failure ought to be so very massive and widespread that it is hardly credible.

It seems that the rejection of (2) is largely unacceptable in the presence of a better option.

And there is considerably a better alternative in the denial of (3) or the thesis that classical mereology is a perfectly comprehensive theory. The use of perfectly unrestricted quantification is an audacious enterprise. It takes some temerity to put forward basic axioms of mereology as perfectly general hypotheses that comprehend absolutely all there is. For we have seen that such ambitious claims impose stringent structural constraints on the universe of objects. Perhaps we should retreat, then, to more modest claims restricted to some subdomain of the universe. When one thinks of the axioms of mereology as restricted to a domain of spatiotemporally extended material objects, for example, one avoids imposing global constraints on the size of the universe of all objects.

This response has its costs. For one reason, it will not sit well with Compositional Monism. One of the fundamental projects of ontology is to draw and explain the distinction between *simple* objects and *complexes*. This distinction is topic-neutral and applies across ontological categories; we all agree that symphonies and novels as well as groups, sets and mereological sums are complexes somehow made out of simpler constituents. But the compositional monist takes a further step to identify mereological composition as the only mode of composition by which the generation of any complexes is to be explained. Because a compositional monist views classical mereology as *the the* theory of composition, she will want its quantifiers to be wide open. And she will hardly be

[31]This is because if κ is inaccessible and λ (the number of mereological atoms) $< \kappa$, then for any $\gamma < \kappa$ $2^\lambda.\gamma < \kappa$.

moved by our predicament to retreat to a more modest view of classical mereology as a set of more parochial principles concerned with a limited parcel of reality.

Another concern is methodological. The combination of (1) – (3) with the No Gunk Thesis is a distinctively metaphysical package. In contrast, the combination of (4) – (6) with the Maximality Thesis is primarily a mathematical package. But is it that obvious that our intent to rescue a primarily mathematical package should take precedence over our overall metaphysical outlook?

The second option open to us is (ii) or the rejection of the second package: (4) – (6). I assume that (4) is out of the question, which leaves (5) or the thesis that the axioms of ZFCU adequately characterize membership and (6) or the thesis that ZFCU is a perfectly comprehensive theory whose quantifiers are genuinely unrestricted. The problem with the first alternative of course is that it is highly revisionary. ZFCU is an incredibly successful mathematical theory with an impressive track record. The apparent incompatibility of such a mathematical theory with a competing metaphysical package is not likely to move a practicing mathematicians to amend their ways and give up one or more of the axioms of ZFCU.

Much more promising is to reject the thesis that ZFCU is a perfectly general theory and, in particular, the view that all objects are members. This is the line taken in [6], where David Lewis effectively rejects (6) or the thesis that ZFCU takes absolutely all objects of whatever kind into its domain. For according to Lewis there are, in addition to individuals and sets, proper classes that are not sets. Sets are some but not all of the classes there are. The distinction is this. Whereas sets are classes that are members of some classes, proper classes never are members of other classes. Not only are proper classes not sets, they are not individuals either. And they lie outside of the domain of ZFCU, which is exclusively concerned with individuals and sets. The advantages of this move should be apparent by now. In exchange for (6), Lewis is in a position to combine all of (1) – (3) (with or without the No Gunk Thesis) and the Main Thesis.

The package is neat. We avoid the problem outlined in section 2 by denying that every object has a singleton; proper classes, after all, are not members. And we are in a position to vindicate Compositional Monism by identifying sets with mereological sums of singletons, which, in the presence of the Main Thesis, are identified with atoms.

Unfortunately, there are at least two reasons to remain dissatisfied.

First there is the question of what it takes to be individual. The answer is trivial when all objects are members: to be an individual is not to be a set. Unfortunately, proper classes are not sets, but, on the present view, they are not individuals either. Otherwise, they too would have singletons and the problem raised in section 2 would re-emerge. So individuals have singletons but proper classes do not. But unless we offer some principled explanation of what makes individuals special in permitting the formation of singletons, it will seem justified to complain that it relies on a mysterious restriction on the singleton operation. Perhaps we should not be surprised by the restriction on account of the fact that, as Lewis acknowledges in [6], the making of singletons is ill-understood to begin with. But that does little to alleviate the mystery.

But even if one is prepared to accept the mystery, the proposal remains somewhat revisionary. For practicing mathematicians standardly conceive of set theory as the most comprehensive theory of collections. It is not that set theorists never indulge in talk of proper classes; sometimes they do in the course of heuristic motivations for some axioms. But they tend to regard such uses of classes as largely eliminable in favor of substitutional quantification over formulas of the language of ZFCU or plural quantification over sets or by some other means entirely. Since, according to orthodoxy, no collections exist that are not sets, no theory of collections is more general than ZFCU.

Unfortunately, the proposal now on the table asks us to sacrifice this in favor of a distinction between two fundamentally different kinds of collections: sets and proper classes. Because proper classes lie outside of the scope of set theory, ZFCU is no longer the most general theory of collections: class theory is. But this is a loss from the standpoint of practicing mathematicians. As a matter of fact, we are not even entitled to the claim that set theory comprises the predominant part of reality, for, as mentioned above, there are far more proper classes in existence than there are individuals and sets. So set theory is thus concerned with a somewhat impoverished parcel of reality.

Two choices remain: (iii) to reject the Maximality Thesis or (iv) to reject the No Gunk Thesis.

As for (iii), it seems reasonable to complain that it is revisionary. To be sure, the Maximality Principle is not entailed by the axioms of ZFCU but it quickly falls out of each of the two thoughts standardly claimed to underlie those axioms. Moreover, whatever its merits, one would have thought we should expect the fate of the Maximality Principle to be decided on the basis of considerations that are internal to mathematical practice and not on the basis of its seeming incompatibility with a metaphysical outlook endorsed by some philosophers.

Alas, there are considerations internal to mathematical practice that speak in favor of the Maximality Principle. For one of the points of set theory is to supply pure set-theoretic surrogates for the greater number of structures exemplified in reality. Of course we should not expect too much. There is no pure (or otherwise) set-theoretic surrogate for the structure of the pure sets under membership on pain of contradiction. And, in general, there is no set-theoretic surrogate for structures whose domain is in 1-1 correspondence with the universe of all objects. But surrendering the Maximality Principle would make the failure of the universe of pure sets to deliver surrogates for structures exemplified in reality much greater than it needs to be. The point of the Maximality Principle is to guarantee that the universe of pure sets is sufficiently rich and varied in order to guarantee the existence of isomorphic copies of the greater possible range of structures.

Our last option is (iv) or the rejection of the No Gunk Thesis. By itself, this option does not tell us where to locate the alleged atomless gunk. It may be that individuals are in the end made of atomless gunk; or it may be that sets themselves are made of atomless gunk; or, even less plausibly, it may be that atomless gunk makes up individuals and sets alike. But it seems unpalatable to conclude from our predicament at least some individuals are made of atomless gunk. For whatever the merits of that view, it seems it should not be established merely in virtue of global considerations on the size of the universe. One would have thought it is not the business of set theory to discern the nature of individuals. Set theorists are exclusively concerned with the structure of membership and should defer to ontology to tell us what individuals there are to begin with and what is their nature. In the absence of independent metaphysical considerations for the thesis that individuals themselves are made of atomless gunk, we should be reluctant to accept that conclusion merely for the purpose of securing the compatibility of classical mereology with applied set theory.

Perhaps (iv) could be made more plausible by viewing our predicament as evidence that sets themselves are made of atomless gunk (even if no individuals are). What we learn from the difficulty is that the mereological structure of the set-theoretic universe is very different from that outlined by the Main Thesis. To make this alternative more vivid, consider what would happen if we replaced the First Thesis by what would otherwise strike us as a crazy thesis:

> **Converse First Thesis**: One set is part of another if and only if the first is a superset of the second, i.e., the second is a subset of the first.

So $\{a, b, c\}$ is part of $\{a, b\}$, which is itself part of $\{a\}$, according to the Converse First Thesis. Two sets x and y overlap iff they have some part in common, i.e., there is a set z such that x is a subset of z and y is a subset of z. It follows that any two sets overlap. Now: interestingly, given some sets, a set x would be their mereological sum just in case (a) x is a subset of each of them, and (b) for any z, z is part of x just in case z overlaps one of the given sets. The last clause is redundant in view of the fact that any two sets overlap. So given some sets, their intersection will qualify as a mereological sum of them. The picture that emerges is one in which every part of a set has further proper parts and hence sets themselves are made of atomless gunk regardless of whether individuals like a or b are. One might even be tempted to take one step further and adopt:

> **Converse Main Thesis**: The parts of a set are all and only its supersets,

Unfortunately, there is one last formal obstacle to identifying the part-whole relation on sets with the superset relation. Although this relation satisfies most of the axioms of classical mereology, Uniqueness of Composition fails. For a cursory look at the definitions of 'overlap' and 'sum' shows that when we take some sets and ask what a sum of them is, we find that their intersection is not the only set to qualify as a sum of them: any subset of their intersection will do just as well.

The Converse Main Thesis *almost* saves the day by reconciling the rejection of the No Gunk Thesis with our pre-theoretic conviction that set theory should remain silent with respect to the character of individuals (and their mereological structure) in particular. So perhaps there is reason to think some close variant of the Converse Main Thesis will manage to recover all of the axioms of classical mereology while making sure sets themselves are made of atomless gunk. Perhaps there is such a variant indeed, but the point I would like to make is that the account of how sets enter into the part-whole relation that would emerge would most likely be at least as far removed from ordinary usage as the account embodied by the Converse Main Thesis. So far removed, in fact, that it is hardly credible.

5. Each of the four responses discussed thus far is very costly. Some may be tempted to question one or more of the background assumptions made at the outset. In particular, some may insist we should deny unrestricted quantification is ever available. For if we had done that, then we could insist that there are no truly universal theories and thus we would reject each of (3) and (6). Moreover, we could deny that the axioms of either classical mereology or applied set theory ever place any constraints on the global structure of the universe. (For there is no such universe to begin with.) Instead, the moral that would have emerged is just that no two models of classical mereology plus the No Gunk Thesis and ZFCU plus the Maximality Principle, respectively, share their domain of quantification. But that is hardly a reason to be concerned.

But if we surrender unrestricted quantification, have we not given up much of philosophical discourse and, in particular, ontology? When we advance an ontological hypothesis we intend it to be more than a parochial account of what surrounds us in an obscure corner of an unimpressive galaxy. Instead, we set out to comprehend absolutely all there is. To declare unrestricted quantification unavailable would seem pose an insurmountable obstacle to the development of perfectly general inquiry. And such a draconian measure would raise more perplexing questions and problems than it would help solve.

My own inclination, however, is to view the predicament as evidence that it takes considerable temerity to indulge in perfectly general inquiry. When we lay down perfectly comprehensive hypotheses taking absolutely all objects in their domain, we risk imposing highly non-trivial constraints on the structure of the universe of all objects. So we should exercise extreme caution before using genuinely unrestricted quantification in the formulation of our hypotheses.

This by itself doesn't adjudicate the issue of whether to restrict the quantifiers employed in our formulation of applied set theory or the quantifiers used in our development of classical mereology. In the case of applied set theory, however, it is particularly difficult to motivate a restriction on the range of our quantifiers. For when we conceive of individuals as whatever objects that are not sets, it seems undeniable that all objects are individuals or sets, and, in particular, all objects are members. The alternative of course is to artificially restrict the domain of individuals to include non-sets that fail to enter into the membership relation to other objects. But then we owe an account of the mysterious distinction between objects that do and objects that fail to enter into the membership relation.

Unless we are prepared to accept the mystery, one should probably question the assumption that the axioms of classical mereology take absolutely all objects into its domain. One may, for example, be more inclined to do this partly because one is tempted to think of the part-whole relation as a spatiotemporal relation among material objects, in which case its field would be exhausted by material objects with spatiotemporal location. If sets are not material objects, then there would be no reason to think they enter into the part-whole relation and our problem would seem to vanish.

Far from me, however, to suggest that this conclusion is forced upon us by the predicament. Instead, the situation appears to be this. Because I am prepared to countenance unmereological modes of composition, I have no use for Compositional Monism. But when one's antecedent theoretical commitments fit well with the restriction of the scope of classical mereology, one should be disposed to pay the price in exchange for the compatibility of the two packages.

None of this, however, will help a compositional monist with an independently motivated investment on the view that the field of the part-whole relation is genuinely unrestricted. Such a philosopher might prefer to embrace what I have called the mysterious distinction between sets and proper classes. Or she might prefer to abandon one of the ingredients needed for the confict. I am aware that nothing I have said will make a compositional monist, for example, feel obligated to restrict the scope of classical mereology. But at least I have issued a challenge. Anyone accepting the combination of (1) – (3) with the No Gunk Thesis will have to face the predicament I discussed in the last two sections. And as far as I can tell, all of the available responses both lack independent motivation and come at a considerable cost, a different one in each case.

References

[1] George Boolos. To be is to be a value of a variable (or to be some values of some variables). *The Journal of Philosophy*, 81:430–49, 1984. Reprinted in [3].

[2] George Boolos. Iteration again. *Philosophical Topics*, 17:5–21, 1989. Reprinted in [3].

[3] George Boolos. *Logic, Logic and Logic*. Harvard, Cambridge, Massachusetts, 1998.

[4] Georg Cantor. *Contributions to the Founding of the Theory of Transfinite Numbers*. Dover, New York, NY, 1955.

[5] S. Koppelberg. General theory of Boolean algebras. 1981. in [10].

[6] David Lewis. *Parts of Classes*. Blackwell, Oxford, 1991.

[7] Kris McDaniel. Modal realism with overlap. *Australasian Journal of Philosophy*, 82:137–152, 2004.

[8] Vann McGee. How we learn mathematical language. *Philosophical Review*, 106:35–68, 1997.

[9] Vann McGee. 'Everything'. 2000. In [14].

[10] Donald Monk. *Handbook of Boolean algebras*, volume 1. North Holland, Amsterdam, 1989.

[11] Agustín Rayo and Gabriel Uzquiano. Toward a theory of second-order consequence. *The Notre Dame Journal of Formal Logic*, 40:315–325, 1999.

[12] Gideon Rosen. Armstrong on classes as states of affairs. *Australasian Journal of Philosophy*, 73(4):613–625, 1995.

[13] Stewart Shapiro. *Foundations Without Foundationalism: A Case for Second-Order Logic*. Clarendon Press, Oxford, 1991.

[14] Gila Sher and Richard Tieszen, editors. *Between Logic and Intuition*. Cambridge University Press, New York and Cambridge, 2000.

[15] Gabriel Uzquiano. Categoricity theorems and conceptions of set. *The Journal of Philosophical Logic*, 31(2):181–196, 2002.

[16] Jean Van Heijenoort, editor. *From Frege to Gödel*. Harvard University Press, Cambridge, MA, 1967.

[17] Timothy Williamson. Everything. *Philosophical Perspectives*, 2003.

[18] Ernst Zermelo. Untersuchungen über die grundlagen der mengenlehre i. *Mathematische Annalen*, 65:261–81, 1908. English translation by Stefan Bauer-Mengelberg, 'Investigations in the Foundations of Set Theory I', in [16].

[19] Dean Zimmerman. Could extended objects be made of simple parts? an argument for atomless gunk. *Philosophy and Phenomenological Research*, 56:1–29, 1996.