Solutions to Some MWG Problems

Christian Roessler
U Melbourne

- 3.D.1, 3.G.8, 3.G.15
- 5.C.2, 5.C.7, 5.C.9
- 6.B.2, 6.B.4, 6.C.1, 6.C.9, 6.C.16, 6.C.20, 6.E.1
- 7.E.1
- 8.B.1, 8.B.4, 8.C.4, 8.D.9
- 9.C.2
- 13.B.2, 13.B.4, 13.B.5, 13.C.4, 13.D.1, 13.D.2
- 14.C.2, 14.C.9
- 15.B.1, 15.B.6, 15.B.7, 15.C.2, 15.D.8, 15.D.9

3.D.1

The Cobb-Douglas utility function

$$u(\mathbf{x}) = x_1^{\alpha} x_2^{1-\alpha}$$

is homogeneous of degree one:

$$\begin{aligned} u(k\mathbf{x}) &= (kx_1)^{\alpha} (kx_2)^{1-\alpha} \\ &= k^{\alpha} x_1^{\alpha} k^{1-\alpha} x_2^{1-\alpha} \\ &= k x_1^{\alpha} x_2^{1-\alpha}. \end{aligned}$$

Walras' law holds since u is locally nonsatiated. (The Cobb-Douglas utility function is strictly increasing in its arguments, which is stronger than local nonsatiation.)

Because u is strictly concave, it has a unique maximum, which is clearly a convex set. Concavity follows from $x_1 > 0$ and $x_2 > 0$ (implied by $\lim_{x_1 \to 0} \frac{\partial u(\mathbf{x})}{\partial x_1} = \lim_{x_2 \to 0} \frac{\partial u(\mathbf{x})}{\partial x_2} = \infty$) and

$$\begin{aligned} \nabla u(\mathbf{x}) &= \begin{bmatrix} \alpha \left(\frac{x_2}{x_1}\right)^{1-\alpha} \\ (1-\alpha)\left(\frac{x_1}{x_2}\right)^{\alpha} \end{bmatrix} > \mathbf{0} \\ D\nabla u(\mathbf{x}) &= \begin{bmatrix} -\alpha\,(1-\alpha)\,\frac{x_1^2 x_2}{(x_1 x_2)^{\alpha}} & \alpha\,(1-\alpha)\,\frac{x_1^{\alpha}}{x_1 x_2^{\alpha}} \\ \alpha\,(1-\alpha)\,\frac{x_1^{\alpha}}{x_1 x_2^{\alpha}} & -\alpha\,(1-\alpha)\,\frac{x_1^{\alpha}}{x_2^{\alpha+1}} \end{bmatrix}. \end{aligned}$$

The Hessian is negative definite, since the principal minor is negative and $|D\nabla u(\mathbf{x})| = \alpha^2\,(1-\alpha)^2 \left(\frac{x_1^4 + x_1^{2\alpha}}{x_1^2 x_2^{2\alpha}}\right) > 0$

3.G.8

If v is logarithmically homogeneous, then

$$v(\mathbf{p}, \alpha w) = \ln \alpha + v(\mathbf{p}, w),$$

where

$$v(\mathbf{p}, w) = \ln \widetilde{v}(\mathbf{p}, w),$$

for some $\widetilde{v}(\mathbf{p}, w)$ that is homogeneous of degree 1. (Thus $v(\mathbf{p}, \alpha w) = \ln \alpha \widetilde{v}(\mathbf{p}, w) = \ln \alpha + v(\mathbf{p}, w)$.)

We have

$$\begin{aligned} \nabla_p v(\mathbf{p}, w) &= \frac{\nabla_p \widetilde{v}(\mathbf{p}, w)}{\widetilde{v}(\mathbf{p}, w)} \\ \nabla_p \widetilde{v}(\mathbf{p}, w) &= \widetilde{v}(\mathbf{p}, w) \nabla_p v(\mathbf{p}, w). \end{aligned}$$

By Euler's theorem,

$$\widetilde{v}(\mathbf{p}, w) = \frac{\partial \widetilde{v}(\mathbf{p}, w)}{\partial w} w.$$

Using Roy's identity,

$$\nabla_p \widetilde{v}(\mathbf{p}, w) = -\frac{\partial \widetilde{v}(\mathbf{p}, w)}{\partial w} \mathbf{x}^*.$$

Substituting,

$$\begin{aligned} \nabla_p \widetilde{v}(\mathbf{p}, w) &= -\frac{\widetilde{v}(\mathbf{p}, w)}{w}\mathbf{x}^* \\ \widetilde{v}(\mathbf{p}, w)\nabla_p v(\mathbf{p}, w) &= -\frac{\widetilde{v}(\mathbf{p}, w)}{w}\mathbf{x}^* \end{aligned}$$

Dividing through by $\widetilde{v}(\mathbf{p}, w)$ and letting $w = 1$,

$$\nabla_p v(\mathbf{p}, 1) = -\mathbf{x}^*.$$

3.G.15

(a) The function $u(\mathbf{x}) = 2\sqrt{x_1} + 4\sqrt{x_2}$ is locally nonsatiated; therefore Walras' law holds. The utility maximization problem is:

$$\begin{aligned} & \max_{x_1, x_2} 2\sqrt{x_1} + 4\sqrt{x_2} \\ \text{s.t.} \quad & p_1 x_1 + p_2 x_2 = w \\ & x_1 \geq 0, \ x_2 \geq 0. \end{aligned}$$

Lagrangian formulation (ignoring the nonnegativity constraints, which will hold at the optimum):

$$\max_{x_1, x_2} 2\sqrt{x_1} + 4\sqrt{x_2} + \lambda\left(w - p_1 x_1 - p_2 x_2\right).$$

First-order conditions:

$$\begin{aligned} \frac{\partial u(\mathbf{x}^*)}{\partial x_1} &= \frac{1}{\sqrt{x_1^*}} - \lambda^* p_1 = 0 \\ \frac{\partial u(\mathbf{x}^*)}{\partial x_2} &= \frac{2}{\sqrt{x_2^*}} - \lambda^* p_2 = 0 \\ \frac{\partial u(\mathbf{x}^*)}{\partial \lambda} &= w - p_1 x_1^* - p_2 x_2^* = 0. \end{aligned}$$

Solving for $\mathbf{x}^*(\mathbf{p}, w)$:

$$\begin{aligned} x_1^* &= \frac{p_2}{4p_1^2 + p_1 p_2} w \\ x_2^* &= \frac{4p_1}{4p_1 p_2 + p_2^2} w. \end{aligned}$$

Nonnegativity of x_1, x_2 is met. The second derivatives are negative, indicating a maximum.

(b) The expenditure minimization problem is:

$$\min_{x_1,x_2} p_1 h_1 + p_2 h_2$$
$$\text{s.t.} \quad 2\sqrt{h_1} + 4\sqrt{h_2} \geq \overline{u}$$
$$h_1 \geq 0, \ h_2 \geq 0.$$

Since $f(\mathbf{h}) = p_1 h_1 + p_2 h_2$ is strictly increasing in $\overline{u}$ and $u(\mathbf{h}) = 2\sqrt{h_1} + 4\sqrt{h_2}$ is continuous, the utility threshold constraint must hold with equality. Ignore the nonnegativity constraints for now. Lagrangian formulation:

$$\max_{x_1,x_2} - p_1 h_1 - p_2 h_2 + \lambda \left(2\sqrt{h_1} + 4\sqrt{h_2} - \overline{u} \right).$$

First-order conditions:

$$\frac{\partial f(\mathbf{h}^*)}{\partial h_1} = -p_1 + \frac{1}{\sqrt{h_1^*}} \lambda^* = 0$$
$$\frac{\partial f(\mathbf{h}^*)}{\partial h_2} = -p_2 + \frac{2}{\sqrt{h_2^*}} \lambda^* = 0$$
$$\frac{\partial f(\mathbf{h}^*)}{\partial \lambda} = 2\sqrt{h_1^*} + 4\sqrt{h_2^*} - \overline{u} = 0.$$

Solving for $\mathbf{h}^*(\mathbf{p}, \overline{u})$:

$$h_1^* = \left(\frac{p_2}{8p_1 + 2p_2} \overline{u} \right)^2$$
$$h_2^* = \left(\frac{p_1}{4p_1 + p_2} \overline{u} \right)^2.$$

Nonnegativity of h_1, h_2 is met. The second derivatives are negative, indicating a maximum.

(c) The expenditure function is determined by:

$$e(\mathbf{p}, \overline{u}) = p_1 h_1^* + p_2 h_2^*.$$

Substituting and solving:

$$e(\mathbf{p}, \overline{u}) = \frac{p_1 p_2^2 + 4p_1^2 p_2}{64p_1^2 + 32p_1 p_2 + 4p_2^2} \overline{u}^2.$$

We have:

$$\frac{\partial e(\mathbf{p}, \overline{u})}{\partial p_1} = \frac{p_2^2}{64p_1^2 + 32p_1 p_2 + 4p_2^2} \overline{u}^2 = h_1^*$$
$$\frac{\partial e(\mathbf{p}, \overline{u})}{\partial p_2} = \frac{p_1^2}{16p_1^2 + 8p_1 p_2 + p_2^2} \overline{u}^2 = h_2^*.$$

(d) The indirect utility function is determined by:

$$v(\mathbf{p},w)=2\sqrt{x_1^*}+4\sqrt{x_2^*}.$$

Substituting:

$$\begin{aligned}v(\mathbf{p},w) &= 2\sqrt{\frac{p_2}{4p_1^2+p_1p_2}w}+4\sqrt{\frac{1}{p_2}w-\frac{1}{4p_1+p_2}w}\\ &= \frac{8p_1+2p_2}{\sqrt{4p_1^2p_2+p_1p_2^2}}\sqrt{w}.\end{aligned}$$

We have:

$$\begin{aligned}\frac{\partial v(\mathbf{p},w)}{\partial w} &= \frac{4p_1+p_2}{\sqrt{4p_1^2p_2+p_1p_2^2}}\frac{1}{\sqrt{w}}\\ \frac{\partial v(\mathbf{p},w)}{\partial p_1} &= -\frac{4p_1p_2^2+p_2^3}{\left(4p_1^2p_2+p_1p_2^2\right)\sqrt{4p_1^2p_2+p_1p_2^2}}\sqrt{w}\\ \frac{\partial v(\mathbf{p},w)}{\partial p_2} &= -\frac{4p_1^2p_2+16p_1^3}{\left(4p_1^2p_2+p_1p_2^2\right)\sqrt{4p_1^2p_2+p_1p_2^2}}\sqrt{w}\end{aligned}$$

Thus:

$$\begin{aligned}\frac{\frac{\partial v(\mathbf{p},w)}{\partial p_1}}{\frac{\partial v(\mathbf{p},w)}{\partial w}} &= -\frac{p_2}{4p_1^2+p_1p_2}w=-x_1^*\\ \frac{\frac{\partial v(\mathbf{p},w)}{\partial p_2}}{\frac{\partial v(\mathbf{p},w)}{\partial w}} &= -\frac{4p_1}{4p_1p_2+p_2^2}w=-x_2^*,\end{aligned}$$

which confirms Roy's identity, $\frac{\nabla_p v(\mathbf{p},w)}{\partial v(\mathbf{p},w)/\partial w}=-\mathbf{x}^*$.

5.C.2

To show: The profit function π is convex in the price vector $\mathbf{p}$.

Proof. Consider the price vectors $\mathbf{p}$ and $\mathbf{p}'$ with optimal production plans $\mathbf{y}(\mathbf{p})$ and $\mathbf{y}(\mathbf{p}')$, and associated profits $\pi(\mathbf{p})=\mathbf{p}^T\cdot\mathbf{y}(\mathbf{p})$ and $\pi(\mathbf{p}')=\mathbf{p}'^T\cdot\mathbf{y}(\mathbf{p}')$. The profit associated with a convex combination $\widetilde{\mathbf{p}}=\alpha\mathbf{p}+(1-\alpha)\mathbf{p}'$ of $\mathbf{p}$ and $\mathbf{p}'$ (where $\alpha\in[0,1]$) is:

$$\begin{aligned}\pi(\widetilde{\mathbf{p}}) &= \widetilde{\mathbf{p}}^T\cdot\mathbf{y}(\widetilde{\mathbf{p}})\\ &= \alpha\mathbf{p}^T\cdot\mathbf{y}(\widetilde{\mathbf{p}})+(1-\alpha)\mathbf{p}'^T\cdot\mathbf{y}(\widetilde{\mathbf{p}})\\ &\le \alpha\mathbf{p}^T\cdot\mathbf{y}(\mathbf{p})+(1-\alpha)\mathbf{p}'^T\cdot\mathbf{y}(\mathbf{p}')\\ &= \alpha\pi(\mathbf{p})+(1-\alpha)\pi(\mathbf{p}').\end{aligned}$$

The inequality follows from the optimality of $\mathbf{y}(\mathbf{p})$ at $\mathbf{p}$ and of $\mathbf{y}(\mathbf{p}')$ at $\mathbf{p}'$. The last equation implies that π is convex. ■

5.C.7

The intution behind the results is obvious: an increase in the output price induces more production, and since the production function is strictly concave, it is cost-minimizing to raise production by increasing all inputs. Secondly, an increase in a factor price leads to less production, and strict concavity of the production function implies that it is cost-minimizing to reduce all inputs.

The profit maximization problem

$$\max_{\mathbf{z}} pf(\mathbf{z}) - \mathbf{w}^T\mathbf{z}$$

has the first-order condition

$$p\nabla f(\mathbf{z}^*) = \mathbf{w}.$$

Differentiate with respect to p (recalling that $\mathbf{z}^*$ is a function of p):

$$\nabla f(\mathbf{z}^*) + pD_z\nabla f(\mathbf{z}^*)D_p\mathbf{z}^* = \mathbf{0}.$$

Denoting $D_z\nabla f(\mathbf{z}^*) = H$, the solution of the system of conditions is

$$D_p\mathbf{z}^* = -\frac{H^{-1}\nabla f(\mathbf{z}^*)}{p}.$$

To show that $D_p\mathbf{z}^* > \mathbf{0}$, we need $H^{-1}\nabla f(\mathbf{z}^*) < \mathbf{0}$. In the two-input case,

$$H^{-1} = \frac{\begin{bmatrix} \frac{\partial^2 f}{\partial z_2^2} & -\frac{\partial^2 f}{\partial z_1 \partial z_2} \\ -\frac{\partial^2 f}{\partial z_1 \partial z_2} & \frac{\partial^2 f}{\partial z_1^2} \end{bmatrix}}{\frac{\partial^2 f}{\partial z_1^2}\frac{\partial^2 f}{\partial z_2^2} - \left(\frac{\partial^2 f}{\partial z_1 \partial z_2}\right)^2}$$

Since f is strictly concave, $\frac{\partial^2 f}{\partial z_1^2} < 0$, $\frac{\partial^2 f}{\partial z_2^2} < 0$, and $|H| = \frac{\partial^2 f}{\partial z_1^2}\frac{\partial^2 f}{\partial z_2^2} - \left(\frac{\partial^2 f}{\partial z_1 \partial z_2}\right)^2 > 0$. Since $\nabla f(\mathbf{z}^*) > \mathbf{0}$ (f is increasing) and $\frac{\partial^2 f}{\partial z_1 \partial z_2} > 0$ (given), this implies

$$\frac{\partial^2 f}{\partial z_2^2} < \frac{\partial^2 f}{\partial z_1 \partial z_2}\frac{\frac{\partial f}{\partial z_2}}{\frac{\partial f}{\partial z_1}}$$

and

$$\frac{\partial^2 f}{\partial z_1^2} < \frac{\partial^2 f}{\partial z_1 \partial z_2} \frac{\frac{\partial f}{\partial z_1}}{\frac{\partial f}{\partial z_2}}.$$

It follows that

$$H^{-1}\nabla f(\mathbf{z}^*) = \frac{1}{|H|} \begin{bmatrix} \frac{\partial^2 f}{\partial z_2^2}\frac{\partial f}{\partial z_1} - \frac{\partial^2 f}{\partial z_1 \partial z_2}\frac{\partial f}{\partial z_2} \\ \frac{\partial^2 f}{\partial z_1^2}\frac{\partial f}{\partial z_2} - \frac{\partial^2 f}{\partial z_1 \partial z_2}\frac{\partial f}{\partial z_1} \end{bmatrix} < \mathbf{0},$$

and therefore $D_p \mathbf{z}^* > \mathbf{0}$. The general case involves more tedious notation, but the same argument applies.

Next, differentiate the first-order condition with respect to $\mathbf{w}$ (recalling that $\mathbf{z}^*$ is a function of $\mathbf{w}$):

$$\begin{aligned} pD_z \nabla f(\mathbf{z}^*) D_w \mathbf{z}^* &= \\ pHD_w \mathbf{z}^* &= \mathbf{1}. \end{aligned}$$

The solution of the system of conditions is

$$D_w \mathbf{z}^* = \frac{H^{-1}\mathbf{1}}{p}.$$

To show that $D_w \mathbf{z}^* < \mathbf{0}$, we need $H^{-1}\mathbf{1} < \mathbf{0}$. This amounts to negative row sums of the adjoint matrix of H. In the two-input case, we need:

$$\begin{bmatrix} \frac{\partial^2 f}{\partial z_2^2} - \frac{\partial^2 f}{\partial z_1 \partial z_2} \\ \frac{\partial^2 f}{\partial z_1^2} - \frac{\partial^2 f}{\partial z_1 \partial z_2} \end{bmatrix} < \mathbf{0}.$$

Since $\frac{\partial^2 f}{\partial z_1^2} < 0$ and $\frac{\partial^2 f}{\partial z_2^2} < 0$ by the concavity of f, and $\frac{\partial^2 f}{\partial z_1 \partial z_2} > 0$ by assumption, the condition holds, and we have $D_w \mathbf{z}^* < \mathbf{0}$.

5.C.9

(a)

The firm's problem is

$$\max_{z_1, z_2} p\sqrt{z_1 + z_2} - w_1 z_1 - w_2 z_2.$$

Since z_1 and z_2 are perfect substitutes in the production function, the firm uses only z_1 if $w_1 < w_2$, only z_2 if $w_1 > w_2$, and an arbitrary ratio of z_1 and z_2 if $w_1 = w_2$.

Case 1: $w_1 < w_2$. Then $z_2^* = 0$ and the problem reduces to $\max_{z_1} p\sqrt{z_1} - w_1 z_1$. First-order condition:

$$\frac{p}{2\sqrt{z_1^*}} = w_1.$$

Hence

$$z_1^* = \frac{p^2}{4w_1^2}.$$

Therefore:

$$\begin{aligned} y &= \sqrt{z_1^* + z_2^*} \\ &= \frac{p}{2w_1}. \end{aligned}$$

$$\begin{aligned} \pi &= p\sqrt{z_1^* + z_2^*} - w_1 z_1^* - w_2 z_2^* \\ &= \frac{p^2}{4w_1}. \end{aligned}$$

Case 2: $w_1 > w_2$. Then $z_1^* = 0$ and the problem reduces to $\max_{z_2} p\sqrt{z_2} - w_1 z_2$. Symmetrically to Case 1, we have:Hence

$$z_2^* = \frac{p^2}{4w_2^2}$$

and

$$\begin{aligned} y &= \frac{p}{2w_1} \\ \pi &= \frac{p^2}{4w_2}. \end{aligned}$$

Case 3: $w_1 = w_2$. Denote the amount of the generic input by $\widetilde{z}$ and the generic factor price by $\widetilde{w}$. Then the problem reduces to $\max_{\widetilde{z}} p\sqrt{2\widetilde{z}} - 2\widetilde{w}\widetilde{z}$. First-order condition:

$$\frac{p}{2\sqrt{2\widetilde{z}^*}} = \widetilde{w}$$

Hence

$$\widetilde{z}^* = \frac{p^2}{8\widetilde{w}^2},$$

where $z_1^* + z_2^* = 2\widetilde{z}^*$ and $\widetilde{w} = w_1 + w_2$. There are infinitely many solutions for the optimal input vector $\mathbf{z}^*$. However, profit is unique:

$$\begin{aligned} y &= \sqrt{2\widetilde{z}^*} \\ &= \frac{p}{2\widetilde{w}}. \end{aligned}$$

$$\begin{aligned} \pi &= p\sqrt{2\widetilde{z}^*} - 2\widetilde{w}\widetilde{z}^* \\ &= \frac{p^2}{4\widetilde{w}}. \end{aligned}$$

(b)

The firm's problem is

$$\max_{z_1, z_2} p\sqrt{\min\{z_1, z_2\}} - w_1 z_1 - w_2 z_2.$$

Since z_1 and z_2 are perfect complements, the firm optimizes by using z_1 and z_2 in equal amounts, i.e. $z_1^* = z_2^*$ such that $\min\{z_1^*, z_2^*\} = z_1^* = z_2^*$. Then the problem reduces to $\max_{z_1} p\sqrt{z_1} - (w_1 + w_2) z_1$. First-order condition:

$$\begin{aligned} \frac{p}{2\sqrt{z_1}} &= w_1 + w_2 \\ z_1^* &= z_2^* = \frac{p^2}{4(w_1 + w_2)^2}. \end{aligned}$$

Therefore:

$$\begin{aligned} y &= \sqrt{\min\{z_1^*, z_2^*\}} \\ &= \frac{p}{2(w_1 + w_2)}. \end{aligned}$$

$$\begin{aligned} \pi &= p\sqrt{z_1} - (w_1 + w_2) z_1 \\ &= \frac{p^2}{4(w_1 + w_2)}. \end{aligned}$$

(c)

Assume first that $\rho < 1$ and $\rho \neq 0$; these special cases are discussed later. The firm's problem is:

$$\max_{z_1, z_2} p\left(z_1^\rho + z_2^\rho\right)^{\frac{1}{\rho}} - w_1 z_1 - w_2 z_2.$$

Observe that the CES production function is homogeneous of degree 1 (i.e. it exhibits constant returns to scale):

$$((\mu z_1)^\rho + (\mu z_2)^\rho)^{\frac{1}{\rho}} = \mu\left(z_1^\rho +^\rho z_2^\rho\right)^{\frac{1}{\rho}}.$$

This implies that, given a constant price vector (p, w_1, w_2), the firm either produces nothing (hence demands $z_1^* = z_2^* = 0$) or produces an infinite quantity (hence demands $z_1^* = z_2^* = \infty$) or produces an arbitrary quantity, employing the inputs in a fixed proportion. Which case obtains depends on the relationship between p and the factor prices w_1 and w_2.

Taking first-order conditions with respect to z_1,

$$\begin{aligned} p\frac{\left(z_1^{*\rho} + z_2^{*\rho}\right)^{\frac{1-\rho}{\rho}}}{z_1^{*1-\rho}} &= w_1 \\ \frac{z_1^{*\rho} + z_2^{*\rho}}{z_1^{*\rho}} &= \left(\frac{w_1}{p}\right)^{\frac{\rho}{1-\rho}} \\ \frac{z_1^*}{z_2^*} &= \frac{1}{\left(\left(\frac{w_1}{p}\right)^{\frac{\rho}{1-\rho}} - 1\right)^{\frac{1}{\rho}}} \end{aligned}$$

and symmetrically with respect to z_2,

$$\frac{z_1^*}{z_2^*} = \left(\left(\frac{w_2}{p}\right)^{\frac{\rho}{1-\rho}} - 1\right)^{\frac{1}{\rho}}.$$

Solving for p in terms of w_1 and w_2:

$$p = \frac{w_1 w_2}{\left(w_1^{\frac{\rho}{1-\rho}} + w_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}}}.$$

If this condition holds, then any vector (z_1, z_2) such that $\frac{z_1^*}{z_2^*} = \left(\left(\frac{w_2}{p}\right)^{\frac{\rho}{1-\rho}} - 1\right)^{\frac{1}{\rho}}$ is optimal for the firm. Since the righthand term is constant, all linear scalings of an optimal factor combination are optimal, i.e. there are infinitely many solutions $\mathbf{z}^*$. Profit is, however, unique:

$$\begin{aligned} \pi &= p\left(z_1^{*\rho} + z_2^{*\rho}\right)^{\frac{1}{\rho}} - w_1 z_1^* - w_2 z_2 \\ &= p\left(\left(\left(\frac{w_2}{p}\right)^{\frac{\rho}{1-\rho}} - 1\right) z_2^{*\rho} + z_2^{*\rho}\right)^{\frac{1}{\rho}} - w_1\left(\left(\frac{w_2}{p}\right)^{\frac{\rho}{1-\rho}} - 1\right)^{\frac{1}{\rho}} z_2^* - w_2 z_2^* \\ &= p\left(\frac{w_2}{p}\right)^{\frac{1}{1-\rho}} z_2^* - w_1\left(\left(\frac{w_2}{p}\right)^{\frac{\rho}{1-\rho}} - 1\right)^{\frac{1}{\rho}} z_2^* - w_2 z_2^*. \end{aligned}$$

Substituting for p and simplifying,

$$\begin{aligned}
\pi &= \frac{w_1 w_2}{\left(w_1^{\frac{\rho}{1-\rho}} + w_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}}} \left(\frac{w_1^{\frac{\rho}{1-\rho}} + w_2^{\frac{\rho}{1-\rho}}}{w_1^{\frac{\rho}{1-\rho}}}\right)^{\frac{1}{\rho}} z_2^* - w_1 \left(\frac{w_2}{w_1}\right)^{\frac{1}{1-\rho}} z_2^* - w_2 z_2^* \\
&= \frac{w_1 w_2 \left(w_1^{\frac{\rho}{1-\rho}} + w_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}}{\left(w_1^{\frac{\rho}{1-\rho}} + w_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}} \left(w_1^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}} z_2^* - w_1 \left(\frac{w_2}{w_1}\right)^{\frac{1}{1-\rho}} z_2^* - w_2 z_2^* \\
&= w_1^{\frac{-\rho}{1-\rho}} w_2 \left(w_1^{\frac{\rho}{1-\rho}} + w_2^{\frac{\rho}{1-\rho}}\right) z_2^* - w_1 \left(\frac{w_2}{w_1}\right)^{\frac{1}{1-\rho}} z_2^* - w_2 z_2^* \\
&= \left(w_2 + w_1^{\frac{-\rho}{1-\rho}} w_2^{\frac{1}{1-\rho}}\right) z_2^* - \left(w_1^{\frac{-\rho}{1-\rho}} w_2^{\frac{1}{1-\rho}} + w_2\right) z_2^* \\
&= 0.
\end{aligned}$$

In the extreme cases, the firm either earns zero profit by producing nothing if

$$p < \frac{w_1 w_2}{\left(w_1^{\frac{\rho}{1-\rho}} + w_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}}}$$

or infinite profit by producing an infinite quantity if

$$p > \frac{w_1 w_2}{\left(w_1^{\frac{\rho}{1-\rho}} + w_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}}}.$$

Now let $\rho = 1$. In this case the production function is linear, and the threshold condition is easily seen to be $p = \min\{w_1, w_2\}$. If this holds, then the firm uses the cheaper input to produce an arbitrary quantity (or if $w_1 = w_2$, the firm employs the factors in an arbitrary ratio). Profit is $(p - \widetilde{w})\,\widetilde{z}^* = 0$. If $p > \min\{w_1, w_2\}$, the firm uses the cheaper input to produce an infinite quantity (or if $w_1 = w_2$, the firm employs the factors in an arbitrary ratio); profit is infinite. Finally, if $p < \min\{w_1, w_2\}$, then the firm produces nothing and earns no profit.

Lastly, if $\rho = 0$,the CES production function is undefined, but it can be shown that as ρ approaches zero, the CES function is the function $\sqrt{z_1 z_2}$ in the limit. Taking logs,

$$\begin{aligned}
\ln f &= \ln\left(z_1^\rho + z_2^\rho\right)^{\frac{1}{\rho}} \\
&= \frac{\ln\left(z_1^\rho + z_2^\rho\right)}{\rho},
\end{aligned}$$

we can apply L'Hôpital's rule to get

$$\begin{aligned}
\lim_{\rho\to 0}\ln f &= \lim_{\rho\to 0}\frac{\frac{\partial}{\partial\rho}\ln\left(z_1^\rho+z_2^\rho\right)}{\frac{\partial}{\partial\rho}\rho}\\
&= \lim_{\rho\to 0}\frac{\frac{z_1^\rho\ln z_1+z_2^\rho\ln z_2}{z_1^\rho+z_2^\rho}}{1}\\
&= \frac{\ln z_1+\ln z_2}{2}\\
&= \frac{1}{2}\ln\left(z_1z_2\right).
\end{aligned}$$

Raising both sides to base e,

$$f=\sqrt{z_1z_2}.$$

In this case, the firm's maximization problem is:

$$\max_{z_1,z_2} p\sqrt{z_1z_2}-w_1z_1-w_2z_2.$$

Notice that the production function continues to be homogeneous of degree 1. The first-order conditions are:

$$\begin{aligned}
\frac{pz_2^*}{2\sqrt{z_1^*z_2^*}} &= w_1\\
\frac{pz_1^*}{2\sqrt{z_1^*z_2^*}} &= w_2.
\end{aligned}$$

Solving these separately for the factor demand ratio:

$$\begin{aligned}
\frac{z_1^*}{z_2^*} &= \frac{1}{4\left(\frac{w_1}{p}\right)^2}\\
\frac{z_1^*}{z_2^*} &= 4\left(\frac{w_2}{p}\right)^2.
\end{aligned}$$

Then the "stable" condition is

$$\begin{aligned}
4\left(\frac{w_1}{p}\right)^2 4\left(\frac{w_2}{p}\right)^2 &= 1\\
p &= 2\sqrt{w_1w_2}.
\end{aligned}$$

If this holds, the firm produces an arbitrary quantity of output, using the factors in the ratio $\frac{z_1^*}{z_2^*}=4\left(\frac{w_2}{p}\right)^2$. Since every linear scaling of an optimal demand vector is also optimal, there are infinitely many solutions. Profit is zero:

$$\begin{aligned}
\pi &= p\sqrt{z_1^* z_2^*} - w_1 z_1^* - w_2 z_2^* \\
&= p\sqrt{4\left(\frac{w_2}{p}\right)^2 z_2^{*2}} - w_1 4\left(\frac{w_2}{p}\right)^2 z_2^* - w_2 z_2^* \\
&= 2w_2 z_2^* - w_1 4\left(\frac{w_2}{2\sqrt{w_1 w_2}}\right)^2 z_2^* - w_2 z_2^* \\
&= 2w_2 z_2^* - w_2 z_2^* - w_2 z_2^* \\
&= 0.
\end{aligned}$$

The remaining cases are $p > 2\sqrt{w_1 w_2}$, in which case the firm produces infinite output and earns infinite profit, and $p < 2\sqrt{w_1 w_2}$, which gives zero production and zero profit.

6.B.2

If U representing $\succsim$ on $\mathfrak{L}$ has the expected utility form, then $\succsim$ satisfies the independence axiom.

Proof. A preference relation $\succsim$ on the set $\mathfrak{L}$ of simple lotteries satisfies the independence axiom if and only if

$$L \succsim L' \Longleftrightarrow \alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''$$

for all $L, L', L'' \in \mathfrak{L}$ and all $\alpha \in (0,1)$. Since this is an "if and only if" statement, we have to show that if U has the expected utility form (equivalently, U is linear), then (if)

$$\alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L'' \Rightarrow L \succsim L'.$$

and (only if)

$$L \succsim L' \Rightarrow \alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''.$$

(If) Suppose $[\alpha L + (1-\alpha) L''] \succsim [\alpha L' + (1-\alpha) L'']$. Since U represents $\succsim$, it follows that

$$U(\alpha L + (1-\alpha) L'') \geq U(\alpha L' + (1-\alpha) L'').$$

Since U is linear, this implies

$$\begin{aligned}
\alpha U(L) + (1-\alpha) U(L'') &\geq \alpha U(L') + (1-\alpha) U(L'') \\
U(L) &\geq U(L').
\end{aligned}$$

But then $L \succsim L'$ as required.

(Only if) Suppose $L \succsim L'$. Then $U(L) \geq U(L')$. Multiplying by α and adding $(1-\alpha)U(L'')$ gives

$$\alpha U(L) + (1-\alpha)U(L'') \geq \alpha U(L') + (1-\alpha)U(L''),$$

By linearity of U,

$$U(\alpha L + (1-\alpha)L'') \geq U(\alpha L' + (1-\alpha)L''),$$

which implies

$$\alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''.$$

■

6.B.4

(a)

It is given that $A \succsim D$, hence if U represents $\succsim$, we must have $U(A) \geq U(D)$. Since utility functions are unique up to monotone transformations, we can assign arbitrary values to $U(A)$ and $U(D)$ such that $U(A) \geq U(D)$. Say $U(A) = 1$ and $U(D) = 0$. Next, it is given that

$$B \sim pA + (1-p)D$$

and

$$C \sim qB + (1-q)D.$$

These imply (using the linearity of U):

$$\begin{aligned} U(B) &= pU(A) + (1-p)U(D) \\ U(C) &= qU(B) + (1-q)U(D). \end{aligned}$$

Substituting $U(A) = 1$ and $U(D) = 0$,

$$\begin{aligned} U(B) &= p \\ U(C) &= pq. \end{aligned}$$

A linear utility function that represents $\succsim$ is $U : \mathfrak{L} \longrightarrow \mathbb{R}$ such that

$$\begin{aligned} U(A) &= 1 \\ U(B) &= p \\ U(C) &= pq \\ U(D) &= 0. \end{aligned}$$

(b)

The event that flooding occurs contains outcomes C and D. The event that no flooding occurs contains outcomes A and B. The conditional probability of an evacuation in the event of flooding is:

$$\frac{\Pr(C)}{\Pr(C \cup D)} = \frac{\Pr(C)}{\Pr(C) + \Pr(D)}$$

(since $C \cap D = \varnothing$). This implies:

$$\Pr(D) = \frac{1 - \frac{\Pr(C)}{\Pr(C \cup D)}}{\frac{\Pr(C)}{\Pr(C \cup D)}} \Pr(C).$$

We know the probability of flooding:

$$\Pr(C \cup D) = \Pr(C) + \Pr(D) = 0.01.$$

Therefore,

$$\begin{aligned} \Pr(D) &= \frac{1 - \frac{\Pr(C)}{\Pr(C \cup D)}}{\frac{\Pr(C)}{\Pr(C \cup D)}} (0.01 - \Pr(D)) \\ &= 0.01 \frac{\frac{\Pr(C)}{\Pr(C \cup D)} - \left(\frac{\Pr(C)}{\Pr(C \cup D)}\right)^2}{\frac{\Pr(C)}{\Pr(C \cup D)}}. \end{aligned}$$

If criterion 1 is used, $\frac{\Pr(C)}{\Pr(C \cup D)} = 0.9$. If criterion 2 is used, $\frac{\Pr(C)}{\Pr(C \cup D)} = 0.95$. Hence

$$\begin{aligned} \Pr(D)^1 &= 0.001 \\ \Pr(D)^2 &= 0.0005 \end{aligned}$$

and, since $\Pr(C) = \Pr(D) \frac{\frac{\Pr(C)}{\Pr(C \cup D)}}{1 - \frac{\Pr(C)}{\Pr(C \cup D)}}$,

$$\begin{aligned} \Pr(C)^1 &= 0.009 \\ \Pr(C)^2 &= 0.0095. \end{aligned}$$

The conditional probability of an evacuation in the event of no flooding is:

$$\frac{\Pr(B)}{\Pr(A \cup B)} = \frac{\Pr(B)}{\Pr(A) + \Pr(B)}.$$

(since $A \cap B = \varnothing$). This implies:

$$\Pr(B) = \frac{\frac{\Pr(B)}{\Pr(A\cup B)}}{1 - \frac{\Pr(B)}{\Pr(A\cup B)}} \Pr(A).$$

By an elementary property of probability,

$$\Pr(A) + \Pr(B) + \Pr(C) + \Pr(D) = 1.$$

Substituting for $\Pr(B)$, we have

$$\Pr(A) = (1 - \Pr(C) - \Pr(D)) \left(1 - \frac{\Pr(B)}{\Pr(A \cup B)}\right).$$

If criterion 1 is used, $\frac{\Pr(B)}{\Pr(A\cup B)} = 0.1$. If criterion 2 is used, $\frac{\Pr(B)}{\Pr(A\cup B)} = 0.05$. Thus:

$$\begin{aligned} \Pr(A)^1 &= 0.891 \\ \Pr(A)^2 &= 0.9405 \end{aligned}$$

and

$$\begin{aligned} \Pr(B)^1 &= 0.099 \\ \Pr(B)^2 &= 0.0495. \end{aligned}$$

In summary, the probability distributions are, for criterion 1,

$$\begin{aligned} \Pr(A)^1 &= 0.891 \\ \Pr(B)^1 &= 0.099 \\ \Pr(C)^1 &= 0.009 \\ \Pr(D)^1 &= 0.001. \end{aligned}$$

For criterion 2,

$$\begin{aligned} \Pr(A)^2 &= 0.9405 \\ \Pr(B)^2 &= 0.0495 \\ \Pr(C)^2 &= 0.0095 \\ \Pr(D)^2 &= 0.0005. \end{aligned}$$

Under the two criteria the agency has, respectively, utility

$$\begin{aligned} U^1(\cdot) &= \Pr(A)^1 U(A) + \Pr(B)^1 U(B) + \Pr(C)^1 U(C) + \Pr(D)^1 U(D) \\ &= 0.891 + 0.099p + 0.009pq. \end{aligned}$$

and

$$\begin{aligned} U^2(\cdot) &= \Pr(A)^2 U(A) + \Pr(B)^2 U(B) + \Pr(C)^2 U(C) + \Pr(D)^2 U(D) \\ &= 0.9405 + 0.0495p + 0.0095pq. \end{aligned}$$

Since $p < 1$,

$$U^1(\cdot) - U^2(\cdot) = -0.0495 + 0.0495p - 0.0005pq < 0,$$

so criterion 2 is preferable.

6.C.1

Suppose, contrary to the claim, that when $q > \pi$ the individual insures completely: $\alpha \geq D$. Then $\alpha > 0$ and the agent's problem

$$\max_{\alpha \geq 0} [\pi u(w - q\alpha - (D - \alpha)) + (1 - \pi) u(w - q\alpha)]$$

has first order condition

$$(1 - q)\pi \frac{\partial}{\partial \alpha} u(w - q\alpha - (D - \alpha)) = q(1 - \pi) \frac{\partial}{\partial \alpha} u(w - q\alpha)$$

(sufficient under strict concavity).

Moreover, $\alpha \geq D$ and strict concavity of u imply

$$\frac{\partial}{\partial \alpha} u(w - q\alpha - (D - \alpha)) \leq \frac{\partial}{\partial \alpha} u(w - q\alpha),$$

therefore

$$\begin{aligned} (1 - q)\pi &\geq q(1 - \pi) \\ \pi &\geq q, \end{aligned}$$

a contradiction.

6.C.9

(a)

The agent's problem without uncertainty

$$\max_x [u(w - x) + v(x)]$$

has first order condition

$$\frac{\partial}{\partial x} u(w - x_0) = \frac{\partial}{\partial x} v(x_0).$$

The agent's problem with uncertainty

$$\max_x \left[u\left(w-x\right)+\mathrm{E}\left[v\left(x+y\right)\right]\right]$$

has first order condition

$$\frac{\partial}{\partial x}u\left(w-x^*\right)=\mathrm{E}\left[\frac{\partial}{\partial x}v\left(x^*+y\right)\right].$$

Suppose, contrary to the claim, that $x^* \leq x_0$. Then, by concavity of v,

$$\mathrm{E}\left[\frac{\partial}{\partial x}v\left(x^*+y\right)\right]\geq \mathrm{E}\left[\frac{\partial}{\partial x}v\left(x_0+y\right)\right],$$

and

$$\mathrm{E}\left[\frac{\partial}{\partial x}v\left(x_0+y\right)\right]>\frac{\partial}{\partial x}v\left(x_0\right)$$

implies

$$\mathrm{E}\left[\frac{\partial}{\partial x}v\left(x^*+y\right)\right]>\frac{\partial}{\partial x}v\left(x_0\right).$$

Using the first order conditions, this leads to

$$\frac{\partial}{\partial x}u\left(w-x^*\right)>\frac{\partial}{\partial x}u\left(w-x_0\right),$$

where concavity of u implies $x^* > x_0$, a contradiction.

(b)

Let

$$-\frac{\frac{\partial^3}{\partial x_1^3}v_1\left(x_1\right)}{\frac{\partial^2}{\partial x_1^2}v_1\left(x_1\right)}\leq-\frac{\frac{\partial^3}{\partial x_2^3}v_2\left(x_2\right)}{\frac{\partial^2}{\partial x_2^2}v_2\left(x_2\right)},$$

or equivalently,

$$\begin{aligned}-\frac{\frac{\partial^2}{\partial x_1^2}\frac{\partial}{\partial x_1}v_1\left(x_1\right)}{\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1}v_1\left(x_1\right)} &\leq -\frac{\frac{\partial^2}{\partial x_2^2}\frac{\partial}{\partial x_2}v_2\left(x_2\right)}{\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2}v_2\left(x_2\right)}\\ r_A\left(\frac{\partial}{\partial x_1}v_1\left(x_1\right)\right) &\leq r_A\left(\frac{\partial}{\partial x_2}v_2\left(x_2\right)\right),\end{aligned}$$

where $r_A(\cdot)$ is the Arrow-Pratt coefficient of absolute risk aversion. Therefore

$$c\left(\frac{\partial}{\partial x_1}v_1(x_1)\right) \geq c\left(\frac{\partial}{\partial x_2}v_2(x_2)\right)$$

(by proposition 6.C.2(iii)), where $c(x+y)$ denotes the certainty equivalent of lottery $x+y$. We will derive the certainty equivalents of the risky prospects $x_1^* + y$ and $x_2^* + y$ (i.e. the optimal choices under v_1 and v_2) and use

$$\mathrm{E}\left[\frac{\partial}{\partial x_1}v_1(x_0+y)\right] > \frac{\partial}{\partial x_1}v_1(x_0)$$

to show that, given the relationship of the certainty equivalents,

$$\mathrm{E}\left[\frac{\partial}{\partial x_2}v_2(x_0+y)\right] > \frac{\partial}{\partial x_2}v_2(x_0).$$

The respective first-order conditions for the problems in v_1 and v_2 are:

$$\frac{\partial}{\partial x_1}u(w-x_1^*) = \mathrm{E}\left[\frac{\partial}{\partial x_1}v_1(x_1^*+y)\right]$$

and

$$\frac{\partial}{\partial x_2}u(w-x_2^*) = \mathrm{E}\left[\frac{\partial}{\partial x_2}v_2(x_2^*+y)\right].$$

Let

$$\mathrm{E}\left[\frac{\partial}{\partial x_1}v_1(x_0+y)\right] > \frac{\partial}{\partial x_1}v_1(x_0)$$

and denote by $\widetilde{x}_1$ the level of saving such that

$$\mathrm{E}\left[\frac{\partial}{\partial x_1}v_1(x_0+y)\right] = \frac{\partial}{\partial x_1}v_1(\widetilde{x}_1).$$

If we interpret $\frac{\partial}{\partial x_1}v_1$ as a Bernoulli utility function, then $\widetilde{x}_1$ is the certainty equivalent of $x_0 + y$ under v_1. Concavity of v implies that $\widetilde{x}_1 = x_0 - z$ for some $z > 0$. Similarly, denote by $\widetilde{x}_2$ the level of saving such that

$$\mathrm{E}\left[\frac{\partial}{\partial x_2}v_2(x_0+y)\right] = \frac{\partial}{\partial x_2}v_2(\widetilde{x}_2).$$

Again, $\widetilde{x}_2$ is the certainty equivalent of $x_0 + y$ under v_2. From our initial argument, we have $\widetilde{x}_1 \geq \widetilde{x}_2$, which implies

$$\begin{aligned} x_0 - \widetilde{x}_2 &\geq z > 0 \\ \widetilde{x}_2 &\leq x_0 - z < x_0. \end{aligned}$$

But then (by concavity of v_2)

$$\mathrm{E}\left[\frac{\partial}{\partial x_2} v_2\left(x_0+y\right)\right] > \frac{\partial}{\partial x_2} v_2\left(\widetilde{x}_0\right).$$

As demonstrated in part (a),

$$\mathrm{E}\left[\frac{\partial}{\partial x} v\left(x_0+y\right)\right] > \frac{\partial}{\partial x} v\left(x_0\right)$$

implies

$$x^* > x_0.$$

Thus we have shown that, if the coefficient of absolute prudence is weakly smaller for utility function v_1 than for v_2, then $x_1^* > x_0$ only if $x_2^* > x_0$. More succinctly, $x_2^* \geq x_1^*$ (the more "prudent" agent will save more).

(c)

Strict concavity must be assumed for parts (c) and (d). If $\frac{\partial^3}{\partial x^3} v(x) = \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial x} v(x) > 0$ and (by strict concavity of v) $\frac{\partial^2}{\partial x^2} v(x) = \frac{\partial}{\partial x}\frac{\partial}{\partial x} v(x) < 0$, the function $\frac{\partial}{\partial x} v(x)$ is strictly convex. Applying Jensen's inequality,

$$\mathrm{E}\left[\frac{\partial}{\partial x} v\left(x+y\right)\right] > \frac{\partial}{\partial x} v\left(\mathrm{E}\left[x+y\right]\right).$$

Using $\mathrm{E}[y] = 0$, hence $\mathrm{E}[x+y] = \mathrm{E}[x] = x$,

$$\mathrm{E}\left[\frac{\partial}{\partial x} v\left(x+y\right)\right] > \frac{\partial}{\partial x} v\left(x\right).$$

(d)

From

$$\frac{\partial r_A\left(v\left(W\right)\right)}{\partial W} = -\frac{\partial}{\partial W}\frac{\frac{\partial^2 v(W)}{\partial^2 W}}{\frac{\partial v(W)}{\partial W}} = -\frac{\frac{\partial^3 v(W)}{\partial^3 W}\frac{\partial v(W)}{\partial W} - \left(\frac{\partial^2 v(W)}{\partial^2 W}\right)^2}{\left(\frac{\partial v(W)}{\partial W}\right)^2} < 0,$$

we have

$$\begin{aligned}
\frac{\partial^3 v\left(W\right)}{\partial^3 W}\frac{\partial v\left(W\right)}{\partial W} &> \left(\frac{\partial^2 v\left(W\right)}{\partial^2 W}\right)^2 \\
\frac{\frac{\partial^3 v(W)}{\partial^3 W}}{\frac{\partial^2 v(W)}{\partial^2 W}}\frac{\partial v\left(W\right)}{\partial W} &< \frac{\partial^2 v\left(W\right)}{\partial^2 W} \\
-\frac{\frac{\partial^3 v(W)}{\partial^3 W}}{\frac{\partial^2 v(W)}{\partial^2 W}} &> -\frac{\frac{\partial^2 v(W)}{\partial^2 W}}{\frac{\partial v(W)}{\partial W}}.
\end{aligned}$$

Since $\frac{\partial^2 v(W)}{\partial^2 W} < 0$ and $\frac{\partial v(W)}{\partial W} > 0$ (by strict concavity), $\frac{\partial^3 v(W)}{\partial^3 W} > 0$.

6.C.16

(a)

If the individual owns the lottery, the expected utility in case of no action is

$$pu(w+G) + (1-p)u(w+B),$$

and the expected utility upon selling the lottery at price q is

$$u(w+q).$$

Selling is rational if

$$u(w+q) \geq pu(w+G) + (1-p)u(w+B)$$

(equality at the minimum acceptable price). Provided that u is bijective, there exists an inverse mapping u^{-1} and we can solve explicitly for q,

$$q = u^{-1}(pu(w+G) + (1-p)u(w+B)) - w.$$

(b)

If the individual does not own the lottery, the expected utility in case of no action is

$$u(w),$$

and the expected utility upon buying the lottery at price r is

$$pu(w-r+G) + (1-p)u(w-r+B).$$

Buying is rational if

$$pu(w-r+G) + (1-p)u(w-r+B) \geq u(w)$$

(equality at the maximum acceptable price). For general functional forms of u, this expression has no explicit solution for r. All that can be said is that r is implicitly determined by

$$pu(w-r+G) + (1-p)u(w-r+B) = u(w).$$

(c)

In general, q (the minimum sale price of the lottery) and r (the maximum purchase price of the lottery) are not equal. The reason is a wealth effect. If the lottery is initially owned, the sale adds to the payoff that is not state contingent; in the opposite case, the purchase subtracts from fixed wealth. In case of risk aversion that is decreasing in wealth, the lottery has more value to a seller (since the buyer's wealth is reduced by paying the fixed price, causing him to discount the lottery). On the other hand, if risk aversion is increasing in wealth, the lottery becomes more valuable to a buyer. A sufficient condition for the (minimum) selling and (maximum) buying price to be the same is therefore that u has a constant absolute risk aversion form, i.e. it belongs to the CARA class.

To see this, observe that at the minimum sale / maximum purchase prices, the expected utility of owning the lottery equals the utility of the certainty equivalent. Denote the certainty equivalent of the lottery in part (a) as c_a and the certainty equivalent of the lottery in part (b) as c_b, hence $c_a = w + q$ and $c_b = w$. Now, c_a (the seller's certainty equivalent) reflects the addition of risk L to fixed wealth w, whereas c_b (the buyer's certainty equivalent) reflects the addition of the same risk to fixed wealth $w - r$.

By an analogous argument to that in proposition 6.C.3(iii), $w - c_a$ and $w - r - c_b$ are constant in w and $w - r$ (respectively) if (and only if) u has the CARA property. We can also conclude that

$$w - c_a = w - r - c_b.$$

(If $w - c_a = k$ and $w - r - c_b = l$, then scaling fixed wealth from w to $w - r$ and k's invariance to scaling imply $w - r - c_b = k$, hence $k = l$.) Substituting for c_a and c_b, we have

$$q = r.$$

(d)

If $u = \sqrt{x}$ and $G = 10$, $B = 5$, $w = 10$,

$$q = \left(p\sqrt{20} + (1-p)\sqrt{15}\right)^2 - 10,$$

and r is determined by

$$p\sqrt{20 - r} + (1-p)\sqrt{15 - r} = \sqrt{10}.$$

This expression can be transformed into a quadratic

$$\left(\frac{2p-1}{p^2}r-20+\frac{10}{p^2}+\left(\frac{1-p}{p}\right)^2 15\right)^2=\left(2\frac{\sqrt{10}}{p}\frac{1-p}{p}\right)^2(15-r)$$

and solved for a function in p by straightforward but tedious calculations. (From the solution to the quadratic formula, eliminate the negative price.)

6.C.20

The certainty equivalent $c(\varepsilon)$ is defined by

$$u(c(\varepsilon))=\frac{1}{2}u(x+\varepsilon)+\frac{1}{2}u(x-\varepsilon).$$

Differentiating with respect to ε:

$$\begin{aligned} u'(c(\varepsilon))c'(\varepsilon) &= \frac{1}{2}u'(x+\varepsilon)-\frac{1}{2}u'(x-\varepsilon) \\ u''(c(\varepsilon))c'(\varepsilon)+u'(c(\varepsilon))c''(\varepsilon) &= \frac{1}{2}u''(x+\varepsilon)+\frac{1}{2}u''(x-\varepsilon). \end{aligned}$$

(where f' is shorthand for $\frac{\partial f}{\partial \varepsilon}$). As $\varepsilon \longrightarrow 0$, this converges to

$$u''(c(0))c'(0)+u'(c(0))c''(0)=u''(x)$$

in the limit.

Solve the first derivative for $c'(\varepsilon)$ and evaluate at $\varepsilon=0$:

$$\begin{aligned} c'(\varepsilon) &= \frac{u'(x+\varepsilon)-u'(x-\varepsilon)}{2u'(c(\varepsilon))} \\ \lim_{\varepsilon\longrightarrow 0} c'(\varepsilon) &= 0. \end{aligned}$$

Moreover, observe that

$$\lim_{\varepsilon\longrightarrow 0} c(\varepsilon)=x,$$

since there is no uncertainty at $\varepsilon=0$.

Substituting for $c'(0)$ and $c(0)$ in the second derivative, we have

$$u'(x)c''(0)=u''(x),$$

i.e.

$$c''(0)=\frac{u''(x)}{u'(x)}=-r_A(x).$$

6.E.1

The expected regret function is

$$\begin{aligned} R(x,x') &= \sum_{s=1}^{S} \pi_s h\left(\max\left[0, x'_s - x_s\right]\right) \\ &= \frac{1}{3}\sum_{s=1}^{S} \sqrt{\max\left[0, x'_s - x_s\right]}. \end{aligned}$$

Define $\succsim$ by $x \succsim x'$ if and only if $R(x,x') \leq R(x',x)$.

For $x = (0,-2,1)$, $x' = (0,2,-2)$, and $x'' = (2,-3,-1)$, we can show: $x' \succsim x$ since

$$R(x,x') = \frac{2}{3} \geq \frac{\sqrt{3}}{3} = R(x',x),$$

$x \succsim x''$ since

$$R(x,x'') = \frac{\sqrt{2}}{3} \leq \frac{\sqrt{2}+1}{3} = R(x'',x),$$

$x'' \succsim x'$ since

$$R(x',x'') = \frac{\sqrt{2}+1}{3} = \frac{5+2\sqrt{2}}{9} \geq \frac{5}{9} = \frac{\sqrt{5}}{3} = R(x'',x').$$

But transitivity requires that $x' \succsim x$ and $x \succsim x''$ only if $x' \succsim x''$, so $x'' \succsim x'$ violates transitivity.

7.E.1

(a)

Player 1's pure strategies specify: 1's move at the first information set (L, M, or R), 1's move at the third information set (x or y), given that M was played initially, and 1's move at the third information set (x or y), given that R was played initially. Listing actions in that order, the pure strategies are:

$$A_1 = \left\{ \begin{matrix} L,x,x & L,x,y & L,y,x & L,y,y \\ M,x,x & M,x,y & M,y,x & M,y,y \\ R,x,x & R,x,y & R,y,x & R,y,y \end{matrix} \right\}.$$

Player 1's strategies are the mixed profiles $\sigma_1 \in \times_{A_1}[0,1]$ such that $\sum_{k\in A_1} \sigma_1^k = 1$.

Player 2's pure strategies are:

$$A_2 = \{l, r\}.$$

Player 2's strategies are the mixed profiles $\sigma_2 \in [0,1] \times [0,1]$ such that $\sigma_2^l + \sigma_2^r = 1$.

(b)

Suppose player 2 adopts mixed strategy $\sigma_2 \in \{\pi\} \times \{1-\pi\}$, i.e. player 2 assigns some frequency π to action l and frequency $(1-\pi)$ to action r. Let player 1's behavioral strategy be as follows. At the first information set, play L with probability u_1, M with probability u_2, and R with probability $1-u_1-u_2$. At the information set reached by playing M initially, play x with probability v and y with probability $(1-v)$. At the information set reached by playing R initially, play x with probability w and y with probability $(1-w)$. The set of player 1's behavioral strategies consists of triple lists of probabilities (one list for each information set):

$$B_1 = \{(u_1, u_2, 1-u_1-u_2), (v, 1-v), (w, 1-w) : u_1, u_2, v, w \in [0,1]\}.$$

Given a behavioral strategy for player 1, the probability that the game reaches a given terminal node is:

$$\begin{aligned}
\Pr(T_0 \mid s_1 \in B_1) &= u_1 \\
\Pr(T_1 \mid s_1 \in B_1) &= u_2 \pi v \\
\Pr(T_2 \mid s_1 \in B_1) &= u_2 \pi (1-v) \\
\Pr(T_3 \mid s_1 \in B_1) &= u_2 (1-\pi) v \\
\Pr(T_4 \mid s_1 \in B_1) &= u_2 (1-\pi)(1-v) \\
\Pr(T_5 \mid s_1 \in B_1) &= (1-u_1-u_2) \pi w \\
\Pr(T_6 \mid s_1 \in B_1) &= (1-u_1-u_2) \pi (1-w) \\
\Pr(T_7 \mid s_1 \in B_1) &= (1-u_1-u_2)(1-\pi) w \\
\Pr(T_8 \mid s_1 \in B_1) &= (1-u_1-u_2)(1-\pi)(1-w).
\end{aligned}$$

If player 1 instead plays a mixed strategy, the probability distribution over terminal nodes is:

$$
\begin{aligned}
\Pr\left(T_0 \mid s_1 \in A_1\right) &= \sigma_1^{Lxx} + \sigma_1^{Lxy} + \sigma_1^{Lyx} + \sigma_1^{Lyy} \\
\Pr\left(T_1 \mid s_1 \in A_1\right) &= \left(\sigma_1^{Mxx} + \sigma_1^{Mxy}\right)\pi \\
\Pr\left(T_2 \mid s_1 \in A_1\right) &= \left(\sigma_1^{Myx} + \sigma_1^{Myy}\right)\pi \\
\Pr\left(T_3 \mid s_1 \in A_1\right) &= \left(\sigma_1^{Mxx} + \sigma_1^{Mxy}\right)(1-\pi) \\
\Pr\left(T_4 \mid s_1 \in A_1\right) &= \left(\sigma_1^{Myx} + \sigma_1^{Myy}\right)(1-\pi) \\
\Pr\left(T_5 \mid s_1 \in A_1\right) &= \left(\sigma_1^{Rxx} + \sigma_1^{Ryx}\right)\pi \\
\Pr\left(T_6 \mid s_1 \in A_1\right) &= \left(\sigma_1^{Rxy} + \sigma_1^{Ryy}\right)\pi \\
\Pr\left(T_7 \mid s_1 \in A_1\right) &= \left(\sigma_1^{Rxx} + \sigma_1^{Ryx}\right)(1-\pi) \\
\Pr\left(T_8 \mid s_1 \in A_1\right) &= \left(\sigma_1^{Rxy} + \sigma_1^{Ryy}\right)(1-\pi).
\end{aligned}
$$

Setting

$$
\Pr\left(T \mid s_1 \in A_1\right) = \Pr\left(T \mid s_1 \in B_1\right)
$$

for all T, we obtain conditions on the mixed strategy profile for player 1 that leads to the same distribution of realizations as a given behavioral strategy:

$$
\begin{aligned}
\sigma_1^{Lxx} + \sigma_1^{Lxy} + \sigma_1^{Lyx} + \sigma_1^{Lyy} &= u_1 \\
\left(\sigma_1^{Mxx} + \sigma_1^{Mxy}\right)\pi &= u_2 \pi v \\
\left(\sigma_1^{Myx} + \sigma_1^{Myy}\right)\pi &= u_2 \pi (1-v) \\
\left(\sigma_1^{Mxx} + \sigma_1^{Mxy}\right)(1-\pi) &= u_2 (1-\pi) v \\
\left(\sigma_1^{Myx} + \sigma_1^{Myy}\right)(1-\pi) &= u_2 (1-\pi)(1-v) \\
\left(\sigma_1^{Rxx} + \sigma_1^{Ryx}\right)\pi &= (1-u_1-u_2)\pi w \\
\left(\sigma_1^{Rxy} + \sigma_1^{Ryy}\right)\pi &= (1-u_1-u_2)\pi (1-w) \\
\left(\sigma_1^{Rxx} + \sigma_1^{Ryx}\right)(1-\pi) &= (1-u_1-u_2)(1-\pi) w \\
\left(\sigma_1^{Rxy} + \sigma_1^{Ryy}\right)(1-\pi) &= (1-u_1-u_2)(1-\pi)(1-w).
\end{aligned}
$$

Simplifying and eliminating redundant conditions, we end up with

$$\begin{aligned}
\sigma_1^{Lxx} + \sigma_1^{Lxy} + \sigma_1^{Lyx} + \sigma_1^{Lyy} &= u_1 \\
\sigma_1^{Mxx} + \sigma_1^{Mxy} &= u_2 v \\
\sigma_1^{Myx} + \sigma_1^{Myy} &= u_2 (1 - v) \\
\sigma_1^{Rxx} + \sigma_1^{Ryx} &= (1 - u_1 - u_2) w.
\end{aligned}$$

(The reduction to five conditions is straightforward; a bit of algebra shows that one more can be eliminated since they are linearly dependent with $\sum_{k \in A_1} \sigma_1^k = 1$.) Any mixed strategy that satisfies this set of equations is realization-equivalent to the behavioral strategy. Clearly, the set of solutions is nonempty (in fact, it is infinite), hence there exist realization-equivalent mixed strategies for every behavioral strategy.

(c)

Suppose player 1 adopts mixed strategy $s_1 = \sigma_1 = \left(\sigma_1^{Lxx} \cdots \sigma_1^{Ryy}\right)$. A realization-equivalent behavioral strategy $s_1 = ((u_1, u_2, 1 - u_1 - u_2), (v, 1 - v), (w, 1 - w))$ must satisfy the conditions just derived in part (b). Solve the set of equations for u_1, u_2, v, and w:

$$\begin{aligned}
u_1 &= \sigma_1^{Lxx} + \sigma_1^{Lxy} + \sigma_1^{Lyx} + \sigma_1^{Lyy} \\
u_2 &= \sigma_1^{Mxx} + \sigma_1^{Mxy} + \sigma_1^{Myx} + \sigma_1^{Myy} \\
v &= \frac{\sigma_1^{Mxx} + \sigma_1^{Mxy}}{\sigma_1^{Mxx} + \sigma_1^{Mxy} + \sigma_1^{Myx} + \sigma_1^{Myy}} \\
w &= \frac{\sigma_1^{Rxx} + \sigma_1^{Ryx}}{\sigma_1^{Rxx} + \sigma_1^{Rxy} + \sigma_1^{Ryx} + \sigma_1^{Ryy}}.
\end{aligned}$$

The solution is nonempty (and unique); hence there exists a realization-equivalent behavioral strategy for every mixed strategy.

(d)

Intuitively, the game is no longer one of perfect recall because, at the second information set, player 1 does not remember his initial move (M or R). Formally, there exist now decision nodes x and x' for player 1 where $H(x) = H(x')$, and the only predecessor of x and x' (the initial node) has an action leading to x but not to x'.

Player 1's set of pure strategies has changed to

$$A_1' = \begin{Bmatrix} L, x & L, y \\ M, x & M, y \\ R, x & R, y \end{Bmatrix}$$

since player 1 can no longer distinguish between the initial moves M and R when choosing x or y. Player 2's strategies are unaltered.

Suppose again that player 2 adopts mixed strategy $\sigma_2 \in \{\pi\} \times \{1-\pi\}$. Player 1's behavioral strategy is as follows. At the first information set, play L with probability u_1, M with probability u_2, and R with probability $1-u_1-u_2$. At the second information set, play x with probability t and y with probability $(1-t)$. The set of behavioral strategies consists of pairs of probability lists:

$$B_1' = \{(u_1, u_2, 1-u_1-u_2), (t, 1-t) : u_1, u_2, t \in [0,1]\}.$$

Given behavioral strategy $((u_1, u_2, 1-u_1-u_2), (t, 1-t))$ for player 1, the probability distribution over terminal nodes is:

$$\begin{aligned}
\Pr(T_0 \mid s_1 \in B_1') &= u_1 \\
\Pr(T_1 \mid s_1 \in B_1') &= u_2 \pi t \\
\Pr(T_2 \mid s_1 \in B_1') &= u_2 \pi (1-t) \\
\Pr(T_3 \mid s_1 \in B_1') &= u_2 (1-\pi) t \\
\Pr(T_4 \mid s_1 \in B_1') &= u_2 (1-\pi)(1-t) \\
\Pr(T_5 \mid s_1 \in B_1') &= (1-u_1-u_2) \pi t \\
\Pr(T_6 \mid s_1 \in B_1') &= (1-u_1-u_2) \pi (1-t) \\
\Pr(T_7 \mid s_1 \in B_1') &= (1-u_1-u_2)(1-\pi) t \\
\Pr(T_8 \mid s_1 \in B_1') &= (1-u_1-u_2)(1-\pi)(1-t).
\end{aligned}$$

If player 1 adopts a mixed strategy, the probability distribution over terminal nodes is:

$$\begin{aligned}
\Pr(T_0 \mid s_1 \in A_1') &= \sigma_1^{Lx} + \sigma_1^{Ly} \\
\Pr(T_1 \mid s_1 \in A_1') &= \sigma_1^{Mx} \pi \\
\Pr(T_2 \mid s_1 \in A_1') &= \sigma_1^{My} \pi \\
\Pr(T_3 \mid s_1 \in A_1') &= \sigma_1^{Mx} (1-\pi) \\
\Pr(T_4 \mid s_1 \in A_1') &= \sigma_1^{My} (1-\pi) \\
\Pr(T_5 \mid s_1 \in A_1') &= \sigma_1^{Rx} \pi \\
\Pr(T_6 \mid s_1 \in A_1') &= \sigma_1^{Ry} \pi \\
\Pr(T_7 \mid s_1 \in A_1') &= \sigma_1^{Rx} (1-\pi) \\
\Pr(T_8 \mid s_1 \in A_1') &= \sigma_1^{Ry} (1-\pi).
\end{aligned}$$

Setting

$$\Pr(T \mid s_1 \in A_1') = \Pr(T \mid s_1 \in B_1')$$

for all T and eliminating redundancies:

$$\begin{aligned}\sigma_1^{Lx}+\sigma_1^{Ly} &= u_1\\ \sigma_1^{Mx} &= u_2 t\\ \sigma_1^{My} &= u_2(1-t)\\ \sigma_1^{Rx} &= (1-u_1-u_2)\,t.\end{aligned}$$

There are infinitely many solutions for σ_1, hence there still exist realization-equivalent mixed strategies for every behavioral strategy.

The converse fails in the presence of perfect recall. If we solve for u_1, u_2, and t, there are four linearly independent equations (and $\sum_{k\in A_1}\sigma_1^k=1$) in three unknowns. Using the first three conditions,

$$\begin{aligned}u_1 &= \sigma_1^{Lx}+\sigma_1^{Ly}\\ u_2 &= \sigma_1^{Mx}+\sigma_1^{My}\\ t &= \frac{\sigma_1^{Mx}}{\sigma_1^{Mx}+\sigma_1^{My}}.\end{aligned}$$

But the fourth condition is not generally compatible:

$$\begin{aligned}t &= \frac{\sigma_1^{Rx}}{1-u_1-u_2}\\ &= \frac{\sigma_1^{Rx}}{\sigma_1^{Rx}+\sigma_1^{Ry}}.\end{aligned}$$

Unless $\frac{\sigma_1^{Rx}}{\sigma_1^{Rx}+\sigma_1^{Ry}}=\frac{\sigma_1^{Mx}}{\sigma_1^{Mx}+\sigma_1^{My}}$ (i.e. unless the mixed strategy is such that the conditional probability of x given initial move M equals the conditional probability of x given initial move R), the solution is empty and there is no realization-equivalent behavioral strategy.

8.B.1

A strictly dominant strategy yields the highest payoff among all available strategies, regardless of other players' actions. Formally, strategy $s_i\in S_i$ is strictly dominant if $u_i(s_i,s_{-i})>u_i(s_i',s_{-i})$ for all $s_i'\neq s_i$ and any $s_{-i}\in S_{-i}$. To show that a strictly dominant strategy exists, it is enough to demonstrate that the strategy s_i which maximizes i's payoff does not depend on s_{-i}, i.e. in the context of the problem, the benefit-maximizing choice of effort h_i does not depend on other h_{-i} (hence the optimal strategy s_i is a constant effort h_i with respect to h_{-i}).

Supposing that the subsidy would be distributed equally among the I firms in the industry (this assumption is merely for convenience; it does not affect the validity of the argument), each firm's payoff is:

$$u_i(s_i, s_{-i}) = \frac{\alpha \sum_i h_i + \beta \prod_i h_i}{I} - w_i^2.$$

The first-order condition for payoff maximization is:

$$\frac{\partial u_i(s_i, s_{-i})}{\partial h_i} = \frac{\alpha + \beta \prod_{-i} h_i}{I} - 2w_i \frac{\partial w_i}{\partial h_i} = 0.$$

(If) For $\beta = 0$, the first-order condition reduces to $\frac{\alpha}{I} - 2w_i \frac{\partial w_i}{\partial h_i} = 0$. Clearly, the optimal choice of h_i does not depend on h_{-i}; hence h_i^* that satisfies the first-order condition is a strictly dominant strategy.

(Only if) For $\beta \neq 0$, h_i^* that satisfies first-order condition is a function of h_{-i}, so there is no strictly dominant strategy.

8.B.4

The set of strategies surviving iterated deletion of strictly dominated strategies is invariant to the order of deletion.

Proof. Let Γ be an arbitrary game with initial strategy set S_0. Suppose iterated deletion of strictly dominated strategies, in some order, maximally reduces the strategy set to S. Now consider another order of deletion that maximally reduces $\tilde{S}_0 = S_0$ to $\tilde{S}$. The claim is that $S = \tilde{S}$. So we need to prove two inclusions: $S \subseteq \tilde{S}$ and $\tilde{S} \subseteq S$.

First we dispose of the trivial case that there are no strictly dominated strategies in S_0. Then $S = S_0 = \tilde{S}_0 = \tilde{S}$, and we are done. Now suppose S_0 does contain strictly dominated strategies, so $S \subset S_0$ and $\tilde{S}_0 \subset \tilde{S}$.

For $S \subseteq \tilde{S}$, the argument is by induction. Let $\left\{\tilde{S}_i\right\}_{i=0}^{n}$ be a sequence of strategy sets associated with a sequence of games $\left\{\tilde{\Gamma}_i\right\}_{i=0}^{n}$ such that $\tilde{S}_{i+1} = \tilde{S}_i \setminus s_i$, where s_i is a strictly dominated strategy in game $\tilde{\Gamma}_i$, and $\tilde{S}_n = \tilde{S}$. The inductive hypothesis is that $S \subseteq \tilde{S}_i$. The inductive hypothesis holds for $i = 0$, since $\tilde{S}_0 = S_0$ and $S \subseteq S_0$. Generalizing, say that the inductive hypothesis holds for i, and consider $i+1$. Given that $S \subseteq \tilde{S}_i$ (by assumption) and $\tilde{S}_{i+1} = \tilde{S}_i \setminus s_i$ (by definition), $S \subseteq \tilde{S}_{i+1}$ if and only if $s_i \notin S$.

We know that there exists $s_i' \in \tilde{S}_i$ that dominates s_i in $\tilde{\Gamma}_i$. (Write this as $s_i \prec s_i'$.) Because $S \subseteq \tilde{S}_i$, $s_i \prec s_i'$ in $\tilde{\Gamma}_i$ implies $s_i \prec s_i'$ in Γ. So if $s_i' \in S$, then $s_i \notin S$, since it is dominated. If $s_i' \notin S$, then there exists $s_i'' \in S$ that transitively dominates s_i' and also s_i. To make this explicit, define the transitive closure of the strict domination relation $\prec$ as follows: $s_i' \prec\prec s_i''$ if there exists a sequence $s_i', \ldots, s_i''$ such that every element strictly dominates the element preceding it. Notice that $s_i' \prec\prec s_i''$ implies $s_i' \prec s_i''$. Because $s_i' \notin S$ only if there exists $s_i'' \in S$ such that $s_i' \prec\prec s_i''$, and since $s_i \prec s_i'$, we can construct such a sequence from s_i to s_i'', and as $s_i'' \in S$, it is necessary that s_i is strictly dominated in S. From this, we have $s_i \notin S$ and consequently $S \subseteq \tilde{S}_{i+1}$ for all i, which proves inductively that $S \subseteq \tilde{S}$.

For the reverse inclusion, it suffices to observe that S was chosen arbitrarily, so $S \subseteq \tilde{S}$ also implies $\tilde{S} \subseteq S$ (we could just rename the sets, and the proof would be identical). Hence $S = \tilde{S}$, which confirms the claim that any order of deletion leads to the same set of strategies. (It suffices to consider pure strategies, since a mixed strategy is dominated if and only if all pure strategies in its support are dominated.) ■

8.C.4
(a)

Suppose player 2 assigns probability p to action U (and $1-p$ to D), and player 3 assigns probability q to action l (and $1-q$ to r). Then player 1's expected utility from pure strategies L, M, and R is:

$$\begin{aligned}
\mathrm{E}[U(L)] &= q(\pi + 4\varepsilon) + (1-q)(\pi - 4\varepsilon) \\
&= \pi + (2q-1)4\varepsilon \\
\mathrm{E}[U(M)] &= (pq + (1-p)(1-q))(\pi - \eta) + (p(1-q) + (1-p)q)\left(\pi + \frac{\eta}{2}\right) \\
&= \pi + (3p - 6pq + 3q - 2)\frac{\eta}{2} \\
\mathrm{E}[U(R)] &= q(\pi - 4\varepsilon) + (1-q)(\pi + 4\varepsilon) \\
&= \pi + (1-2q)4\varepsilon
\end{aligned}$$

If $q = \frac{1}{2}$,

$$\mathrm{E}[U(M)] = \pi - \frac{\eta}{4} < \pi = \mathrm{E}[L] = \mathrm{E}[R].$$

If $q < \frac{1}{2}$, player 1's payoff is increasing in p, since $\frac{\partial \mathrm{E}[M]}{\partial p} = (3-6q)\frac{\eta}{2} > 0$. So it suffices to show that $\mathrm{E}[M]$ is not a best response if $p = 1$. In this case,

$$\mathrm{E}[U(M)] = \pi + (1-3q)\frac{\eta}{2}.$$

Since $\eta < 4\varepsilon$,

$$\begin{aligned} \mathrm{E}\left[U\left(R\right)\right] = \pi + \left(1-2q\right)4\varepsilon > & \quad \pi + \left(1-2q\right)\eta \\ = & \quad \pi + \left(1-3q\right)\frac{\eta}{2} + \frac{\eta\left(1-q\right)}{2} \\ > & \quad \mathrm{E}\left[U\left(M\right)\right]. \end{aligned}$$

If $q > \frac{1}{2}$, player 1's payoff is increasing in p, since $\frac{\partial \mathrm{E}[M]}{\partial p} = (3-6q)\frac{\eta}{2} < 0$. It suffices to show that $\mathrm{E}\left[M\right]$ is not a best response if $p = 0$. In this case,

$$\mathrm{E}\left[U\left(M\right)\right] = \pi + \left(3q-2\right)\frac{\eta}{2}.$$

Since $\eta < 4\varepsilon$,

$$\begin{aligned} \mathrm{E}\left[U\left(L\right)\right] = \pi + \left(2q-1\right)4\varepsilon > & \quad \pi + \left(2q-1\right)\eta \\ = & \quad \pi + \left(3q-2\right)\frac{\eta}{2} + \frac{\eta q}{2} \\ > & \quad \mathrm{E}\left[U\left(M\right)\right]. \end{aligned}$$

We have shown that, for any randomization that players 2 and 3 may adopt, there is a pure strategy for player 1 that outperforms M, so M is never a best response.

(b)

If player 1 assigns probability z to action L (and $1-z$ to R), we show that there always exists a randomization for players 2 and 3 such that M has a higher expected payoff than $\left\{\left(L,z\right),\left(R,1-z\right)\right\}$.

Suppose $z \geq \frac{1}{2}$. Then if $p = 1$ and $q = 0$,

$$\mathrm{E}\left[U\left(M\right)\right] = \pi + \frac{\eta}{2}$$

and

$$\begin{aligned} \mathrm{E}\left[U\left(\left\{\left(L,z\right),\left(R,1-z\right)\right\}\right)\right] &= z\,\mathrm{E}\left[U\left(L\right)\right] + \left(1-z\right)\mathrm{E}\left[U\left(R\right)\right] \\ &= z\left(\pi-4\varepsilon\right) + \left(1-z\right)\left(\pi+4\varepsilon\right) \\ &= \pi + 4\varepsilon - 8\varepsilon z \\ &\leq \pi \\ &< \mathrm{E}\left[U\left(M\right)\right]. \end{aligned}$$

Alternatively, suppose $z < \frac{1}{2}$. Then if $p = 0$ and $q = 1$,

$$\mathrm{E}\left[U\left(M\right)\right] = \pi + \frac{\eta}{2}$$

and

$$\begin{aligned}
\mathrm{E}\left[U\left(\left\{(L,z),(R,1-z)\right\}\right)\right] &= z\,\mathrm{E}\left[U(L)\right]+(1-z)\,\mathrm{E}\left[U(R)\right] \\
&= z(\pi+4\varepsilon)+(1-z)(\pi-4\varepsilon) \\
&= \pi-4\varepsilon+8\varepsilon z \\
&< \pi \\
&< \mathrm{E}\left[U(M)\right].
\end{aligned}$$

(c)

If players 2 and 3 assign equal frequencies to $\{U,r\}$ and $\{D,l\}$, then

$$\mathrm{E}\left[U(M)\right]=\pi+\frac{\eta}{2}$$

and

$$\begin{aligned}
\mathrm{E}\left[U\left(\left\{(L,z),(R,1-z)\right\}\right)\right] &= \frac{1}{2}\left(z(\pi-4\varepsilon)+(1-z)(\pi+4\varepsilon)\right) \\
&\quad+\frac{1}{2}\left(z(\pi+4\varepsilon)+(1-z)(\pi-4\varepsilon)\right) \\
&= \pi \\
&< \mathrm{E}\left[U(M)\right].
\end{aligned}$$

Hence there exists a randomization for players 2 and 3 over correlated strategies such that M is outperforms any randomization for player 1 over L and R.

8.D.9

(a)

Without any other information, it seems reasonable for player 2 to choose M, which gives a certain payoff of 0. Player 1's action is not predictable, since there is no dominant strategy. The slight potential gains from other actions player 2 may take are offset by possibilities of significant loss.

(b)

There are two Nash equilibria in pure strategies: $\{U,LL\}$ and $\{D,R\}$. In a mixed-strategy equilibrium, let player 1 assign probability p to action U and $1-p$ to action D. Player 2 can mix in several different ways, but only a best response to $\{(U,p),(D,1-p)\}$ constitutes a Nash equilibrium.

The optimal probabilities are determined by the condition that the expected payoff for player i must be the same for all actions in the support of i's equilibrium mixed strategy.

Some observations on the support of player 2's equilibrium mixed strategy:

- LL must be one of the actions over which player 2 randomizes; otherwise D is strictly dominant for player 1, and then R is the unique best response for player 2.

- Player 2 can only randomize over LL and one of L or M or R. For if player 2 randomizes over LL and L, then $p = \frac{51}{52}$ from

  $$2p - 100(1-p) = p - 49(1-p).$$

  But if player 2 randomizes over LL and M, then $p = \frac{50}{51}$ from

  $$2p - 100(1-p) = 0.$$

  And if player 2 randomizes over LL and R, then $p = \frac{1}{2}$ from

  $$2p - 100(1-p) = -100p + 2(1-p).$$

  Since the three conditions for p are incompatible, no two actions of L, M, and R could be simultaneously in the support of an equilibrium mixed strategy.

- Since player 2 has a pure strategy that delivers a certain payoff of zero (M), a mixed strategy that has negative expected payoff can never be optimal. But if $p = \frac{1}{2}$, the expected payoffs to player 2 of LL and R are clearly negative, hence the expected payoff of randomizing over LL and R is negative. Therefore $\{(LL, q), (R, 1-q)\}$ is not a best response to $\left\{\left(U, \frac{1}{2}\right), \left(D, 1-\frac{1}{2}\right)\right\}$ for any q.

- If $p = \frac{50}{51}$, player 2 has a pure strategy that delivers a positive expected payoff:

  $$U_2\left(L, \left\{\left(U, \frac{50}{51}\right), \left(D, \frac{1}{51}\right)\right\}\right) = p - 49(1-p) = \frac{1}{51} > 0.$$

  On the other hand,

  $$\begin{aligned} U_2\left(LL, \left\{\left(U, \frac{50}{51}\right), \left(D, \frac{1}{51}\right)\right\}\right) &= U_2\left(M, \left\{\left(U, \frac{50}{51}\right), \left(D, \frac{1}{51}\right)\right\}\right) \\ &= 0, \end{aligned}$$

  so randomizing over LL and M has zero expected payoff. But then $\{(LL, q), (M, 1-q)\}$ is not a best response to $\{(U, p), (D, 1-p)\}$.

The only Nash equilibrium in pure strategies is one where player 2 randomizes over LL and L. We have already seen that $p = \frac{51}{52}$. Similarly, $q = \frac{1}{2}$ follows from

$$100q - 100(1-q) = -100q + 100(1-q).$$

The expected payoffs are

$$\begin{aligned} U_1(\cdot) &= p(100q - 100(1-q)) + (1-p)(-100q + 100(1-q)) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} U_2(\cdot) &= q(2p - 100(1-p)) + (1-q)(p - 49(1-p)) \\ &= \frac{1}{26}, \end{aligned}$$

so the mixed strategies are really best responses. The Nash equilibrium in pure strategies is completely described by

$$\left\{\left\{\left(U, \frac{51}{52}\right), \left(D, \frac{1}{52}\right)\right\}, \left\{\left(LL, \frac{1}{2}\right), \left(L, \frac{1}{2}\right)\right\}\right\}.$$

(c)

The pure strategy chosen in (a), namely M, is not part of any Nash equilibrium profile. But it is rationalizable: e.g. if player 1 randomizes between U and D with $p = \frac{1}{2}$, then M is the best response.

(d)

With pre-play communication, any of the three Nash equilibria is implementable. Since the pure strategy equilibria Pareto-dominate the mixed strategy equilibrium, it may be expected that either $\{U, LL\}$ or $\{D, R\}$ is played.

9.C.2

As in MWG, example 9.C.3, any belief such that $(\mu_1, \mu_2) \in \left\{\left(\left[0, \frac{2}{3}\right), 1 - \mu_1\right)\right\}$ causes the incumbent to accommodate with probability 1, in which case the entrant plays In_1 with probability 1 and the belief is inconsistent. Hence, in any weak PBE, $\mu_1 \geq \frac{2}{3}$.

In case $\mu_1 > \frac{2}{3}$, the incumbent will certainly fight, since

$$\begin{aligned} \mathrm{E}\left[U_I\left(F\right)\right] &= -1 \\ &> -2\mu_1 + (1-\mu_1) = \mathrm{E}\left[U_I\left(A\right)\right]. \end{aligned}$$

This makes it optimal for the entrant to stay out because, with $\gamma \in (-1, 0)$,

$$\begin{aligned} \mathrm{E}\left[U_E\left(0\right)\right] &= 0 \\ &> \gamma = \mathrm{E}\left[U_E\left(2\right)\right] \\ &> -1 = \mathrm{E}\left[U_E\left(1\right)\right]. \end{aligned}$$

A set of weak PBE is therefore characterized by $(\sigma_0, \sigma_1, \sigma_2) = (1, 0, 0)$, $(\sigma_F, \sigma_A) = (1, 0)$, and unique beliefs $(\mu_1, \mu_2) \in \left\{\left(\left(\frac{2}{3}, 1\right], 1-\mu_1\right)\right\}$. Here the incumbent's beliefs need not be consistent with the entrant's strategy, since the incumbent's information set is never reached.

If $\mu_1 = \frac{2}{3}$, the incumbent is indifferent between fighting and accomodating:

$$\begin{aligned} \mathrm{E}\left[U_I\left(F\right)\right] &= -1 \\ &= -2\mu_1 + (1-\mu_1) = \mathrm{E}\left[U_I\left(A\right)\right]. \end{aligned}$$

Therefore, any randomization is a best response for the incumbent. But the particulars of the incumbent's randomization determine the entrant's willingness to randomize. Observe from $\mathrm{E}\left[U_E\left(0\right)\right] = 0$, $\mathrm{E}\left[U_E\left(1\right)\right] = -\sigma_F + 3\left(1-\sigma_F\right)$, and $\mathrm{E}\left[U_E\left(2\right)\right] = \gamma\sigma_F + 2\left(1-\sigma_F\right)$:

$$\mathrm{E}\left[U_E\left(0\right)\right] = \mathrm{E}\left[U_E\left(1\right)\right] \Longleftrightarrow \sigma_F = \frac{3}{4}$$

($\mathrm{E}\left[U_E\left(0\right)\right] \geq \mathrm{E}\left[U_E\left(1\right)\right] \Longleftrightarrow \sigma_F \geq \frac{3}{4}$ and $\mathrm{E}\left[U_E\left(0\right)\right] \leq \mathrm{E}\left[U_E\left(1\right)\right] \Longleftrightarrow \sigma_F \leq \frac{3}{4}$),

$$\mathrm{E}\left[U_E\left(0\right)\right] = \mathrm{E}\left[U_E\left(2\right)\right] \Longleftrightarrow \sigma_F = \frac{2}{2-\gamma}$$

($\mathrm{E}\left[U_E\left(0\right)\right] \geq \mathrm{E}\left[U_E\left(2\right)\right] \Longleftrightarrow \sigma_F \geq \frac{2}{2-\gamma}$ and $\mathrm{E}\left[U_E\left(0\right)\right] \leq \mathrm{E}\left[U_E\left(2\right)\right] \Longleftrightarrow \sigma_F \leq \frac{2}{2-\gamma}$), and

$$\mathrm{E}\left[U_E\left(1\right)\right] = \mathrm{E}\left[U_E\left(2\right)\right] \Longleftrightarrow \sigma_F = \frac{1}{\gamma+2}$$

($\mathrm{E}\left[U_E\left(1\right)\right] \geq \mathrm{E}\left[U_E\left(2\right)\right] \Longleftrightarrow \sigma_F \leq \frac{1}{\gamma+2}$ and $\mathrm{E}\left[U_E\left(1\right)\right] \leq \mathrm{E}\left[U_E\left(2\right)\right] \Longleftrightarrow \sigma_F \geq \frac{1}{\gamma+2}$).

The entrant randomizes over all three actions only if $\mathrm{E}[U_E(0)] = \mathrm{E}[U_E(1)] = \mathrm{E}[U_E(2)]$, i.e. if $\sigma_F = \frac{3}{4} = \frac{2}{2-\gamma} = \frac{1}{\gamma+2}$. This is only possible if $\gamma = -\frac{2}{3}$. Note that, since the incumbent's information set is reached with positive probability, beliefs must be consistent in equilibrium. In other words, the entrant must play 1 and 2 with probabilities $\frac{2}{3}$ and $\frac{1}{3}$ at the information set (play 1 twice as often as 2). This leads to another set of weak PBE equilibria in the event that $\gamma = -\frac{2}{3}$, characterized by $\{(\sigma_0, \sigma_1, \sigma_2)\} = \left\{\left(x, \frac{2}{3}(1-x), \frac{1}{3}(1-x)\right)\right\}$, $(\sigma_F, \sigma_A) = \left(\frac{3}{4}, \frac{1}{4}\right)$, and $(\mu_1, \mu_2) = \left(\frac{2}{3}, \frac{1}{3}\right)$, where $x \in (0, 1)$.

Next, consider equilibria where the entrant randomizes over two actions. Since the incumbent's beliefs must be consistent with the entrant's actions, we can rule out randomizations over 0 and 1 and over 0 and 2 (which would assign probability zero to either 1 or 2). Randomization over 1 and 2 requires $\mathrm{E}[U_E(1)] = \mathrm{E}[U_E(2)]$ and $\mathrm{E}[U_E(1)] \geq \mathrm{E}[U_E(0)]$, i.e. $\sigma_F = \frac{1}{\gamma+2}$ and $\sigma_F \leq \frac{3}{4}$, or $\gamma \geq -\frac{2}{3}$. Therefore, in the event that $\gamma \geq -\frac{2}{3}$, there is another weak PBE, characterized by $(\sigma_0, \sigma_1, \sigma_2) = \left(0, \frac{2}{3}, \frac{1}{3}\right)$, $(\sigma_F, \sigma_A) = \left(\frac{1}{\gamma+2}, \frac{\gamma+1}{\gamma+2}\right)$, and $(\mu_1, \mu_2) = \left(\frac{2}{3}, \frac{1}{3}\right)$.

Finally, the entrant plays 0 with certainty (pure strategies 1 and 2 are ruled out by the consistency condition) if $\mathrm{E}[U_E(0)] > \mathrm{E}[U_E(1)]$ and $\mathrm{E}[U_E(0)] > \mathrm{E}[U_E(2)]$, i.e. $\sigma_F > \frac{3}{4}$ and $\sigma_F > \frac{2}{2-\gamma}$, or $\sigma_F > \max\left[\frac{3}{4}, \frac{2}{2-\gamma}\right]$. If $\gamma \geq -\frac{2}{3}$, then $\frac{2}{2-\gamma} \geq \frac{3}{4}$; if $\gamma \leq -\frac{2}{3}$, then $\frac{2}{2-\gamma} \leq \frac{3}{4}$. Hence, in the event that $\gamma \geq -\frac{2}{3}$, there is a set of weak PBE characterized by $(\sigma_0, \sigma_1, \sigma_2) = (1, 0, 0)$, $\{(\sigma_F, \sigma_A)\} \in \left\{\left(\left[\frac{2}{2-\gamma}, 1\right], 1 - \sigma_F\right)\right\}$, and $(\mu_1, \mu_2) = \left(\frac{2}{3}, \frac{1}{3}\right)$. In the event that $\gamma \leq -\frac{2}{3}$, there is a set of weak PBE characterized by $(\sigma_0, \sigma_1, \sigma_2) = (1, 0, 0)$, $\{(\sigma_F, \sigma_A)\} \in \left\{\left(\left[\frac{3}{4}, 1\right], 1 - \sigma_F\right)\right\}$, and $(\mu_1, \mu_2) = \left(\frac{2}{3}, \frac{1}{3}\right)$.

In summary, the weak PBE of the game are:

- For $\gamma \in (-1, 0)$: $(\sigma_0, \sigma_1, \sigma_2) = (1, 0, 0)$, $(\sigma_F, \sigma_A) = (1, 0)$, $\{(\mu_1, \mu_2)\} \in \left\{\left(\left(\frac{2}{3}, 1\right], 1 - \mu_1\right)\right\}$.
- For $\gamma \in \left[-\frac{2}{3}, 0\right)$: $(\sigma_0, \sigma_1, \sigma_2) = \left(0, \frac{2}{3}, \frac{1}{3}\right)$, $(\sigma_F, \sigma_A) = \left(\frac{1}{\gamma+2}, \frac{\gamma+1}{\gamma+2}\right)$, $(\mu_1, \mu_2) = \left(\frac{2}{3}, \frac{1}{3}\right)$ and $(\sigma_0, \sigma_1, \sigma_2) = (1, 0, 0)$, $\{(\sigma_F, \sigma_A)\} \in \left\{\left(\left[\frac{2}{2-\gamma}, 1\right], 1 - \sigma_F\right)\right\}$, $(\mu_1, \mu_2) = \left(\frac{2}{3}, \frac{1}{3}\right)$.
- For $\gamma = -\frac{2}{3}$: $\{(\sigma_0, \sigma_1, \sigma_2)\} = \left\{\left(x, \frac{2}{3}(1-x), \frac{1}{3}(1-x)\right)\right\}$, $(\sigma_F, \sigma_A) = \left(\frac{3}{4}, \frac{1}{4}\right)$, $(\mu_1, \mu_2) = \left(\frac{2}{3}, \frac{1}{3}\right)$, where $x \in (0, 1)$.
- For $\gamma \in \left(0, -\frac{2}{3}\right]$: $(\sigma_0, \sigma_1, \sigma_2) = (1, 0, 0)$, $\{(\sigma_F, \sigma_A)\} \in \left\{\left(\left[\frac{3}{4}, 1\right], 1 - \sigma_F\right)\right\}$, $(\mu_1, \mu_2) = \left(\frac{2}{3}, \frac{1}{3}\right)$.

13.B.2

A competitive equilibrium with unobservable types is characterized by

$$\begin{aligned} \Theta^* &= \{\theta : r(\theta) \leq w^*\} \\ w^* &= \mathrm{E}[\theta \mid \theta \in \Theta^*], \end{aligned}$$

where Θ^* is the set of types that accept employment at the wage w^* offered by firms. In a Pareto optimal allocation of labor, workers accept employment if $\theta \geq r(\theta)$ and decline employment if $\theta < r(\theta)$. By assumption, there exists $\hat{\theta} \in [\underline{\theta}, \overline{\theta}]$ such that $\theta > r(\theta)$ if $\hat{\theta} > \theta$ and $\theta < r(\theta)$ if $\hat{\theta} < \theta$. Hence, Pareto optimality involves that workers accept employment if $\hat{\theta} \geq \theta$ and decline employment if $\hat{\theta} < \theta$.

This implies $\Theta^* = \left\{\theta : r(\theta) \leq \hat{\theta}\right\}$, so in a Pareto optimal competitive equilibrium we must have $w^* = \hat{\theta}$. But since $f(\theta)$ has full support, there exist workers with θ strictly below $\hat{\theta}$ in $\Theta^* = \left[\underline{\theta}, \hat{\theta}\right]$. Then

$$\mathrm{E}[\theta \mid \theta \in \Theta^*] < \hat{\theta} = w^*,$$

which is inconsistent with competitive equilibrium. It follows that the Pareto optimal allocation is not feasible in a competitive equilibrium in this case.

13.B.4

The no-trade theorem requires the additional assumptions that prior beliefs and the distribution of signals are, respectively, the same for both agents and common knowledge. Since both agents care only about the expected value of the asset, they value it equally before signals are observed.

The valuation of the asset is a random variable $V(\omega)$ on a state space Ω that includes the actual state $\bar{\omega}$. Given that $\bar{\omega}$ occurs, 1 receives a random signal θ_1 and updates $\mathrm{E}[V(\omega)]$ to $\mathrm{E}[V(\omega) \mid \theta_1(\bar{\omega})]$, whereas 2 receives a random signal θ_2 and updates $\mathrm{E}[V(\omega)]$ to $\mathrm{E}[V(\omega) \mid \theta_2(\bar{\omega})]$. If trade took place, it would become common knowledge that signals have been observed which cause both sides to wish to trade. Thus, it would be common knowledge that $\bar{\omega}$ belongs to the event

$$S = \{\omega : \mathrm{E}[V(\omega) \mid \theta_1(\omega)] < p, \mathrm{E}[V(\omega) \mid \theta_2(\omega)] > p\}.$$

With common priors, the implication is that $\mu(\omega \mid \theta_1(\bar{\omega})) = \mu(\omega \mid \theta_2(\bar{\omega}))$, so

$$\begin{aligned}
\mathrm{E}[V(\omega) \mid \theta_1(\bar{\omega})] &= \int_S V(\omega)\,\mu(\omega \mid \theta_1(\bar{\omega}))\,d\omega \\
&= \int_S V(\omega)\,\mu(\omega \mid \theta_2(\bar{\omega}))\,d\omega = \mathrm{E}[V(\omega) \mid \theta_2(\bar{\omega})].
\end{aligned}$$

Therefore, S is in fact empty; there is no state in which the mutual willingness to trade could be common knowledge.

Intuitively, the fact that 2 is willing to buy at some price p informs 1 that 2 has received a more favorable signal about the likely value of the asset, so 1 updates her expectation and rejects p. Similarly, the fact that 1 is willing to sell at p informs 2 that 1 has received a less favorable signal about the likely value of the asset, so 2 updates her expectation and rejects p. If offers can be renegotiated, expectations are updated until both value the asset equally, so (by assumption) they will never trade.

The key step is clearly the equality of the posteriors when readiness to trade is common knowledge. This is the "agreeing to disagree" result. Suppose that, after receiving private signals at $\bar{\omega}$, 1 knows that $\bar{\omega} \in P_1(\theta_1(\bar{\omega}))$, and 2 knows that $\bar{\omega} \in P_2(\theta_2(\bar{\omega}))$, where $P_1, P_2 \subseteq 2^{\Omega}$ are sets of possible states. Moreover, 1 holds belief $\mu(\omega \mid \theta_1(\bar{\omega}))$ on $P_1(\theta_1(\bar{\omega}))$, and 2 holds belief $\mu(\omega \mid \theta_2(\bar{\omega}))$ on $P_2(\theta_2(\bar{\omega}))$, and this is commonly known. Precisely, the event

$$F = \left\{\omega : \mu\left(E \mid \tilde{\theta}_1\right) = \bar{\mu}_1, \mu\left(E \mid \tilde{\theta}_2\right) = \bar{\mu}_2\right\}.$$

is common knowledge. The signals $\tilde{\theta}_1$ with the property $\mu\left(E \mid \tilde{\theta}_1(\omega)\right) = \bar{\mu}_1$ (i.e., from the perspective of 2, who observes $\bar{\mu}_1$ but not θ_1, the candidate signals that could have lead 1 to conclude $\bar{\mu}_1$) induce a partition of F into sets $P_1\left(\tilde{\theta}_1(\omega)\right)$ of states where 1 believes $\bar{\mu}_1$. An analogous statement holds for 2. This implies $\mu\left(E \mid \tilde{\theta}_1(F)\right) = \bar{\mu}_1$ and $\mu\left(E \mid \tilde{\theta}_2(F)\right) = \bar{\mu}_2$. But $\tilde{\theta}_1(F) = \tilde{\theta}_2(F)$, by the partitional nature of information, so $\bar{\mu}_1 = \bar{\mu}_2$. Since E was chosen arbitrarily, the posterior beliefs indeed agree.

13.B.5

(a)

Let $\mathrm{E}[\theta] \geq r > \underline{\theta}$ and $\mathrm{E}[\theta \mid \theta \in \varnothing] = \underline{\theta}$. To see that $w = \mathrm{E}[\theta]$ and $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$ is a competitive equilibrium, notice that, since $w \geq r(\theta)$ for all θ,

$$\Theta = \{\theta : r(\theta) \leq w\} = \left[\underline{\theta}, \overline{\theta}\right]$$

(i.e. all workers accept employment), and then

$$w = \mathrm{E}\left[\theta \mid \theta \in \left[\underline{\theta}, \overline{\theta}\right]\right] = \mathrm{E}\left[\theta\right].$$

To see that $w = \underline{\theta}$ and $\Theta = \varnothing$ is a competitive equilibrium, observe that, since $w < r(\theta)$ for all θ,

$$\Theta = \{\theta : r(\theta) \leq w\} = \varnothing$$

(i.e. all workers decline employment), and then (by assumption)

$$w = \mathrm{E}\left[\theta \mid \theta \in \varnothing\right] = \underline{\theta}.$$

There are no other competitive equilibria. If Θ is a strict, nonempty subset of $\left[\underline{\theta}, \overline{\theta}\right]$, there exists $\hat{\theta} \in \left(\underline{\theta}, \overline{\theta}\right)$ such that workers with $\theta < \hat{\theta}$ decline employment. Then $\mathrm{E}\left[\theta \mid \theta \in \Theta\right] > \mathrm{E}\left[\theta\right] = r$, and so $w^* > r$, implying $\Theta^* = \{\theta : r(\theta) \leq w\} = \left[\underline{\theta}, \overline{\theta}\right] \neq \Theta$.

Let $\underline{\theta} \geq r$. To see that $w = \mathrm{E}\left[\theta\right]$ and $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$ is a competitive equilibrium, notice that, since $\mathrm{E}\left[\theta\right] \geq \underline{\theta}$, we have $w \geq r(\theta)$ for all θ, so

$$\Theta = \{\theta : r(\theta) \leq w\} = \left[\underline{\theta}, \overline{\theta}\right]$$

(i.e. all workers accept employment), and then

$$w = \mathrm{E}\left[\theta \mid \theta \in \left[\underline{\theta}, \overline{\theta}\right]\right] = \mathrm{E}\left[\theta\right].$$

There are no other competitive equilibria. If Θ is a strict, nonempty subset of $\left[\underline{\theta}, \overline{\theta}\right]$, the argument is as above. If $\Theta = \varnothing$, then $w^* = \mathrm{E}\left[\theta \mid \theta \in \varnothing\right] = \underline{\theta} \geq r$ (by assumption), so $\Theta^* = \{\theta : r(\theta) \leq w\} = \left[\underline{\theta}, \overline{\theta}\right] \neq \Theta$.

Let $r(\theta) > \mathrm{E}\left[\theta\right]$ for all θ, and $\mathrm{E}\left[\theta \mid \theta \in \varnothing\right] = \underline{\theta}$. To see that $w = \underline{\theta}$ and $\Theta = \varnothing$ is a competitive equilibrium, observe that, by $\mathrm{E}\left[\theta\right] \geq \underline{\theta}$, we have $r > \underline{\theta} = w$ for all θ, so

$$\Theta = \{\theta : r(\theta) \leq w\} = \varnothing$$

(i.e. all workers decline employment), and then

$$w = \mathrm{E}\left[\theta \mid \theta \in \left[\underline{\theta}, \overline{\theta}\right]\right] = \mathrm{E}\left[\theta\right]$$

by assumption. But contrary to the claim, the equilibrium may be nonunique, since $w^* = \mathrm{E}\left[\theta \mid \theta \in \Theta\right] > \mathrm{E}\left[\theta\right]$ if workers with $\theta \geq \hat{\theta}$ (for some $\hat{\theta} \in \left(\underline{\theta}, \overline{\theta}\right)$) accept employment. Depending on r and the distribution of θ, it is possible that $w^* \geq r$ and $\Theta^* = \{\theta : r(\theta) \leq w\} = \left[\hat{\theta}, \overline{\theta}\right] \neq \varnothing$.

(b)

Since the firms' profit is always zero in competitive equilibrium, Pareto dominance can be discussed only with reference to workers. In the no-employment equilibrium, all workers earn the reservation payoff r. In the full-employment equilibrium, workers earn $\mathrm{E}[\theta]$. Clearly, if $\mathrm{E}[\theta] > r$, the full-employment equilibrium Pareto dominates.

(c)

For $\mathrm{E}[\theta] > r$, the claims follow directly from MWG proposition 13.B.1(i). If $\mathrm{E}[\theta] = r$, the two competitive equilibria from part (a), i.e. $w = \mathrm{E}[\theta]$, $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$ and $w = \underline{\theta}$, $\Theta = \varnothing$, are SPNE by MWG proposition 13.B.1(ii). ($\mathrm{E}[\theta] = r$ implies that, in the full-employment as in the no-employment competitive equilibrium, all workers earn r, the payoff of the high-wage competitive equilibrium.) If $\mathrm{E}[\theta] < r$, there are many SPNE, in all of which firms offer $w < r$, and no worker accepts employment.

(d)

As in (b), we may discuss the Pareto dominance of competitive equilibria without reference to firms, since they necessarily earn zero profit. The highest-wage competitive equilibrium is therefore trivially a Pareto optimum, given the information constraints (i.e. given that θ is not directly observed). The constraint strictly binds whenever some worker with type $\theta > r$ declines employment, or some worker with type $\theta < r$ accepts employment. This is typically the case with unobservable types, where every worker is paid the same equilibrium wage $\mathrm{E}[\theta \mid \theta \in \Theta]$.

13.C.4

In a separating PBE, firms and workers play best responses, and beliefs of firms about workers' types are correct. After observing their types $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$, workers pick an education level e that maximizes $u(\theta) = w(e(\theta)) - c(e(\theta) \mid \theta)$. Since the type distribution is continuous (and has positive probability everywhere), the payoff function is everywhere differentiable, so workers' first-order condition can be derived by differentiating u with respect to e:

$$\frac{\partial w(e^*(\theta))}{\partial e} = \frac{\partial c(e^*(\theta) \mid \theta)}{\partial e} = \frac{2e^*(\theta)}{\theta}.$$

A firm's optimal wage offer is $w^*(e(\theta)) = \mathrm{E}[\theta \mid e(\theta)] = \theta$ in a separating equilibrium. Differentiating both sides with respect to θ, we have $\frac{\partial w^*(e(\theta))}{\partial e}\frac{\partial e(\theta)}{\partial \theta} = 1$, so

$$\frac{\partial w^*(e(\theta))}{\partial e} = \frac{1}{\frac{\partial e(\theta)}{\partial \theta}}.$$

By equating the expressions for $\frac{\partial w^*(e(\theta))}{\partial e}$, the condition for joint optimization can be written as:

$$2e^*(\theta)\frac{\partial e^*(\theta)}{\partial \theta} = \theta.$$

Rearranging this to $2e^*(\theta)\partial e^* = \theta\partial\theta$, integration with respect to θ is straightforward and gives

$$e^*(\theta)^2 = \frac{\theta^2}{2} + k,$$

where k is the constant of integration. From $e^*(\underline{\theta}) = \sqrt{\frac{\underline{\theta}^2}{2} + k} = 0$, it follows that $k = -\frac{\underline{\theta}^2}{2}$, so

$$e^*(\theta) = \sqrt{\frac{\theta^2 - \underline{\theta}^2}{2}}.$$

Solving this expression for θ and using the condition for a separating equilibrium, $w^*(e) = \theta$,

$$w^*(e) = \sqrt{2e_i^{*2} + \underline{\theta}^2}.$$

The education choice function $e^*(\theta)$, the wage offer function $w^*(e)$, and a consistent system of beliefs $\mu(e)$, which assigns to every signal e a probability distribution over Θ, constitute the PBE. In a separating equilibrium, the wages are conditioned on the signal such that workers have an incentive to reveal their type in choosing an education level. Therefore $\mu(e)$ is consistent if firms associate education levels correctly with types, i.e. $\mu(e)$ assigns probability 1 to θ and probability to 0 to $\tilde{\theta} \neq \theta$, for all e.

13.D.1

The zero-profit trait of SPNE means, in the full-information case, that firms pay wages equal to a type's product:

$$w_H \equiv w(\theta_H) = \theta_H(1 + \mu t_H^{**})$$

and

$$w_L \equiv w(\theta_L) = \theta_L(1+\mu t_L^{**}).$$

Workers choose effort levels that maximize their respective objectives,

$$u_H = w_H - c(t_H, \theta_H) = \theta_H(1+\mu t_H) - c(t_H, \theta_H)$$

and

$$u_L = w_L - c(t_L, \theta_L) = \theta_L(1+\mu t_L) - c(t_L, \theta_L).$$

The first-order conditions show that t_H^{**} and t_L^{**} are determined by

$$\frac{\partial c(t_H^{**}, \theta_H)}{\partial t_H} = \mu\theta_H$$

and

$$\frac{\partial c(t_L^{**}, \theta_L)}{\partial t_L} = \mu\theta_L.$$

Since $c(\cdot)$ is strictly convex in t, the full-information equilibrium is unique for given μ, θ, and functional form for $c(\cdot)$.

If types are unobservable, the full-information optimal wages are not feasible strategies for the firms. In principle, there could be pooling and separating equilibria, i.e. either equal wages for all types or wages that vary with type and provide an incentive to "reveal" types. However, this model does not admit pooling equilibria, since it is profitable for firms to deviate from uniform wage offers and design contracts that screen out low types. In a separating equilibrium, low types must be paid their full-information wage, $w(t_L^{**}) = \theta_L(1+\mu t_L^{**})$. Clearly, $w(t_L) \nless \theta_L(1+\mu t_L)$ in equilibrium, since every worker has at least this level of productivity, so firms always find it profitable to attract a worker who is paid less by offering a slightly higher wage. The zero-profit condition also implies $w(t_L) \ngtr \theta_L(1+\mu t_L)$, because else we must have $w(t_H) < \theta_H(1+\mu t_H)$, so that firms just break even. But then there exists a profitable contract for a rival firm that screens out low types and attracts all high types; thus $w(t_L) > \theta_L(1+\mu t_L)$ would result in negative expected profit for the firm that offers the contract.The low types choose the efficient level of effort t_L^{**} (as in the full-information case) in response to this wage function.

The equilibrium offer to the high type, $(w(t_H^*), t_H^*)$, must be incentive compatible, so that

$$u_L(t_H) = w(t_H^*) - c(t_H^*, \theta_L) \leq w(t_L^{**}) - c(t_L^{**}, \theta_L) = u_L(t_L^{**})$$

holds, inducing t_L^{**} from the low type. Equilibrium contracts must satisfy the additional condition that the high type prefers $(w(t_H^*), t_H^*)$ to any contract that offers $w(t_H^*) \leq \mathrm{E}_\Theta[\theta(1+\mu t^*(\theta))]$. Otherwise, firms have a positive-profit incentive to deviate to a pooling contract. Since $\mathrm{E}_\Theta[\theta(1+\mu t^*(\theta))]$ decreases in the ratio of low types to high types in the economy, the set of SPNE increases in this ratio. One separating equilibrium corresponds to the full-information equilibrium: $w(t_H^*) = \theta_H(1+\mu t_H^*)$, and the high type optimally chooses t_H^* that fulfills the first-order condition.

13.D.2

(a)

With an insurance policy (M, R), wealth is $W - M$ in case of no loss (where W is initial wealth), and $W - M + R - L$ (where L is the magnitude of the loss). Given W, the policy therefore determines state-contingent wealth.

(b)

The argument given here is heuristic; for more detail, refer to the graphical derivation and compare with the monopolistic insurance model. A SPNE is a separating equilibrium with zero profits and truthful "reporting" of types, i.e. incentive compatibility. The high-risk type must be fully insured at the actuarially fair price, $r(t_H)$. Firms are willing to provide full insurance that is actuarially fair to high-risk types, since there are no types with higher p_H, so firms always profit by attracting those who pay more than $r(t_H)$ by offering slightly cheaper insurance. Given that actuarially fair insurance is available, high-risk types accept it in equilibrium. The zero-profit condition also implies that insurance for high-risk types cannot sell for less than the actuarially fair price; else insurance for low-risk types must exceed the actuarially fair price, $r(t_L)$, so that firms break even. But then there exists a profitable contract for a rival insurer that screens out high risks and attracts all low risks; hence the firm that insures high risks would earn negative expected profit. This argument shows that in any SPNE actuarially fair insurance is offered to high-risk types, and purchased.

The equilibrium offer of insurance to the low-risk type must be incentive compatible, so that

$$\mathrm{E}[u_H(t_H)] \geq \mathrm{E}[u_H(t_L)]$$

holds, inducing t_H from the high-risk type. Since insurance is valuable to the low risks and (at fair odds) costless to the principals, low risks bid for the maximum coverage they can get within this constraint. Thus, incentive compatibility for high risks must bind. But the constraint does not allow low risks to insure fully, for such a policy would have to be cheaper than the high risk policy (given that low risks value insurance less) and thus attract the high risks.

Since equilibrium insurance for the high-risk type is actuarially fair, it is linear in p_L, and therefore induces a unique SPNE, if any. For a SPNE to exist, low-risk types must prefer low-risk insurance to any pooling contract. This implies that existence of the SPNE depends on having a high enough ratio of high-risk to low-risk types in the economy.

14.C.2

If the state of the world is publically observed, then the principal solves:

$$\begin{array}{cl} \max\limits_{\substack{w_H, e_H \geq 0 \\ w_L, e_L \geq 0}} & \left[\lambda\left(\pi\left(e_H\right)-w_H\right)+(1-\lambda)\left(\pi\left(e_L\right)-w_L\right)\right] \\ \text{s.t.} & \lambda v\left(w_H-g\left(e_H \mid \theta_H\right)\right)+(1-\lambda) v\left(w_L-g\left(e_L \mid \theta_L\right)\right) \geq \bar{u}. \end{array}$$

Denote the optimal mechanism that satisfies the first-order conditions,

$$w_i=v^{-1}(\bar{u})+g\left(e_i \mid \theta_i\right)$$

and

$$\frac{\partial \pi\left(e_i\right)}{\partial e_i}=\frac{\partial g\left(e_i \mid \theta_i\right)}{\partial e_i},$$

by $\left(w_i^{**}, e_i^{**}\right)$. The principal's expected payoff is

$$\begin{aligned} \hat{\pi}^{**} &= \lambda\left(\pi\left(e_H^{**}\right)-w_H^{**}\right)+(1-\lambda)\left(\pi\left(e_L^{**}\right)-w_L^{**}\right) \\ &= \lambda\left(\pi\left(e_H^{**}\right)-g\left(e_H^{**} \mid \theta_H\right)\right)+(1-\lambda)\left(\pi\left(e_L^{**}\right)-g\left(e_L^{**} \mid \theta_L\right)\right)-v^{-1}(\bar{u}). \end{aligned}$$

If, on the other hand, the state of the world is privately observed by the agent, then the principal solves:

$$\begin{array}{cl} \max\limits_{\substack{w_H, e_H \geq 0 \\ w_L, e_L \geq 0}} & \left[\lambda\left(\pi\left(e_H\right)-w_H\right)+(1-\lambda)\left(\pi\left(e_L\right)-w_L\right)\right] \\ \text{s.t.} & \lambda v\left(w_H-g\left(e_H \mid \theta_H\right)\right)+(1-\lambda) v\left(w_L-g\left(e_L \mid \theta_L\right)\right) \geq \bar{u} \\ & w_H-g\left(e_H \mid \theta_H\right) \geq w_L-g\left(e_L \mid \theta_H\right) \\ & w_L-g\left(e_L \mid \theta_L\right) \geq w_H-g\left(e_H \mid \theta_L\right). \end{array}$$

We are to show that, when the agent is risk-neutral, she accepts a wage $w^* (\pi) = \pi - \alpha$ which gives the principal

$$\begin{aligned} \hat{\pi}^* &= \lambda (\pi (e_H^*) - w_H^*) + (1 - \lambda) (\pi (e_L^*) - w_L^*) \\ &= \alpha = \hat{\pi}^{**}, \end{aligned}$$

or in other words, the agent is willing to "rent the firm" at the price $\alpha = \hat{\pi}^{**}$. (This is intuitively quite obvious, since a risk-neutral agent does not value insurance, so she requires no premium for incurring all the risk and provide optimal effort under perfectly aligned incentives.)

It is straightforward to show that the agent's participation and incentive constraints are satisfied if $\alpha = \hat{\pi}^{**}$, by subsitution of

$$\begin{aligned} w_H &= \pi (e_H^{**}) - \hat{\pi}^{**} \\ &= \pi (e_H^{**}) - \lambda (\pi (e_H^{**}) - g (e_H^{**} \mid \theta_H)) - (1 - \lambda) (\pi (e_L^{**}) - g (e_L^{**} \mid \theta_L)) - v^{-1} (\bar{u}) \\ &= (1 - \lambda) \pi (e_H^{**}) + \lambda g (e_H^{**} \mid \theta_H) - (1 - \lambda) \pi (e_L^{**}) + (1 - \lambda) g (e_L^{**} \mid \theta_L) + v^{-1} (\bar{u}) \end{aligned}$$

and

$$\begin{aligned} w_L &= \pi (e_L^{**}) - \hat{\pi}^{**} \\ &= \pi (e_L^{**}) - \lambda (\pi (e_H^{**}) - g (e_H^{**} \mid \theta_H)) - (1 - \lambda) (\pi (e_L^{**}) - g (e_L^{**} \mid \theta_L)) - v^{-1} (\bar{u}) \\ &= -\lambda \pi (e_H^{**}) + \lambda g (e_H^{**} \mid \theta_H) + \lambda \pi (e_L^{**}) + (1 - \lambda) g (e_L^{**} \mid \theta_L) + v^{-1} (\bar{u}). \end{aligned}$$

into each constraint. E.g., by the linearity of v, the participation constraint may be rewritten as follows:

$$\begin{aligned} & v (\lambda (w_H - g (e_H \mid \theta_H)) + (1 - \lambda) (w_L - g (e_L \mid \theta_L))) \geq \bar{u} \\ \Longrightarrow\ & \lambda (w_H - g (e_H \mid \theta_H)) + (1 - \lambda) (w_L - g (e_L \mid \theta_L)) - v^{-1} (\bar{u}) \geq 0 \\ \Longrightarrow\ & \lambda w_H + (1 - \lambda) w_L - \lambda g (e_H \mid \theta_H) - (1 - \lambda) g (e_L \mid \theta_L) - v^{-1} (\bar{u}) \geq 0. \end{aligned}$$

Substituting

$$\lambda w_H + (1 - \lambda) w_L = \lambda g (e_H^{**} \mid \theta_H) + (1 - \lambda) g (e_L^{**} \mid \theta_L) + v^{-1} (\bar{u}),$$

we find that the participation constraint holds with equality.

To verify the incentive constraints, note simply that the agent's optimization problem (given the proposed contract),

$$\begin{aligned} \max_{e \in \{e_H, e_L\}} \quad & [\lambda (w (\pi (e)) - g (e_H \mid \theta_H)) + (1 - \lambda) (w (\pi (e)) - g (e_L \mid \theta_L))] \\ \text{s.t.} \quad & \lambda v (w_H - g (e_H \mid \theta_H)) + (1 - \lambda) v (w_L - g (e_L \mid \theta_L)) \geq \bar{u} \\ & w_H - g (e_H \mid \theta_H) \geq w_L - g (e_L \mid \theta_H) \\ & w_L - g (e_L \mid \theta_L) \geq w_H - g (e_H \mid \theta_L), \end{aligned}$$

reduces to the full-information problem of the principal, since

$$w\left(\pi\left(e\right)\right)-g\left(e_i \mid \theta_i\right)=\pi\left(e\right)-\alpha+v^{-1}\left(\bar{u}\right)-w_i$$

and $-\alpha+v^{-1}\left(\bar{u}\right)$ is constant, so does not enter decision making. This implies that the agent chooses e_i^{**} in each state.

The revelation mechanism which implements the full-information outcome in the limited-information case is the pair $\left(w_i, e_i\right)=\left(\pi\left(e_i\right)-\alpha, e_i^{**}\right)$, with $\alpha=\hat{\pi}^{**}$, which - as we just argued - induces the agent to "disclose" the true state of the world i.

14.C.9

(a)

The full-information optimal insurance contract solves:

$$\begin{array}{ll}\max\limits_{c_1,c_2} & \displaystyle\int_i \left[\theta^i\left(W-L-c_2^i\right)+\left(1-\theta^i\right)\left(W-c_1^i\right)\right] f\left(\theta^i\right) di \\ \text{s.t.} & \theta^i u\left(c_2^i\right)+\left(1-\theta^i\right) u\left(c_1^i\right)\geq \bar{u}_i \text{ for all } i.\end{array}$$

There is a unique participation constraint for every i because the contract must be acceptable to the agent whatever his type (the type is assumed to be observed by the agent before deciding on the contract). The participation constraint holds with equality, since the insurer is a monopolist and extracts all surplus. First-order conditions are, for every i, $\mu^i=\frac{1}{u'\left(c_1^{i**}\right)}$ and $\mu^i=\frac{1}{u'\left(c_2^{i**}\right)}$ (where μ^i is the Lagrange multiplier of type i's participation constraint), hence $u'\left(c_1^{i**}\right)=u'\left(c_2^{i**}\right)$ and $c_1^{i**}=c_2^{i**}\equiv\bar{c}^i$ (the risk-averse agent is fully insured, given his type). Then the participation constraint reduces to

$$\theta^i u\left(\bar{c}^i\right)+\left(1-\theta^i\right) u^i\left(\bar{c}^i\right)=u\left(\bar{c}^i\right)=\bar{u},$$

so $\bar{c}^i=u^{-1}\left(\bar{u}\right)$.

The agent's reservation utility is the expected wealth if uninsured:

$$\bar{u}=\theta u\left(W-L\right)+\left(1-\theta\right) u\left(W\right).$$

Therefore, the optimal insurance contract specifies $\left(c_1^{i**}, c_2^{i**}\right)$ for all i, where

$$\begin{aligned} c_1^{i**} &= u^{-1}\left(\theta^i u\left(W-L\right)+\left(1-\theta^i\right) u\left(W\right)\right) \\ c_2^{i**} &= u^{-1}\left(\theta^i u\left(W-L\right)+\left(1-\theta^i\right) u\left(W\right)\right). \end{aligned}$$

(b)

In case of accident, the insurer receives the liability $W - L - c_2$ and the agent has wealth c_2; if there is no accident, the insurer receives a premium $W - c_1$ and the agent has wealth c_1. The principal solves:

$$\begin{array}{lll} \max_c & & \left[\begin{array}{c} \lambda\left(\theta_L\left(W - L - c_2^L\right) + (1-\theta_L)\left(W - c_1^L\right)\right) \\ +(1-\lambda)\left(\theta_H\left(W - L - c_2^H\right) + (1-\theta_H)\left(W - c_1^H\right)\right) \end{array} \right] \\ \text{s.t.} & \text{(PL)} & \theta_L u\left(c_2^L\right) + (1-\theta_L) u\left(c_1^L\right) \geq \bar{u}_L \\ & \text{(PH)} & \theta_H u\left(c_2^H\right) + (1-\theta_H) u\left(c_1^H\right) \geq \bar{u}_H \\ & \text{(IL)} & \theta_L u\left(c_2^L\right) + (1-\theta_L) u\left(c_1^L\right) \geq \theta_L u\left(c_2^H\right) + (1-\theta_L) u\left(c_1^H\right) \\ & \text{(IH)} & \theta_H u\left(c_2^H\right) + (1-\theta_H) u\left(c_1^H\right) \geq \theta_H u\left(c_2^L\right) + (1-\theta_H) u\left(c_1^L\right). \end{array}$$

The reservation utilities derive from the uninsured scenario:

$$\begin{aligned} \bar{u}_L &= \theta_L u(W-L) + (1-\theta_L) u(W) \\ \bar{u}_H &= \theta_H u(W-L) + (1-\theta_H) u(W). \end{aligned}$$

Substituting:

$$\begin{aligned} W - c_1 &= W - u^{-1}(u(c_1)) \\ W - L - c_2 &= W - L - u^{-1}(u(c_2)), \end{aligned}$$

and assuming, subject to verification, that (PH) and (IL) are redundant, and (PL) and (IH) bind, the principal's reduces to:

$$\begin{array}{lll} \max_u & & \left[\begin{array}{c} \lambda\left(W - \theta_L\left(L + u^{-1}u\left(c_2^L\right)\right) - (1-\theta_L) u^{-1}u\left(c_1^L\right)\right) \\ +(1-\lambda)\left(W - \theta_H\left(L + u^{-1}u\left(c_2^H\right)\right) - (1-\theta_H) u^{-1}u\left(c_1^H\right)\right) \end{array} \right] \\ \text{s.t.} & \text{(PL)} & \theta_L u\left(c_2^L\right) + (1-\theta_L) u\left(c_1^L\right) = \bar{u}_L \\ & \text{(IH)} & \theta_H u\left(c_2^H\right) + (1-\theta_H) u\left(c_1^H\right) = \theta_H u\left(c_2^L\right) + (1-\theta_H) u\left(c_1^L\right). \end{array}$$

To make the problem tractable, think of the principal as choosing utility levels for the agents that maximize expected profit. Call the Lagrange mutlipliers for the constraints μ_{PL} and μ_{IH}. First-order conditions with respect to (i) $u\left(c_1^L\right)$, (ii) $u\left(c_2^L\right)$, (iii) $u\left(c_1^H\right)$, and (iv) $u\left(c_2^H\right)$ are:

$$\begin{array}{ll} \text{(i)} & -\lambda(1-\theta_L)\left(u^{-1}\right)'\left(u\left(c_1^L\right)\right) + \mu_{PL}(1-\theta_L) - \mu_{IH}(1-\theta_H) = 0 \\ \text{(ii)} & -\lambda\theta_L\left(u^{-1}\right)'\left(u\left(c_2^L\right)\right) + \mu_{PL}\theta_L - \mu_{IH}\theta_H = 0 \\ \text{(iii)} & -(1-\lambda)(1-\theta_H)\left(u^{-1}\right)'\left(u\left(c_1^H\right)\right) + \mu_{IH}(1-\theta_H) = 0 \\ \text{(iv)} & -(1-\lambda)\theta_H\left(u^{-1}\right)'\left(u\left(c_2^H\right)\right) + \mu_{IH}\theta_H = 0. \end{array}$$

Rearrange (iii) and (iv) for μ_{IH}:

$$\begin{array}{ll} \text{(iii)} & \mu_{IH} = (1-\lambda)\left(u^{-1}\right)'\left(u\left(c_1^H\right)\right) \\ \text{(iv)} & \mu_{IH} = (1-\lambda)\left(u^{-1}\right)'\left(u\left(c_2^H\right)\right). \end{array}$$

Equating, we find that profit maximization implies full insurance for type H:

$$c_1^H = c_2^H \equiv c^H.$$

Moreover, notice that, since u is strictly concave (because the agent is risk-averse), u^{-1} is strictly convex. In particular, $\left(u^{-1}\right)'(\cdot) > 0$. This implies, from either (iii) or (iv), that $\mu_{IH} > 0$, i.e. the IH constraint binds. Next, sum (i) through (iv) and cancel out the μ_{IH} terms:

$$\begin{aligned}
& -\lambda\left(1-\theta_L\right)\left(u^{-1}\right)'\left(u\left(c_1^L\right)\right)+\mu_{PL}\left(1-\theta_L\right)-\mu_{IH}\left(1-\theta_H\right) \\
& -\lambda\theta_L\left(u^{-1}\right)'\left(u\left(c_2^L\right)\right)+\mu_{PL}\theta_L-\mu_{IH}\theta_H \\
& -\left(1-\lambda\right)\left(1-\theta_H\right)\left(u^{-1}\right)'\left(u\left(c_1^H\right)\right)+\mu_{IH}\left(1-\theta_H\right) \\
& -\left(1-\lambda\right)\theta_H\left(u^{-1}\right)'\left(u\left(c_2^H\right)\right)+\mu_{IH}\theta_H \\
= & -\lambda\left(1-\theta_L\right)\left(u^{-1}\right)'\left(u\left(c_1^L\right)\right)+\mu_{PL}\left(1-\theta_L\right)-\lambda\theta_L\left(u^{-1}\right)'\left(u\left(c_2^L\right)\right)+\mu_{PL}\theta_L \\
& -\left(1-\lambda\right)\left(1-\theta_H\right)\left(u^{-1}\right)'\left(u\left(c_1^H\right)\right)-\left(1-\lambda\right)\theta_H\left(u^{-1}\right)'\left(u\left(c_2^H\right)\right) \\
= & \ 0.
\end{aligned}$$

Solving for μ_{PL}, we find

$$\begin{aligned}
\mu_{PL} &= \lambda\left(1-\theta_L\right)\left(u^{-1}\right)'\left(u\left(c_1^L\right)\right)+\lambda\theta_L\left(u^{-1}\right)'\left(u\left(c_2^L\right)\right) \\
&\quad +\left(1-\lambda\right)\left(1-\theta_H\right)\left(u^{-1}\right)'\left(u\left(c_1^H\right)\right)+\left(1-\lambda\right)\theta_H\left(u^{-1}\right)'\left(u\left(c_2^H\right)\right) \\
&> 0.
\end{aligned}$$

Therefore, the PL and IH constraints indeed bind, as projected. Simplify the constraints as follows (using $u\left(c_2^L\right)-u\left(c_1^L\right) \equiv v^L$ and $c_1^H = c_2^H \equiv c^H$):

$$\begin{array}{ll}
\text{(PL)} & \theta_L\left(u\left(c_2^L\right)-u\left(c_1^L\right)\right)+u\left(c_1^L\right) \equiv \theta_L v^L + u\left(c_1^L\right) = \bar{u}_L \\
\text{(IH)} & u\left(c^H\right) = \theta_H\left(u\left(c_2^L\right)-u\left(c_1^L\right)\right)+u\left(c_1^L\right) \equiv \theta_H v^L + u\left(c_1^L\right).
\end{array}$$

Also, use PL to write:

$$\begin{aligned}
\bar{u}_L &= \theta_L u\left(c_2^L\right)+\left(1-\theta_L\right)u\left(c_1^L\right) \\
&= \theta_L u\left(c_2^L\right)+u\left(c_2^L\right)-u\left(c_2^L\right)+\left(1-\theta_L\right)u\left(c_1^L\right) \\
&= u\left(c_2^L\right)-\left(1-\theta_L\right)\left(u\left(c_2^L\right)-u\left(c_1^L\right)\right) \\
&= u\left(c_2^L\right)-\left(1-\theta_L\right)v^L.
\end{aligned}$$

Substitute $u\left(c_2^L\right) = \bar{u}_L+\left(1-\theta_L\right)v^L$ and $u\left(c_1^L\right) = \bar{u}_L-\theta_L v^L$ into the principal's objective to get a problem in v^L:

$$\begin{array}{lll}
\max_{v^L} & & \left[\begin{array}{c} \lambda\left(W-\theta_L\left(L+u^{-1}\left(\bar{u}_L+\left(1-\theta_L\right)v^L\right)\right)-\left(1-\theta_L\right)u^{-1}\left(\bar{u}_L-\theta_L v^L\right)\right) \\ +\left(1-\lambda\right)\left(W-\theta_H L-u^{-1}u\left(c^H\right)\right) \end{array}\right] \\
\text{s.t.} & \text{(PL)} & \theta_L\left(\bar{u}_L+\left(1-\theta_L\right)v^L\right)+\left(1-\theta_L\right)\left(\bar{u}_L-\theta_L v^L\right) = \bar{u}_L \\
& \text{(IH)} & u\left(c^H\right) = \theta_H\left(\bar{u}_L+\left(1-\theta_L\right)v^L\right)+\left(1-\theta_H\right)\left(\bar{u}_L-\theta_L v^L\right).
\end{array}$$

Think of the principal now as choosing the utility difference for type L between the two states in a profit-maximizing contract. (It will be zero if and only if L is fully insured.) Maximizing the Lagrangian with respect to v^L:

$$\begin{aligned} & \lambda\theta_L(1-\theta_L)\left(u^{-1}\right)'\left(\bar{u}_L-\theta_L v^L\right)-\lambda\theta_L(1-\theta_L)\left(u^{-1}\right)'\left(\bar{u}_L+(1-\theta_L)v^L\right) \\ & -(1-\lambda)\gamma \\ & +\mu_{PL}\left[\theta_L(1-\theta_L)-(1-\theta_L)\theta_L\right] \\ & +\mu_{IH}\left[(1-\theta_H)\theta_L-\theta_H(1-\theta_L)\right] \\ = & \ 0, \end{aligned}$$

where

$$\gamma \equiv \left(u^{-1}\right)'\left(u\left(c^H\right)\right)u'\left(c^H\right) > 0,$$

since u^{-1} is strictly convex, u is strictly concave. The μ_{PL} term cancels out, and $\mu_{IH} = (1-\lambda)\left(u^{-1}\right)'\left(u\left(c^H\right)\right)$ from above. Dividing through by $\lambda\theta_L(1-\theta_L)$, and rearranging:

$$\begin{aligned} & \left(u^{-1}\right)'\left(\bar{u}_L-\theta_L v^L\right)-\left(u^{-1}\right)'\left(\bar{u}_L+(1-\theta_L)v^L\right) \\ = & \ \frac{1-\lambda}{\lambda}\frac{1}{\theta_L(1-\theta_L)}\gamma\left(\left(c^H\right)'\left(v^L\right)+(\theta_H-\theta_L)\frac{1}{u'\left(c^H\right)}\right). \end{aligned}$$

The righthand side is strictly positive, since $\left(c^H\right)'\left(v^L\right)>0$; c^H must increase in insurance for type L because the incentive compatibility constraint for H binds. Hence

$$\left(u^{-1}\right)'\left(\bar{u}_L-\theta_L v^L\right) > \left(u^{-1}\right)'\left(\bar{u}_L+(1-\theta_L)v^L\right),$$

which implies $-\theta_L v^L > (1-\theta_L)v^L$, or

$$v^L < 0.$$

Recall that $v^L = u\left(c_2^L\right)-u\left(c_1^L\right)$. This means that

$$c_1^L > c_2^L;$$

in other words, L is not fully insured.

We should check that the two ignored constraints are really satisfied. If $c_1^L > c_2^L$, then $\theta_H > \theta_L$ implies

$$\theta_L u\left(c_2^L\right)+(1-\theta_L)u\left(c_1^L\right) > \theta_H u\left(c_2^L\right)+(1-\theta_H)u\left(c_1^L\right),$$

which is the IL constraint. PH is not so easy to verify - given that the other constraints hold, PH may fail when the parameters of the problem are such that no equilibrium exists. We can use

$$\begin{aligned}\bar{u}_L &= \theta_L u(W-L) + (1-\theta_L) u(W) \\ &> \theta_H u(W-L) + (1-\theta_H) u(W) \\ &= \bar{u}_H\end{aligned}$$

and PL to show that

$$\begin{aligned}\bar{u}_L &= \theta_L u\left(c_2^L\right) + (1-\theta_L) u\left(c_1^L\right) \\ &= \theta_L\left(u\left(c_2^L\right) - u\left(c_1^L\right)\right) + u\left(c_1^L\right) \\ &< \theta_H\left(u\left(c_2^H\right) - u\left(c_1^H\right)\right) + u\left(c_1^L\right) \\ &= \theta_H u\left(c_2^H\right) + u\left(c_1^L\right) - \theta_H u\left(c_1^H\right).\end{aligned}$$

Hence PH holds if $u\left(c_1^H\right)$ is not too large relative to $u\left(c_1^L\right)$; an obvious statement (saying that the price of insurance for H should not be too high). This condition could be characterized more meaningfully, but a full discussion would lead too far afield into existence conditions for an equilibrium.

Generally, the low type is underinsured and would like to purchase more insurance at better-than-fair odds (from the perspective of the insurer). This is obvious from the fact that L is risk-averse and therefore values insurance, whereas the principal is risk-neutral and can provide fair insurance at no cost. But increasing the L type's coverage, while maintaining incentive compatibility for the H type (remember that IH binds), would require that the insurer decrease the coverage or cost of insurance for high types. This makes the provision of additional insurance to L costly to the principal, and in equilibrium the L type does not value insurance enough to compensate the principal fully.

(c)

Both in 14.C.9 (monopolist principal) and 13.D.2 (competitive principal), high risks are fully insured, and low risks are partially insured. The differences are (i) that the competitive insurer earns zero profit, whereas the monopolist insurer generally earns positive profit, and (ii) that the L type can have a surplus in the competitive case (participation constraint does not bind), but not in the monopolist case (participation constraint binds). Moreover, a monopolist principal is more likely not to offer insurance to low risks at all, if the ratio of H to L types in the economy is small. In this case, the cost of designing the low-risk contract so that it avoids pooling is high, particularly if the principal is able to extract monopoly rents from high risks.

15.B.1

(a)

Nonsatiation implies Walras' law, so

$$\begin{aligned} p_1 x_{1i} + p_2 x_{2i} &= \omega_i \\ &= p_1 \omega_{1i} + p_2 \omega_{2i} \end{aligned}$$

for all i.

Adding the equations for consumer 1 and 2, and rearranging,

$$\begin{aligned} p_1 x_{11} + p_2 x_{21} + p_1 x_{12} + p_2 x_{22} &= p_1 \omega_{11} + p_2 \omega_{21} + p_1 \omega_{12} + p_2 \omega_{22} \\ p_1 \sum_{i=1}^{2} x_{1i} + p_2 \sum_{i=1}^{2} x_{2i} &= p_1 \sum_{i=1}^{2} \omega_{1i} + p_2 \sum_{i=1}^{2} \omega_{2i} \\ p_1 \left(\sum_{i=1}^{2} x_{1i} - \omega_1 \right) + p_2 \left(\sum_{i=1}^{2} x_{2i} - \omega_2 \right) &= 0, \end{aligned}$$

where $\omega_1 = \sum_i \omega_{1i}$ and $\omega_2 = \sum_i \omega_{2i}$.

(b)

If the market for good 1 clears, then $\sum_i x_{1i} - \omega_1 = 0$. This implies, from *(a)*,

$$p_2 \left(\sum_{i=1}^{2} x_{2i} - \omega_2 \right) = 0,$$

i.e. $\sum_i x_{2i} - \omega_2 = 0$ (the market for good 2 clears).

15.B.6

The equilibrium is determined by the agent's objectives,

$$\begin{aligned} u_1 (x_{11}, x_{21}) &= \frac{1}{\sqrt{\frac{1}{x_{11}^2} + \left(\frac{12}{37}\right)^3 \frac{1}{x_{21}^2}}} \\ u_2 (x_{12}, x_{22}) &= \frac{1}{\sqrt{\left(\frac{12}{37}\right)^3 \frac{1}{x_{12}^2} + \frac{1}{x_{22}^2}}}, \end{aligned}$$

the agent's budget constraints,

$$\begin{aligned} p_1 x_{11} + p_2 x_{21} &= p_1 \omega_{11} + p_2 \omega_{21} = p_1 \\ p_1 x_{12} + p_2 x_{22} &= p_1 \omega_{12} + p_2 \omega_{22} = p_2, \end{aligned}$$

and the market clearing condition,

$$x_{11}^{*}+x_{12}^{*}=1.$$

(The second market will clear if the first market clears.)

The objective functions are strictly increasing in x_1 and x_2, so that Walras' law holds. The first-order conditions of the first agent's maximization problem,

$$\max_{x_{11}.x_{21}} u_1+\lambda_1\left(p_1-p_1x_{11}-p_2x_{21}\right),$$

are:

$$\begin{aligned}\left(x_{11}^{*-2}+\left(\frac{12}{37}\right)^3 x_{21}^{*-2}\right)^{-\frac{3}{2}} x_{11}^{*-3} &= \lambda_1^{*}p_1\\ \left(x_{11}^{*-2}+\left(\frac{12}{37}\right)^3 x_{21}^{*-2}\right)^{-\frac{3}{2}}\left(\frac{12}{37}\right)^3 x_{21}^{*-3} &= \lambda_1^{*}p_2.\end{aligned}$$

Second-order conditions for a maximum hold for all $\mathbf{x}^{*}$.

Thus:

$$x_{11}^{*}=\left(\frac{p_1}{p_2}\right)^{-\frac{1}{3}}\left(\frac{12}{37}\right)^{-1} x_{21}^{*}.$$

Using $x_{21}=\frac{p_1}{p_2}\left(1-x_{11}\right)$ from the budget constraint,

$$\begin{aligned}x_{11}^{*} &= \frac{1}{\frac{12}{37}\left(\frac{p_1}{p_2}\right)^{-\frac{2}{3}}+1}\\ x_{21}^{*} &= \frac{\frac{p_1}{p_2}}{\left(\frac{12}{37}\right)^{-1}\left(\frac{p_1}{p_2}\right)^{\frac{2}{3}}+1}.\end{aligned}$$

The first-order conditions of the second agent's maximization problem,

$$\max_{x_{12}.x_{22}} u_2+\lambda_2\left(p_2-p_1x_{12}-p_2x_{22}\right),$$

are:

$$\begin{aligned}\left(\left(\frac{12}{37}\right)^3 x_{12}^{*-2}+x_{22}^{*-2}\right)^{-\frac{3}{2}}\left(\frac{12}{37}\right)^3 x_{12}^{*-3} &= \lambda_1^{*}p_1\\ \left(\left(\frac{12}{37}\right)^3 x_{12}^{*-2}+x_{22}^{*-2}\right)^{-\frac{3}{2}} x_{22}^{*-3} &= \lambda_1^{*}p_2.\end{aligned}$$

Thus:

$$x_{12}^* = \left(\frac{p_1}{p_2}\right)^{-\frac{1}{3}} \left(\frac{12}{37}\right) x_{22}^*.$$

Using $x_{22} = 1 - \frac{p_1}{p_2} x_{12}$ from the budget constraint,

$$\begin{aligned} x_{12}^* &= \frac{\left(\frac{p_1}{p_2}\right)^{-1}}{\left(\frac{12}{37}\right)^{-1}\left(\frac{p_1}{p_2}\right)^{-\frac{2}{3}} + 1} \\ x_{22}^* &= \frac{1}{\frac{12}{37}\left(\frac{p_1}{p_2}\right)^{\frac{2}{3}} + 1}. \end{aligned}$$

Now market clearing requires that

$$\begin{aligned} x_{11}^* + x_{12}^* &= 1 \\ \frac{1}{\frac{12}{37}\left(\frac{p_1}{p_2}\right)^{-\frac{2}{3}} + 1} + \frac{\left(\frac{p_1}{p_2}\right)^{-1}}{\left(\frac{12}{37}\right)^{-1}\left(\frac{p_1}{p_2}\right)^{-\frac{2}{3}} + 1} &= 1. \end{aligned}$$

Simplifying:

$$\left(\frac{p_1}{p_2}\right)^{-1} - \frac{37}{12}\left(\frac{p_1}{p_2}\right)^{-\frac{2}{3}} + \frac{37}{12}\left(\frac{p_1}{p_2}\right)^{-\frac{1}{3}} - 1 = 0.$$

After a change of variables, taking $y \equiv \left(\frac{p_1}{p_2}\right)^{-\frac{1}{3}}$, this is a cubic equation in y:

$$y^3 - \frac{37}{12}y^2 + \frac{37}{12}y - 1 = 0.$$

Clearly, $y_1 = 1$ is a solution. The polynomial can be factored as

$$(y-1)\left(y^2 - \frac{25}{12}y + 1\right) = 0,$$

and the remaing roots are found by solving the quadratic equation $y^2 - \frac{25}{12}y + 1 = 0$ for $y_2 = \frac{3}{4}$ and $y_3 = \frac{4}{3}$. Reversing the change of variables, $\frac{p_1}{p_2} = y^{-3}$ and the three solutions are:

$$\begin{aligned} \frac{p_1}{p_2} &= 1 \\ \frac{p_1}{p_2} &= \frac{64}{27} \\ \frac{p_1}{p_2} &= \frac{27}{64}. \end{aligned}$$

The three Walrasian equilibria are:

$$\begin{aligned}\frac{p_1}{p_2} &= 1\\ \mathbf{x}^* &= \left\{\left(\frac{37}{49},\frac{12}{49}\right),\left(\frac{12}{49},\frac{37}{49}\right)\right\}\end{aligned}$$

$$\begin{aligned}\frac{p_1}{p_2} &= \frac{64}{27}\\ \mathbf{x}^* &= \left\{\left(\frac{148}{175},\frac{64}{175}\right),\left(\frac{27}{175},\frac{111}{175}\right)\right\}\end{aligned}$$

$$\begin{aligned}\frac{p_1}{p_2} &= \frac{27}{64}\\ \mathbf{x}^* &= \left\{\left(\frac{111}{175},\frac{27}{175}\right),\left(\frac{64}{175},\frac{148}{175}\right)\right\}.\end{aligned}$$

15.B.7

For strongly montone, continuous, and convex preference relations, the Pareto set in the Edgeworth box is connected.

Proof. The result follows from an application of the maximum theorem, which states that the solution to a parameterized family of optimization problems is continuous in the parameter, if the objective function is continuous and the constraint set is continuous and compact. The trick is to formulate the problem so that it fits the conditions of the maximum theorem.

To that end, define the Pareto set as the set of allocations $\mathbf{x}^*(\overline{u}_2) = (\mathbf{x}_1^*(\overline{u}_2), \mathbf{x}_2^*(\overline{u}_2))$, where $\mathbf{x}_1^*(\overline{u}_2)$ is the solution to

$$\begin{aligned}\mathbf{x}_1^*(\overline{u}_2) = \quad & \arg\max_{\mathbf{x}_1} u_1(\mathbf{x}_1)\\ & \text{s.t. } u_2(\boldsymbol{\omega} - \mathbf{x}_1) \geq \overline{u}_2,\end{aligned}$$

and

$$\mathbf{x}_2^*(\overline{u}_2) = \boldsymbol{\omega} - \mathbf{x}_1^*(\overline{u}_2).$$

In other words, we consider a family of problems in which 2 is given a minimal utility $\overline{u}_2 \in [u_2(\mathbf{0}), u_2(\boldsymbol{\omega})]$, and 1's utility is maximized subject to this constraint. It is easy to see that $P = \{\mathbf{x}^*(\overline{u}_2) : \overline{u}_2 \in [u_2(\mathbf{0}), u_2(\boldsymbol{\omega})]\}$ is indeed the Pareto set. Since $u_2(\mathbf{x})$ is necessarily in the interval for all allocations, and $\mathbf{x}_1^*(\overline{u}_2)$ maximizes $u_1(\mathbf{x}_1 | u_2 = \overline{u}_2)$ in any Pareto-optimal allocation, P

contains the Pareto set. On the other hand, the Pareto set contains P because any allocation where $\mathbf{x}_1^*(\overline{u}_2)$ maximizes $u_1(\mathbf{x}_1|u_2 = \overline{u}_2)$ and $u_2(\mathbf{x})$ is optimal given $\mathbf{x}_1^*$ (it is, due to monotonicity) is Pareto-optimal.

We have: (i) a family of optimization problems valued by $\mathbf{x}^* \in \mathbb{R}^2$, parameterized by $\overline{u}_2$, with a continuous objective function u_1; (ii) a constraint set $C(\overline{u}_2) = \{\mathbf{x}_1 : u_2(\boldsymbol{\omega} - \mathbf{x}_1) \geq \overline{u}_2\}$ that is continuous (by continuity of u_2) and compact (closed and bounded). Under these conditions, the maximum theorem guarantees that the solution correspondence $\mathbf{x}^*(\overline{u}_2)$ is upper-hemicontinuous. Since preferences are convex, $\mathbf{x}^*(\overline{u}_2)$ is in fact a function (there is a unique optimal allocation for every $\overline{u}_2$), and therefore continuous.

What we have shown is that the Pareto set is the continuous image of the interval $I = [u_2(\mathbf{0}), u_2(\boldsymbol{\omega})]$ under the function

$$f(\overline{u}_2) = (\mathbf{x}_1^*(\overline{u}_2), \mathbf{x}_2^*(\overline{u}_2)) = \left(\arg\max_{\mathbf{x}_1 \in C(\overline{u}_2)} u_1(\mathbf{x}_1),\ \boldsymbol{\omega} - \mathbf{x}_1^*(\overline{u}_2)\right)$$

constructed above. Every interval is a connected set, and every continuous image of a connected set is connected. To see this, consider a nonempty open set of allocations $A \subseteq f(I)$ with a nonempty open complement A^c in I. (I.e. suppose $f(I)$ is not connected.) Since I is connected, there exists (by the definition of connectedness) no partition of I into nonempty open sets. By continuity of f, the inverse images of A and A^c (i.e. the sets $f^{-1}(A) = \{\overline{u}_2 \in I : \mathbf{x}^*(\overline{u}_2) \in A\}$ and $f^{-1}(A^c) = \{\overline{u}_2 \in I : \mathbf{x}^*(\overline{u}_2) \in A^c\}$) are disjoint and open. Therefore, one of $f^{-1}(A)$ or $f^{-1}(A^c)$ must be empty, implying that either A or A^c is empty - a contradiction. Hence we argue that the Pareto set is connected, as it is the continuous image of a connected set. ■

Now suppose that the preferences are also homothetic. Then the points of tangency lie on a ray from the origin. But the allocations where the indifference curves are tangent contain the Pareto set. Since the Pareto set is connected, Pareto-optimal allocations lie on both sides of the 45 degree line only if there exists a Pareto-optimal allocation *on* the 45 degree line. But then *all* Pareto-optimal allocations must lie on the 45 degree line, since it is a ray from the origin. Therefore, in the case of homothetic preferences, the Pareto set lies weakly on one side of the 45 degree line.

15.C.2

The equilibrium $(\mathbf{p}, \mathbf{x}) = \left(\frac{w}{p}, x_1, x_2\right)$ is determined by:

(1)	Consumer's objective function:	$u(x_1, x_2) = \ln x_1 + \ln x_2$
(2)	Consumer's budget constraint:	$px_2 \leq w(L - x_1) + \pi$

(3) Firm's objective function: $p\sqrt{z} - wz$
(4) Clearing condition for the goods market: $\sqrt{z^*} = x_2^*$
(5) Clearing condition for the factor market: $z^* = L - x_1^* = 1 - x_1^*$.

The consumer solves

$$\max_{x_1, x_2} \ln x_1 + \ln x_2 + \lambda \left(w \left(L - x_1 \right) + \pi - p x_2 \right).$$

(Since the objective function is concave, the budget constraint holds with equality and first-order conditions are sufficient for a maximum.) Note that π is a function of z, but not of x_1 or x_2. From first-order conditions:

$$\frac{w}{p} = \frac{x_2^*}{x_1^*}.$$

The firm solves

$$\max_z p\sqrt{z} - wz.$$

First-order condition:

$$\frac{w}{p} = \frac{1}{2\sqrt{z^*}}.$$

(Again the firm's objective function is concave, and the first-order condition is sufficient for a maximum.)

Impose the goods market clearing condition to rewrite the firm's first-order condition as

$$\frac{w}{p} = \frac{1}{2x_2^*}.$$

Solve the first-order conditions for x_2^* in terms of x_1^*:

$$x_2^* = \sqrt{\frac{x_1^*}{2}}.$$

The factor market clearing condition (with z^* replaced from the good market condition) gives:

$$\begin{aligned} \sqrt{1 - x_1^*} &= x_2^* \\ x_1^* &= 1 - x_2^{*2} \\ &= 1 - \frac{x_1^*}{2} \\ x_1^* &= \frac{2}{3}, \end{aligned}$$

so

$$x_2^* = \frac{1}{\sqrt{3}}.$$

It remains to find the equilibrium relative price:

$$\begin{aligned}\frac{w}{p} &= \frac{1}{2x_2^*} \\ &= \frac{\sqrt{3}}{2}.\end{aligned}$$

Substituting into the profit function:

$$\begin{aligned}\pi &= p\sqrt{z^*} - wz^* \\ &= p\sqrt{1 - x_1^*} - w\left(1 - x_1^*\right) \\ &= \frac{1}{2\sqrt{3}}p.\end{aligned}$$

15.D.8

After substituting $z_{21} = \overline{z}_1 - z_{11}$ and $z_{22} = \overline{z}_2 - z_{21}$, the objective is:

$$\max_{z_{11}, z_{12}} f(\mathbf{z}) = z_{11}^{\frac{2}{3}} z_{12}^{\frac{1}{3}} + \left(\overline{z}_1 - z_{21}\right)^{\frac{1}{3}} \left(\overline{z}_2 - z_{22}\right)^{\frac{2}{3}}.$$

Because factor markets clear in a competitive economy, costs are constant and do not affect the optimal allocation of inputs to technologies. We must, however, keep nonnegativity constraints in mind:

$$\begin{aligned}z_{11} &\geq 0 \\ z_{11} &\leq \overline{z}_1 \\ z_{12} &\geq 0 \\ z_{12} &\leq \overline{z}_2.\end{aligned}$$

Suppose the solution is interior. Then the nonnegativity constraints do not bind and the first-order conditions are:

$$\begin{aligned}\frac{\partial f}{\partial z_{11}} &= \frac{2}{3} z_{11}^{*-\frac{1}{3}} z_{12}^{\frac{1}{3}} - \frac{1}{3}\left(\overline{z}_1 - z_{11}^*\right)^{-\frac{2}{3}} \left(\overline{z}_2 - z_{12}\right)^{\frac{2}{3}} = 0 \\ \frac{\partial f}{\partial z_{21}} &= \frac{1}{3} z_{11}^{\frac{2}{3}} z_{12}^{*-\frac{2}{3}} - \frac{2}{3}\left(\overline{z}_1 - z_{11}\right)^{\frac{1}{3}} \left(\overline{z}_2 - z_{12}^*\right)^{-\frac{1}{3}} = 0.\end{aligned}$$

These imply, respectively,

$$\begin{aligned} 2z_{11}^{*-\frac{1}{3}} z_{12}^{\frac{1}{3}} &= (\overline{z}_1 - z_{11}^*)^{-\frac{2}{3}} (\overline{z}_2 - z_{12})^{\frac{2}{3}} \\ \frac{z_{11}^*}{z_{12}} &= 8 \left(\frac{\overline{z}_1 - z_{11}^*}{\overline{z}_2 - z_{12}} \right)^2 \end{aligned}$$

and

$$\begin{aligned} z_{11}^{\frac{2}{3}} z_{12}^{*-\frac{2}{3}} &= 2 (\overline{z}_1 - z_{11})^{\frac{1}{3}} (\overline{z}_2 - z_{12}^*)^{-\frac{1}{3}} \\ \frac{z_{11}}{z_{12}^*} &= \sqrt{8} \left(\frac{\overline{z}_1 - z_{11}}{\overline{z}_2 - z_{12}^*} \right)^{\frac{1}{2}}. \end{aligned}$$

Equating the righthand expressions, we obtain:

$$\frac{\overline{z}_1 - z_{11}^*}{\overline{z}_2 - z_{12}^*} = \frac{1}{2}.$$

Alternatively, the first-order conditions can be transformed into

$$\frac{1}{\sqrt{8}} \sqrt{\frac{z_{11}^*}{z_{12}}} = \frac{\overline{z}_1 - z_{11}^*}{\overline{z}_2 - z_{12}}$$

and

$$\frac{1}{8} \left(\frac{z_{11}}{z_{12}^*} \right)^2 = \frac{\overline{z}_1 - z_{11}}{\overline{z}_2 - z_{12}^*}.$$

Equating the lefthand expressions,

$$\frac{z_{11}^*}{z_{12}^*} = 2.$$

Now we can solve for the interior solution in terms of endowments:

$$\begin{aligned} z_{11}^* &= \overline{z}_1 - \frac{1}{2} (\overline{z}_2 - z_{12}^*) \\ &= \overline{z}_1 - \frac{1}{2} \left(\overline{z}_2 - \frac{z_{11}^*}{2} \right) \\ z_{11}^* &= \frac{4}{3}\overline{z}_1 - \frac{2}{3}\overline{z}_2 \\ z_{12}^* &= \frac{z_{11}^*}{2} = \frac{2}{3}\overline{z}_1 - \frac{1}{3}\overline{z}_2 \\ z_{21}^* &= \overline{z}_1 - z_{11}^* = \frac{2}{3}\overline{z}_2 - \frac{1}{3}\overline{z}_1 \\ z_{22}^* &= \overline{z}_2 - z_{12}^* = \frac{4}{3}\overline{z}_2 - \frac{2}{3}\overline{z}_1. \end{aligned}$$

In specialized solutions, either $z_{11} = z_{12} = 0$ or $z_{21} = z_{22} = 0$ (since no output can be produced by either technology if one of the inputs is zero, any other corner allocation would be wasteful). In the first case, $\frac{\partial f}{\partial z_{11}} < 0$ and $\frac{\partial f}{\partial z_{12}} < 0$. In the second case, the inequalities are reversed. Therefore, the first condition for a specialized solution (with $z_{21} = z_{22} = 0$) is:

$$\frac{\overline{z}_1 - z_{11}^*}{\overline{z}_2 - z_{12}} < \frac{1}{2} \Rightarrow \frac{\overline{z}_1}{\overline{z}_2} < \frac{1}{2}.$$

The input vectors is then $\mathbf{z}' = (z_{11}, z_{12}, z_{21}, z_{22}) = (0, 0, \overline{z}_1, \overline{z}_2)$. The second condition for specialized solution (with $z_{11} = \overline{z}_1$, $z_{12} = \overline{z}_2$) is:

$$\frac{z_{11}^*}{z_{12}^*} > 2 \Rightarrow \frac{\overline{z}_1}{\overline{z}_2} > 2,$$

with input vector $\mathbf{z}'' = (z_{11}, z_{12}, z_{21}, z_{22}) = (\overline{z}_1, \overline{z}_2, 0, 0)$. Hence, the interior solution occurs if and only if

$$\frac{1}{2} \leq \frac{\overline{z}_1}{\overline{z}_2} \leq 2.$$

In a competitive economy, input prices equal the marginal value products (which are constant under the constant returns assumption). Thus:

$$\begin{aligned} w_1^* &= p_1 \frac{\partial f(\mathbf{z}^*)}{\partial \overline{z}_1} = \frac{\partial f(\mathbf{z}^*)}{\partial \overline{z}_1} \\ w_2^* &= p_2 \frac{\partial f(\mathbf{z}^*)}{\partial \overline{z}_2} = \frac{\partial f(\mathbf{z}^*)}{\partial \overline{z}_2}. \end{aligned}$$

Since

$$\begin{aligned} f(\mathbf{z}^*) &= z_{11}^{*\frac{2}{3}} z_{12}^{*\frac{1}{3}} + (\overline{z}_1 - z_{11}^*)^{\frac{1}{3}} (\overline{z}_2 - z_{12}^*)^{\frac{2}{3}} \\ &= \left(\frac{4}{3}\overline{z}_1 - \frac{2}{3}\overline{z}_2\right)^{\frac{2}{3}} \left(\frac{2}{3}\overline{z}_1 - \frac{1}{3}\overline{z}_2\right)^{\frac{1}{3}} + \left(\frac{2}{3}\overline{z}_2 - \frac{1}{3}\overline{z}_1\right)^{\frac{1}{3}} \left(\frac{4}{3}\overline{z}_2 - \frac{2}{3}\overline{z}_1\right)^{\frac{2}{3}} \\ &= 2^{\frac{2}{3}} \left(\frac{2}{3}\overline{z}_1 - \frac{1}{3}\overline{z}_2\right)^{\frac{2}{3}} \left(\frac{2}{3}\overline{z}_1 - \frac{1}{3}\overline{z}_2\right)^{\frac{1}{3}} + 2^{\frac{2}{3}} \left(\frac{2}{3}\overline{z}_2 - \frac{1}{3}\overline{z}_1\right)^{\frac{1}{3}} \left(\frac{2}{3}\overline{z}_2 - \frac{1}{3}\overline{z}_1\right)^{\frac{2}{3}} \\ &= 2^{\frac{2}{3}} \left(\frac{2}{3}\overline{z}_1 - \frac{1}{3}\overline{z}_2\right) + 2^{\frac{2}{3}} \left(\frac{2}{3}\overline{z}_2 - \frac{1}{3}\overline{z}_1\right) \\ &= \frac{2^{\frac{2}{3}}}{3} (\overline{z}_1 + \overline{z}_2), \end{aligned}$$

we have

$$w_1^* = w_2^* = \frac{2^{\frac{2}{3}}}{3}$$

in the interior solution. In the specialized solutions, $f(\mathbf{z}') = \overline{z}_1^{\frac{2}{3}}\overline{z}_2^{\frac{1}{3}}$ and $f(\mathbf{z}'') = \overline{z}_1^{\frac{1}{3}}\overline{z}_2^{\frac{2}{3}}$, respectively, so:

$$\begin{aligned} w_1' &= p_1\frac{\partial f(\mathbf{z}')}{\partial \overline{z}_1} = \frac{\partial f(\mathbf{z}')}{\partial \overline{z}_1} = \frac{2}{3}\left(\frac{\overline{z}_2}{\overline{z}_1}\right)^{\frac{1}{3}} \\ w_2' &= p_2\frac{\partial f(\mathbf{z}')}{\partial \overline{z}_2} = \frac{\partial f(\mathbf{z}')}{\partial \overline{z}_2} = \frac{1}{3}\left(\frac{\overline{z}_1}{\overline{z}_2}\right)^{\frac{2}{3}}. \end{aligned}$$

and

$$\begin{aligned} w_1'' &= p_1\frac{\partial f(\mathbf{z}'')}{\partial \overline{z}_1} = \frac{\partial f(\mathbf{z}'')}{\partial \overline{z}_1} = \frac{1}{3}\left(\frac{\overline{z}_2}{\overline{z}_1}\right)^{\frac{2}{3}} \\ w_2'' &= p_2\frac{\partial f(\mathbf{z}'')}{\partial \overline{z}_2} = \frac{\partial f(\mathbf{z}'')}{\partial \overline{z}_2} = \frac{2}{3}\left(\frac{\overline{z}_1}{\overline{z}_2}\right)^{\frac{1}{3}}. \end{aligned}$$

In summary, the complete solution is as follows. If $\frac{1}{2} \leq \frac{\overline{z}_1}{\overline{z}_2} \leq 2$, then the optimal inputs allocation is:

$$\begin{aligned} z_{11}^* &= \frac{4}{3}\overline{z}_1 - \frac{2}{3}\overline{z}_2 \\ z_{12}^* &= \frac{2}{3}\overline{z}_1 - \frac{1}{3}\overline{z}_2 \\ z_{21}^* &= \frac{2}{3}\overline{z}_2 - \frac{1}{3}\overline{z}_1 \\ z_{22}^* &= \frac{4}{3}\overline{z}_2 - \frac{2}{3}\overline{z}_1, \end{aligned}$$

and equilibrium input prices are:

$$w_1^* = w_2^* = \frac{2^{\frac{2}{3}}}{3}.$$

If $\frac{\overline{z}_1}{\overline{z}_2} < \frac{1}{2}$, then the optimal inputs allocation is:

$$\begin{aligned} z_{11}' &= 0 \\ z_{12}' &= 0 \\ z_{21}' &= \overline{z}_1 \\ z_{22}' &= \overline{z}_2, \end{aligned}$$

and equilibrium input prices are:

$$w_1' = \frac{2}{3}\left(\frac{\overline{z}_2}{\overline{z}_1}\right)^{\frac{1}{3}}$$
$$w_2' = \frac{1}{3}\left(\frac{\overline{z}_1}{\overline{z}_2}\right)^{\frac{2}{3}}.$$

If $\frac{\overline{z}_1}{\overline{z}_2} > 2$, then the optimal inputs allocation is:

$$z_{11}^{'} = \overline{z}_1$$
$$z_{12}' = \overline{z}_2$$
$$z_{21}' = 0$$
$$z_{22}' = 0,$$

and equilibrium input prices are:

$$w_1'' = \frac{1}{3}\left(\frac{\overline{z}_2}{\overline{z}_1}\right)^{\frac{2}{3}}$$
$$w_2'' = \frac{2}{3}\left(\frac{\overline{z}_1}{\overline{z}_2}\right)^{\frac{1}{3}}.$$

15.D.9

Heckscher-Ohlin theorem: In a 2x2 model (tradable goods 1 and 2, non-tradable factors A and B), where two economies have identical CRS technologies, identical increasing, concave, and homogeneous utility functions, and good 1 is intensive in factor A, the economy that is relatively abundant in factor A exports good 1.

Proof. Free trade implies uniform goods prices across trading economies. Therefore, if two economies have identical homogeneous preferences, i.e. identical and budget-invariant marginal rates of substitution, each economy in a 2x2 model exports the good of which it produces relatively more than the other economy. Hence it is sufficient to prove that an economy which is relatively abundant in a factor will produce relatively more of the good which is intensive in that factor. Denoting the economy's ouput vector by (y_1, y_2) and the economy's input vector by $(\overline{z}_A, \overline{z}_B)$, we should establish that

$$\frac{\partial \frac{y_1}{y_2}}{\partial \frac{\overline{z}_A}{\overline{z}_B}} > 0$$

if $\frac{a_{1A}}{a_{1B}} > \frac{a_{2A}}{a_{2B}}$ at all factor prices $\frac{w_A}{w_B}$. (Recall that a_{1A} is the optimal quantity of input A in the production of *one* unit of good 1.)

The clearing conditions for the factor markets are:

$$\begin{aligned} \overline{z}_A &= a_{1A}y_1 + a_{2A}y_2 \\ \overline{z}_B &= a_{1B}y_1 + a_{2B}y_2. \end{aligned}$$

Solving for y_1 and y_2,

$$\begin{aligned} y_1 &= \frac{a_{2B}\overline{z}_A - a_{2A}\overline{z}_B}{a_{1A}a_{2B} - a_{2A}a_{1B}} \\ y_2 &= \frac{a_{1A}\overline{z}_B - a_{1B}\overline{z}_A}{a_{1A}a_{2B} - a_{2A}a_{1B}}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{y_1}{y_2} &= \frac{a_{2B}\overline{z}_A - a_{2A}\overline{z}_B}{a_{1A}\overline{z}_B - a_{1B}\overline{z}_A} \\ &= \frac{a_{2B}\frac{\overline{z}_A}{\overline{z}_B} - a_{2A}}{a_{1A} - a_{1B}\frac{\overline{z}_A}{\overline{z}_B}} \\ &= \frac{a_{2B}}{a_{1B}} \frac{\frac{\overline{z}_A}{\overline{z}_B} - \frac{a_{2A}}{a_{2B}}}{\frac{a_{1A}}{a_{1B}} - \frac{\overline{z}_A}{\overline{z}_B}}. \end{aligned}$$

Differentiating:

$$\begin{aligned} \frac{\partial \frac{y_1}{y_2}}{\partial \frac{\overline{z}_A}{\overline{z}_B}} &= \frac{a_{2B}}{a_{1B}} \frac{\frac{a_{1A}}{a_{1B}} - \frac{\overline{z}_A}{\overline{z}_B} + \frac{\overline{z}_A}{\overline{z}_B} - \frac{a_{2A}}{a_{2B}}}{\left(\frac{a_{1A}}{a_{1B}} - \frac{\overline{z}_A}{\overline{z}_B}\right)^2} \\ &= \frac{a_{2B}}{a_{1B}} \frac{\frac{a_{1A}}{a_{1B}} - \frac{a_{2A}}{a_{2B}}}{\left(\frac{a_{1A}}{a_{1B}} - \frac{\overline{z}_A}{\overline{z}_B}\right)^2}. \end{aligned}$$

Clearly, if $\frac{a_{1A}}{a_{1B}} > \frac{a_{2A}}{a_{2B}}$, then $\partial \frac{y_1}{y_2} / \partial \frac{\overline{z}_A}{\overline{z}_B}$ is indeed positive. As we have argued at the outset, this completes the proof of the theorem. To see why, suppose two economies coexist in autarky in a 2x2 world. Good 1 is intensive in input A, and one economy is relatively abundant in input A. As was just shown, this economy will produce relatively more of good 1 (and of course relatively less of good 2). If preferences are identical and scale-invariant, the price of good 1 (in terms of good 2) must be lower in the economy that produces relatively more of good 1. If we allow the economies to trade, the relatively cheaper good will be exported and the relatively more expensive good will be imported. ■