# THE AXIOM OF CHOICE

A *choice function* on a family $S$ of sets is a function $f$ with domain $S$ such that, for each nonempty set $X$ in $S$, $f(X)$ is an element of $X$: figuratively put, $f$ "chooses" an element of each member of $S$. If $S$ is *finite*, the existence of a choice function on $S$ is a straightforward consequence of the basic principles of set formation and the rules of classical logic. When $S$ is *infinite*, however, these principles no longer suffice and so the existence of a choice function on $S$ must be postulated. The assertion that on *any* family of nonempty sets — even if it be infinite —there exists at least one choice function is called the *axiom of choice*. This principle was first explicitly stated (in a different, but equivalent form) by Zermelo in 1904 and employed in his proof that any set can be well ordered — his famous *well-ordering theorem*. Its highly nonconstructive character provoked considerable initial criticism: while it asserts the possibility of making arbitrarily many arbitrary "choices" —or at least of crystallizing such an imagined procedure into a genuine function —it provides no indication whatsoever of how these "choices" are to be made, or how the resulting function is to be defined. For example, the scepticism of the French mathematician Emile Borel concerning such a possibility was sufficient to move him to declare that "any argument where one supposes an *arbitrary choice* a non-denumerably infinite number of times is outside the domain of mathematics." In 1938, however, Gödel established the relative consistency of the axiom of choice with respect to the usual systems of set theory, and this, coupled with its evident indispensability in the proofs of many significant mathematical theorems, eventually led to its acceptance by the majority of mathematicians.

Judged by the number of its mathematical consequences, the axiom of choice is unquestionably the most fertile principle of set theory. Remarkably, many of these consequences turn out to be *formally equivalent* to it: more than 200 of these equivalents are recorded in the book of Rubin and Rubin. Among the most significant of these equivalents are:

*Zermelo's well-ordering theorem*: every set can be well-ordered;

*Trichotomy Principle:* of any pair of cardinal numbers, one is less than the other, or they are equal;

*The Kuratowski-Zorn lemma*: any nonempty set in which each totally ordered subset has an upper bound posseses a maximal element;

*Tychonov's theorem*: the product of any family of compact topological spaces is compact;

*The model existence theorem for first-order logic:* every infinite consistent set $\Sigma$ of first-order sentences has a model of cardinality no greater than that of $\Sigma$;

*The Hamel basis theorem:* every vector space has a basis.

While the (relative) consistency of the axiom of choice was not established until almost four decades after its formulation, the first steps in confirming its *formal independence* of the basic axioms of set theory were taken as early as 1922 by A. Fraenkel. He showed that the axiom is independent of a certain system of set theory allowing the presence of *atoms*, that is, objects possessing no members, yet not identical with the empty set. Remarkable as this advance was, however, it neither answered the question of whether the axiom of choice is independent of the full set-theoretic system of Zermelo-Fraenkel (which includes the axiom of foundation: see the article *Set Theory*), nor did it demonstrate the independence of the most important consequence of Zermelo's original invocation of the axiom of choice: the existence of a well-ordering of the set of real numbers. The issue was finally resolved in 1964 when P.J.Cohen devised his method of *forcing*. Cohen in fact established the independence of a surprisingly weak form of the axiom of choice, namely that asserting the existence of a choice function on a countable family of pairs. Subsequent work by R.M.Solovay and others has established the independence of certain important consequences of the axiom of choice, notably, the Hahn-Banach theorem and the existence of non-Lebesgue measurable sets of real numbers.

As we have remarked, the nonconstructive character of the axiom of choice was realized from the beginning. However, the question of its exact logical status remained unresolved for a long time. Finally, in 1975, Diaconescu showed (in a category-theoretic setting) that the classical law of excluded middle can be derived within a system of intuitionistic set theory augmented by the axiom of choice. Put succinctly, *the axiom of choice implies the law of excluded middle*. It was later shown that this pivotal law of classical logic can be derived just from the classically trivial version of the axiom of choice that the family of all subsets of a two element set has a choice function.

Finally, a word about the "paradoxical" consequences of the axiom of choice. In 1914 Hausdorff derived from it the curious result that the surface $S$ of a sphere can be decomposed into disjoint sets $S = A \cup B \cup C \cup Q$ in such a way that $A,B,C$ and $B \cup C$ are mutually congruent and $Q$ is countable. Put succinctly, tha axiom of choice implies that two-thirds of the surface of a sphere is congruent to one-third of it. This was extended to three dimensions by Banach and Tarski in 1924 when they used the axiom of choice to show that any solid sphere can be decomposed into finitely many (later shown by Raphael Robinson to be reducible to 5) subsets which can themselves be reassembled to form *two* solid spheres, *each of the same size as the original.* They also showed that the axiom of choice yields another version of this, the *Banach-Tarski* paradox, namely, given any pair of solid spheres, either one of them can be decomposed into finitely many subsets which can be reassembled to form a solid sphere of the same size as the other. Thus, to put it dramatically, the axiom of choice implies that *a sphere the size of the sun can be decomposed and the pieces reassembled so as to form a sphere the size of a pea*. Strange as these results are, however, they do not constitute outright contradictions: "paradoxical" sphere decompositions of this kind only become possible in set theory only because continuous geometrical objects have been analyzed into discrete sets of points which the axiom of choice then allows to be rearranged in an arbitrary manner.

## Bibliography

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