

APPENDIX 4

THE PHILOSOPHICAL THOUGHT OF A GREAT MATHEMATICIAN:

HERMANN WEYL

As we have observed, mathematics and philosophy are closely linked, and several great mathematicians who were at the same time great philosophers come to mind—Pythagoras, Descartes and Leibniz, for instance. One great mathematician of the modern era in whose thinking philosophy played a major role was *Hermann Weyl* (1885–1955), whose work encompassed analysis, number theory, topology, differential geometry, relativity theory, quantum mechanics, and mathematical logic. His many writings are informed by a vast erudition, an acute philosophical awareness, and even, on occasion, a certain playfulness. No matter what the subject may be—mathematics, physics, philosophy—Weyl’s writing fascinates both by the depth of insight it reveals and by its startling departures from academic convention. Who else, for instance, would have the daring to liken (as he does in the discussion of Space and Time in his *Philosophy of Mathematics and Natural Science*), a coordinate system to “the residue of the annihilation of the ego”¹? Or then (somewhat further on in the same discussion) to express the belief in the impossibility of a completely objective account of individual consciousness by the assertion “...it is shattered by Judas’ desperate outcry, ‘Why did I have to be Judas?’”².

In this final Appendix we trace Weyl’s philosophical thinking on mathematics and science through quotations from his works. It is a fascinating and instructive journey.

Weyl’s philosophical outlook was strongly influenced by that of *Edmund Husserl* (1859–1938), the creator of the philosophy known as *phenomenology*. The principal tenet of phenomenology is that the only things which are directly given to us, that we can know completely, are objects of consciousness. It is these with which philosophy, and all knowledge, must begin. This is acknowledged by Weyl in the introduction to *Space-Time-Matter* (1919), his famous book on the theory of relativity. Modestly describing his remarks as “a few reflections of a philosophical character,” he observes that the objective world “constructed” by mathematical physics cannot of necessity coincide with the subjective world of qualities given through perception:

¹ Weyl, *Philosophy*, 123. This metaphor seems first to have appeared in *The Continuum*, where Weyl asserts on p. 94:

The coordinate system is the unavoidable residue of the eradication of the ego in that geometrico-physical world which reason sifts from the given using “objectivity” as its standard—a final scanty token in this objective sphere that existence is only given and *can* only be given as the intentional content of the processes of consciousness of a pure, sense-giving ego.

² *Ibid.*, 124–125.

Expressed as a general principle, this means that the real world, and every one of its constituents, are, and can only be, given as intentional objects of acts of consciousness. The immediate data which I receive are the experiences of consciousness in just the form in which I receive them ... we may say that in a sensation an object, for example, is actually physically present for me—to whom that sensation relates—in a manner known to everyone, yet, since it is characteristic, it cannot be described more fully. (Weyl, *Space-Time-Matter*, 4)

His phenomenological orientation is proclaimed still more emphatically when he goes on to say:

...the datum of consciousness is the starting point at which we must place ourselves if we are to understand the absolute meaning of, as well as the right to, the supposition of reality ... “Pure consciousness” is the seat of what is philosophically a priori. (*ibid.*, 5)

Later, he asserts that

Time is the ... form of the stream of consciousness ...and space... the form of external material reality. (*ibid.*, 5)

And in a memorable passage he describes how these two opposed facets of existence ineluctably come to be fused, so leading to the amalgamation of time and space, of which the theory of relativity is (thus far) the deepest expression:

... if the worlds of consciousness and of transcendental reality were totally different from one another, or, rather, if only the passive act of perception bridged the gulf between them, the state of affairs would remain as I have represented it, namely, on the one hand a consciousness rolling on in the form of a lasting present, yet spaceless; on the other, a reality spatially extended, yet timeless, of which the former contains but a varying appearance. Antecedent to all perception there is in us the experience of effort and opposition, of being active and being passive. For a person leading a natural life of activity, perception serves above all to place clearly before his consciousness the definite point of the action he wills, and the source of the opposition to it. As the doer and endurer of actions I become a single individual with a psychical reality attached to a body which has its place in space among the material things of the external world, and by which I am in communication with other similar individuals. Consciousness, without surrendering its immanence, becomes a part of reality, becomes this particular person, myself, who was born and will die. Moreover, as a result of this, consciousness spreads out its web, in the form of time, over reality. Change, motion, elapse of time, coming and ceasing to be, exist in time itself; just as my will acts on the external world through and beyond my body as a motive power, so the external world is in its turn active. Its phenomena are related throughout by a causal connection. In fact, physics shows that cosmic time and physical time cannot be disassociated from one another. The new solution of the problem of amalgamating space and time offered by the theory of relativity brings with it a deeper insight into the harmony of action in the world. (*ibid.*, 6)

Thus, for Weyl, the duality between mind and material reality leads to a unity between space and time.

Weyl believed that, in the nature of things, science was ultimately limited in its capacity to describe the world. Thus in the Preface to his luminous work *The Open World* (1932) he expresses the opinion that:

Modern science, insofar as I am familiar with it through my own scientific work, mathematics and physics make the world appear more and more an open one. ... Science finds itself compelled, at once by the epistemological, the physical and the constructive-mathematical aspect of its own methods and results, to recognize this situation. It remains to be added that science can do no more

than show us this open horizon; we must not by including the transcendental sphere attempt to establish anew a closed (though more comprehensive) world. (Weyl, *Open World*, v)

And in his *Address on the Unity of Knowledge* (1954) he says, in a particularly striking passage lightened by a touch of humour:

The riddle posed by the double nature of the ego certainly lies beyond those limits [i.e., of science]. On the one hand, I am a real individual man, born by a mother and destined to die, carrying out real and psychical acts (far too many, I may think, if boarding a subway during rush hour). On the other hand, I am “vision” open to reason, a self-penetrating light, immanent sense-giving consciousness, or how ever you may call it, and as such unique. (Weyl, *Address*, 3)

In the same address, he reiterates that

...at the basis of all knowledge lies, firstly, intuition, mind’s ordinary act of seeing what is given to it. (*ibid.*, 7)

In particular, Weyl held, like Brouwer, that intuition, or *insight*—rather than *proof*—furnishes the ultimate foundation of *mathematical* knowledge³. Thus in *The Continuum* (1918) he says, with an unusual hint of asperity,

In the Preface to Dedekind (1888) we read that “In science, whatever is provable must not be believed without proof.” This remark is certainly characteristic of the way most mathematicians think. Nevertheless, it is a preposterous principle. As if such an indirect concatenation of grounds, call it a proof though we may, can awaken any “belief” apart from assuring ourselves through immediate insight that each individual step is correct. In all cases, this process of confirmation—and not the proof—remains the ultimate source from which knowledge derives its authority; it is the “experience of truth”. (Weyl, *Continuum*, 119)

Weyl recognized, however, that requiring all mathematical knowledge to possess intuitive immediacy is an unattainable ideal:

The states of affairs with which mathematics deals are, apart from the very simplest ones, so complicated that it is practically impossible to bring them into full givenness in consciousness and in this way to grasp them completely. (*ibid.*, 17)

But Weyl did not think that this fact furnished justification for extending the bounds of mathematics to embrace notions which cannot be given intuitively *even in principle* (e.g., the actual infinite). He believed, rather, that this extension of mathematics into the transcendent—the world beyond immediate consciousness—is made necessary by the fact that mathematics plays an indispensable role in the physical sciences, in which intuitive evidence is *necessarily* transcended:

... if mathematics is taken by itself, one should restrict oneself with Brouwer to the intuitively cognizable truths ... nothing compels us to go farther. But in the natural sciences we are in contact with a sphere which is impervious to intuitive evidence; here cognition necessarily becomes symbolical construction. Hence we need no longer demand that when mathematics is taken into the process of theoretical construction in physics it should be possible to set apart the

³ It is to be noted in this connection that in *The Continuum* Weyl also makes the assertion that ...the idea of iteration, i.e., of the sequence of natural numbers, is an ultimate foundation of mathematical thought ... (p. 48)

mathematical element as a special domain in which all judgements are intuitively certain; from this higher standpoint which makes the whole of science appear as one unit, I consider Hilbert to be right. (Weyl, *Open World*, 82)

In *Consistency in Mathematics* (1929), Weyl characterized the mathematical method as

the *a priori* construction of the possible in opposition to the *a posteriori* description of what is actually given. (Weyl, *Consistency*, 249)

The problem of identifying the limits on constructing “the possible” in this sense occupied Weyl a great deal. He was particularly concerned with the concept of the mathematical *infinite*, which he believed to elude “construction” in the idealized sense of set theory:

No one can describe an infinite set other than by indicating properties characteristic of the elements of the set. ... The notion that a set is a “gathering” brought together by infinitely many individual arbitrary acts of selection, assembled and then surveyed as a whole by consciousness, is nonsensical; “inexhaustibility” is essential to the infinite. (Weyl, *Continuum*, 23)

But the necessity of expanding mathematics into the external world forces it to embody a conception of the actual infinite, as Weyl attests towards the end of *The Open World*:

The infinite is accessible to the mind intuitively in the form of a field of possibilities open to infinity, analogous to the sequence of numbers which can be continued indefinitely, but the completed, the actual infinite as a closed realm of actual existence is forever beyond its reach. Yet the demand for totality and the metaphysical belief in reality inevitably compel the mind to represent the infinite by closed being as symbolical construction. (Weyl, *Open World*, 83)

Another mathematical “possible” to which Weyl gave a great deal of thought is the idea of the *continuum*. During the period 1918–1921 he wrestled unceasingly with the problem of providing it with an exact mathematical formulation free of the taint of the actual infinite. As he saw it, there is an unbridgeable gap between intuitively given continua (e.g. those of time and motion) on the one hand, and the “discrete” exact concepts of mathematics (e.g. that of real number) on the other:

... the conceptual world of mathematics is so foreign to what the intuitive continuum presents to us that the demand for coincidence between the two must be dismissed as absurd (Weyl, *Continuum*, 108);

... the continuity given to us immediately by intuition (in the flow of time and of motion) has yet to be grasped mathematically as a totality of discrete “stages” in accordance with that part of its content which can be conceptualized in an exact way (*ibid.*, 24);

When our experience has turned into a real process in a real world and our phenomenal time has spread itself out over this world and assumed a cosmic dimension, we are not satisfied with replacing the continuum by the exact concept of the real number, in spite of the essential and undeniable inexactness arising from what is given. (*ibid.*, 93).

By 1921 he had joined Brouwer in regarding the continuum as a “medium of free becoming.”; in one of his last papers he observes:

... constructive transition to the continuum of real numbers is a serious affair... and I am bold enough to say that not even to this day are the logical issues involved in that constructive concept completely clarified and settled. (Weyl, *Axiomatic*, 17)

It is greatly to be regretted that Weyl did not live to see the startling developments stemming from nonstandard analysis and, especially, smooth infinitesimal analysis, which provides a mathematical framework within which the concept of a true continuum—that is, not “synthesized” from discrete elements—can be formulated and developed (*q.v.* Appendix 3).

It seems appropriate to conclude with a quotation from *The Open World* which expresses with clarity and eloquence the essence of Weyl's philosophy:

The beginning of all philosophical thought is the realization that the perceptual world is but an image, a vision, a phenomenon of our consciousness; our consciousness does not directly grasp a transcendental real world which is as it appears. The tension between subject and object is no doubt reflected in our conscious acts, for example, in sense perceptions. Nevertheless, from the purely epistemological point of view, no objection can be made to a phenomenalism which would like to limit science to the description of what is “immediately given to consciousness”. The postulation of the real ego, of the thou and of the world, is a metaphysical matter, not judgment, but an act of acknowledgment and belief. But this belief is after all the soul of all knowledge. It was an error of idealism to assume that the phenomena of consciousness guarantee the reality of the ego in an essentially different and somehow more certain manner than the reality of the external world; in the transition from consciousness to reality the ego, the thou and the world rise into existence indissolubly connected and, as it were, at one stroke. (Weyl, *Open World*, 26–27)