Choice Principles in Intuitionistic Set Theory

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We work in intuitionistic Zermelo-Fraenkel set theory **IST** (for a presentation, see [3], where it is called **ZF′I**). Let us begin by fixing some notation. For each set A we write $\mathrm{P}A$ for the power set of A, and $\mathrm{Q}X$ for the set of *inhabited* subsets of A, that is, of subsets X of A for which $\exists x\ (x \in A)$. The set of functions from A to B is denoted by B^A; the class of functions with domain A is denoted by $\mathrm{Fun}(A)$. The empty set is denoted by 0, {0} by 1, and {0, 1} by 2.

We tabulate the following *logical schemes*:

SLEM	$\alpha \vee \neg\alpha$ (α any sentence)
Lin	$(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$ (α, β any sentences)
Stone	$\neg\alpha \vee \neg\neg\alpha$ (α any sentence)
Ex	$\exists x[\exists x\alpha(x) \rightarrow \alpha(x)]$ ($\alpha(x)$ any formula with at most x free)
Un	$\exists x[\alpha(x) \rightarrow \forall x\alpha(x)]$ ($\alpha(x)$ any formula with at most x free)
Dis	$\forall x[\alpha \vee \beta(x)] \rightarrow \alpha \vee \forall x\beta(x)$ (α any sentence, $\beta(x)$ any formula with at most x free)

Over intuitionistic logic, **Lin, Stone** and **Ex** are consequences of **SLEM**; and **Un** implies **Dis**. All of these schemes follow, of course, from the full law of excluded middle, that is **SLEM** for arbitrary formulas.

We formulate the following *choice principles*—here X is an arbitrary set and $\varphi(x,y)$ an arbitrary formula of the language of **IST** with at most the free variables x, y:

$\mathbf{AC}_X$	$\forall x{\in}X\, \exists y\, \varphi(x,y) \rightarrow \exists f{\in}\mathrm{Fun}(X)\, \forall x{\in}X\, \varphi(x,fx)$
$\mathbf{AC}^*_X$	$\exists f{\in}\mathrm{Fun}(X)\, [\forall x{\in}X\, \exists y\, \varphi(x,y) \rightarrow \forall x{\in}X\, \varphi(x,fx)]$
$\mathbf{DAC}_X$	$\forall f{\in}\mathrm{Fun}(X)\, \exists x{\in}X\, \varphi(x,fx) \rightarrow \exists x{\in}X\ \forall y\, \varphi(x,y)$
$\mathbf{DAC}^*_X$	$\exists f{\in}\mathrm{Fun}(X)\, [\exists x{\in}X\, \varphi(x,fx) \rightarrow \exists x{\in}X\ \forall y\, \varphi(x,y)]$

The first two of these are forms of the *axiom of choice* for X; while classically equivalent, in **IST** $\mathbf{AC}^*_X$ implies $\mathbf{AC}_X$, but not conversely. The principles $\mathbf{DAC}_X$ and $\mathbf{DAC}^*_X$ are *dual* forms of the axiom of choice for X: classically they are both equivalent to $\mathbf{AC}_X$ and $\mathbf{AC}^*_X$, but in **IST** $\mathbf{DAC}^*_X$ implies $\mathbf{DAC}_X$, and not conversely.

We also formulate what we shall call the *weak extensional selection principle*, in which $\alpha(x)$ and $\beta(x)$ are any formulas with at most the variable x free:

WESP $\quad \exists x{\in}2\alpha(x) \wedge \exists x{\in}2\beta(x) \rightarrow \exists x{\in}2\exists y{\in}2[\alpha(x) \wedge \beta(y) \wedge [\forall x{\in}2[\alpha(x) \leftrightarrow \beta(x)] \rightarrow x = y]].$

This principle asserts that, for any pair of instantiated properties of members of 2, instances may be assigned to the properties in a manner that depends just on their extensions. **WESP** is a straightforward consequence of $\mathbf{AC}_{Q2}$. For taking $\varphi(u, y)$ to be $y \in u$ in $\mathbf{AC}_{Q2}$ yields the existence of a function f with domain $Q2$ such that $fu \in u$ for every $u \in Q2$. Given formulas $\alpha(x)$, $\beta(x)$, and assuming the antecedent of **WESP,** the sets $U = \{x{\in}2\colon \alpha(x)\}$ and $V = \{x{\in}2\colon \beta(x)\}$ are members of $Q2$, so that $a = fU \in U$, and $b = fV \in V$, whence $\alpha(a)$ and $\beta(b)$. Also, if $\forall x{\in}2[\alpha(x) \leftrightarrow \beta(x)]$, then $U = V$, whence $a = b$; it follows then that the consequent of **WESP** holds.

We are going to show that each of the logical principles tabulated above is equivalent (over **IST**) to a choice principle. Starting at the top of the list, we have first:

- **WESP** and **SLEM** are equivalent over **IST**.

Proof. Assume **WESP.** Let σ be any sentence and define

$$\alpha(x) \equiv x = 0 \vee \sigma \qquad \beta(x) \equiv x = 1 \vee \sigma .$$

With these instances of α and β the antecedent of **WESP** is clearly satisfied, so that there exist members a, b of 2 for which (1) $\alpha(a) \wedge \beta(b)$ and (2) $\forall x\, [[\forall x{\in}2[\alpha(x) \leftrightarrow \beta(x)] \rightarrow a = b$. It follows from (1) that $\sigma \vee (a = 0 \wedge b = 1)$, whence (3) $\sigma \vee a \neq b$. And since clearly $\sigma \rightarrow \forall x{\in}2[\alpha(x) \leftrightarrow \beta(x)]$ we deduce from (2) that $\sigma \rightarrow a = b$, whence $a \neq b \rightarrow \neg\sigma$. Putting this last together with (3) yields $\sigma \vee \neg\sigma$, and **SLEM** follows.

For the converse, we argue informally. Suppose that **SLEM** holds. Assuming the antecedent of **WESP**, choose $a \in 2$ for which $\alpha(a)$. Now (using **SLEM**) define an element $b \in 2$ as follows. If $\forall x{\in}2[\alpha(x) \leftrightarrow \beta(x)]$ holds, let $b = a$; if not, choose b so that $\beta(b)$. It is now easy to see that a and b satisfy $\alpha(a) \wedge \beta(b) \wedge [\forall x{\in}2[\alpha(x) \leftrightarrow \beta(x)] \rightarrow a = b]$. **WESP** follows. ■

Remark. The argument for **WESP** $\rightarrow$ **SLEM** is another "stripped down" version of Diaconescu's theorem that, in a topos, the axiom of choice implies the law of excluded middle. The result may be compared with that of [2] to the effect that the presence of extensional ε-terms renders intuitionistic logic classical.

Next, we observe that, while $\mathbf{AC}_1$ is (trivially) provable in **IST,** by contrast

- $\mathbf{AC}_1^*$ and **Ex** are equivalent over **IST.**

Proof. Assuming $\mathbf{AC}_1^*$, take $\varphi(x,y) \equiv \alpha(y)$ in its antecedent. This yields an $f \in \mathrm{Fun}(1)$ for which $\forall y\alpha(y) \to \alpha(f0)$, giving $\exists y[\exists y\alpha(y) \to \alpha(y)]$, i.e., **Ex.**

Conversely, define $\alpha(y) \equiv \varphi(0,y)$. Then, assuming **Ex**, there is b for which $\exists y\alpha(y) \to \alpha(b)$, whence $\forall x{\in}1\exists y\varphi(x,y) \to \forall x{\in}1\varphi(x,b)$. Defining $f \in \mathrm{Fun}(1)$ by $f = \{\langle 0,b\rangle\}$ gives $\forall x{\in}1\exists y\varphi(x,y) \to \forall x{\in}1\varphi(x,fx)$, and $\mathbf{AC}_1^*$ follows. ■

Further, while $\mathbf{DAC}_1$ is easily seen to be provable in **IST**, we have

- $\mathbf{DAC}_1^*$ and **Un** are equivalent over **IST.**

Proof. Given α, Define $\varphi(x,y) \equiv \alpha(y)$. Then, for $f \in \mathrm{Fun}(1)$, $\exists x{\in}1\varphi(x,fx) \leftrightarrow \alpha(f0)$ and $\exists x{\in}1\forall y\varphi(x,y) \leftrightarrow \forall y\alpha(y)$. $\mathbf{DAC}_1^*$ then gives

$$\exists f{\in}\mathrm{Fun}(1)[\alpha(f0) \to \forall y\alpha(y)],$$

from which **Un** follows easily.

Conversely, given φ, define $\alpha(y) \equiv \varphi(0,y)$. Then from **Un** we infer that there exists b for which $\alpha(b) \to \forall y\alpha(y)$, i.e. $\varphi(0,b) \to \forall y\varphi(0,y)$. Defining $f \in \mathrm{Fun}(1)$ by $f = \{\langle 0,b\rangle\}$ then gives $\varphi(0,f0) \to \exists x{\in}1\forall y\varphi(x,y)$, whence $\exists x{\in}1\varphi(x,fx) \to \exists x{\in}1\forall y\varphi(x,y)$, and **Un** follows. ■

Next, while $\mathbf{AC}_2$ is easily proved in **IST**, by contrast we have

- $\mathbf{DAC}_2$ and **Dis** are equivalent over **IST**.

Proof. The antecedent of $\mathbf{DAC}_2$ is equivalent to the assertion

$$\forall f{\in}\mathrm{Fun}(2)[\varphi(0, f0) \vee \varphi(1, f1)],$$

which, in view of the natural correlation between members of Fun (2) and ordered pairs, is equivalent to the assertion

$$\forall y\forall y'[\varphi(0, y) \vee \varphi(1, y')].$$

The consequent of $\mathbf{DAC}_2$ is equivalent to the assertion

$$\forall y\in Y\varphi(0,y) \vee \forall y'\in Y\varphi(1,y')$$

So $\mathbf{DAC}_2$ itself is equivalent to

$$\forall y\forall y'[\varphi(0,y) \vee \varphi(1,y')] \;\to\; \forall y\varphi(0,y) \vee \forall y'\varphi(1,y').$$

But this is obviously equivalent to the scheme

$$\forall y\forall y'[\alpha(y) \vee \beta(y')] \;\to\; \forall y\alpha(y) \vee \forall y'\beta(y'),$$

where y does not occur free in β, nor y' in α. And this last is easily seen to be equivalent to **Dis**. ■

Now consider $\mathbf{DAC}_2^*$. This is quickly seen to be equivalent to the assertion

$$\exists z\exists z'[\varphi(0,z) \vee \varphi(1,z') \;\to \forall y\varphi(0,y) \vee \;\forall y'\varphi(1,y'),$$

i.e. to the assertion, for arbitrary α(*x*), β(*x*), that

$$\exists z\exists z'[\alpha(z) \vee \beta(z') \;\to\; \forall y\alpha(y) \vee \;\forall y'\beta(y')].$$

This is in turn equivalent to the assertion, for any sentence α,

$$\exists y[\alpha \vee \beta(y) \;\to\; \alpha \vee \;\forall y\beta(y)] \qquad (*)$$

Now (*) obviously entails **Un.** Conversely, given **Un**, there is b for which $\beta(b) \to \forall y\beta(y)$. Hence $\alpha \vee \beta(b) \to \alpha \vee \forall y\beta(y)$, whence (*). So we have shown that

- Over **IST,** $\mathbf{DAC}_2^*$ is equivalent to **Un**, and hence also to $\mathbf{DAC}_1^*$.

In order to provide choice schemes equivalent to **Lin** and **Stone** we introduce

$\mathbf{ac}_X^*$ $\quad \exists f\in 2^X\,[\forall x\in X\,\exists y\in 2\;\varphi(x,y) \to \forall x\in X\;\varphi(x,fx)]$

$\mathbf{wac}_X^*$ $\quad \exists f\in 2^X\,[\forall x\in X\,\exists y\in 2\;\varphi(x,y) \to \forall x\in X\;\varphi(x,fx)]$ provided $\vdash_{\mathbf{IST}} \forall x[\varphi(x,0) \to \neg\varphi(x,1)]$

Clearly $\mathbf{ac}_X^*$ is equivalent to

$$\exists f\in 2^X\,[\forall x\in X[\varphi(x,0) \vee \varphi(x,1)] \to \forall x\in X\;\varphi(x,fx)]$$

and similarly for $\mathbf{wac}_X^*$.

Then

- Over **IST**, $\mathbf{ac}_1^*$ and $\mathbf{wac}_1^*$ are equivalent, respectively, to **Lin** and **Stone.**

Proof. Let α and β be sentences, and define $\varphi(x,y) \equiv x = 0 \wedge [(y = 0 \wedge \alpha) \vee (y = 1 \wedge \beta)]$. Then $\alpha \leftrightarrow \varphi(0,0)$ and $\beta \leftrightarrow \varphi(0,1)$, and so $\forall x{\in}1[\varphi(x,0) \vee \varphi(x,1)] \leftrightarrow \varphi(0,0) \vee \varphi(0,1) \leftrightarrow \alpha \vee \beta$. Therefore

$$\begin{aligned}\exists f{\in}2^1\, [\forall x{\in}1[\varphi(x,0) \vee \varphi(x,1)] \to \forall x{\in}1\ \varphi(x,fx)] &\leftrightarrow\ \exists f{\in}2^1[\alpha \vee \beta \to \varphi(0,f0)]\\ &\leftrightarrow\ [\alpha \vee \beta \to \varphi(0,0)] \vee [\alpha \vee \beta \to \varphi(0,1)]\\ &\leftrightarrow\ [\alpha \vee \beta \to \alpha] \vee [\alpha \vee \beta \to \beta]\\ &\leftrightarrow\ \beta \to \alpha\ \vee\ \alpha \to \beta.\end{aligned}$$

This yields $\mathbf{ac}_1^* \to$ **Lin.** For the converse, define $\alpha \equiv \varphi(0,0)$ and $\beta \equiv \varphi(0,1)$ and reverse the argument.

To establish the second stated equivalence, notice that, when $\varphi(x,y)$ is defined as above, but with β replaced by $\neg\alpha$, it satisfies the provisions imposed in $\mathbf{wac}_1^*$. As above, that principle gives $\neg\alpha \to \alpha \vee \alpha \to \neg\alpha$, that is, $\neg\alpha \vee \neg\neg\alpha$. So **Stone** follows from $\mathbf{wac}_1^*$. Conversely, suppose that φ meets the condition imposed in $\mathbf{wac}_1^*$ Then from $\varphi(0,0) \to \neg\varphi(0,1)$ we deduce $\neg\neg\varphi(0,0) \to \neg\varphi(0,1)$; now, assuming **Stone**, we have $\neg\varphi(0,0) \vee \neg\neg\varphi(0,0)$, whence $\neg\varphi(0,0) \vee \neg\varphi(0,1)$. Since $\neg\varphi(0,0) \to [\varphi(0,0) \to \varphi(0,1)]$ and $\neg\varphi(0,1) \to [\varphi(0,1) \to \varphi(0,0)]$ we deduce $[\varphi(0,0) \to \varphi(0,1)] \vee [\varphi(0,1) \to \varphi(0,0)]$. From the argument above it now follows that $\exists f{\in}2^1\, [\forall x{\in}1[\varphi(x,0) \vee \varphi(x,1)] \to \forall x{\in}1\ \varphi(x,fx)]$. Accordingly $\mathbf{wac}_1^*$ is a consequence of **Stone**.

Connections with term-forming operators.

The ε- and τ-*operators* are term-forming operators yielding, for formulas $\alpha(x)$, terms $\varepsilon_x\alpha$ and $\tau_x\alpha$ in which the variable x is no longer free; they are introduced in conjunction with the axioms—the ε- and τ-*schemes*:

$$\exists x\alpha(x) \to \alpha(\varepsilon_x\alpha) \qquad \alpha(\tau_x\alpha) \to \forall x\alpha(x).$$

It is an easy matter to derive **Un** from the τ-scheme when τ is merely allowed to act on formulas with at most one free variable. When τ's action is extended to formulas with two free variables, the τ-scheme applied in **IST** yields the full dual axiom of choice $\forall X$ $\mathbf{DAC}_X^*$. For under these conditions we have, for any formula $\varphi(x,y)$,

$$\forall x{\in}X[\varphi(x,\tau_y\varphi(x,y)) \to \forall y\varphi(x,y)] \qquad (*)$$

Let $t \in$ Fun(X) be the map $x \mapsto \tau_y\varphi(x,y)$. Assuming that $\forall f \in Y^X \exists x \in X \varphi(x, fx)$, let $a \in X$ satisfy $\varphi(a,ta)$. We deduce from (*) that $\forall y \in Y \varphi(a,y)$, whence $\exists x \in X \forall y \in Y \varphi(x, y)$. The dual axiom of choice follows.

In the case of the ε-operator, the number of free variables in the formulas on which the operator is allowed to act is an even more sensitive matter. If ε is allowed to act only on formulas with at most one free variable (so yielding only closed terms), the corresponding ε-scheme applied in **IST** is easily seen to yield both **Ex** and $\mathbf{ac}_1^*$, and so also **Lin**. But it is (in essence) shown in [1] that, if only closed ε-terms are admitted, **SLEM** is not derivable, and so therefore neither is **WESP.** The situation changes dramatically when ε is permitted to operate on formulas with just *two* free variables. For then from the corresponding ε-scheme it is easy to derive $\mathbf{AC}_X$ for all sets X, and in particular $\mathbf{AC}_{Q2}$, and hence also **SLEM.**

I have found three ways of strengthening, or modifying, the single-variable ε-scheme so as to enable it to yield **SLEM.** The first, presented originally in [ ], is to add to the ε-scheme *Ackermann's Extensionality Principle*, viz.

$$\forall x[\alpha(x) \leftrightarrow \beta(x)] \rightarrow \varepsilon_x\alpha = \varepsilon_x\beta \ .$$

From these **WESP** is easily derived, and so, *a fortiori*, **SLEM.**

The second approach is to take the ε-axiom in the (classically equivalent) form

(*) $$\alpha(\varepsilon_x\alpha) \vee \forall x \neg\alpha(x).$$

From this we can intuitionistically derive **SLEM** as follows:

Given a sentence β, define $\alpha(x)$ to be the formula

$$(x = 0 \wedge \beta) \vee (x = 1 \wedge \neg\beta).$$

Then from (*) we get

$$[(\varepsilon_x\alpha = 0 \wedge \beta) \vee ([(\varepsilon_x\alpha = 1 \wedge \neg\beta)] \vee \forall x\neg[(x = 0 \wedge \beta) \vee (x = 1 \wedge \neg\beta)],$$

which implies

$$[\beta \vee \neg\beta) \vee [\forall x\neg(x = 0 \wedge \beta) \wedge \forall x\neg(x = 1 \wedge \neg\beta)],$$

whence

$$[\beta \vee \neg\beta) \vee [\neg \beta \wedge \neg\neg\beta],$$

winding up with

$$\beta \vee \neg\beta.$$

The third method is to allow ε to act on *pairs* of formulas, each with a *single* free variable. Here, for each pair of formulas $\alpha(x)$, $\beta(x)$ we introduce the "relativized" ε-term $\varepsilon_x\alpha/\beta$ and the "relativized" ε-axioms

$$(1)\ \exists x\, \beta(x) \rightarrow \beta(\varepsilon_x\alpha/\beta) \qquad (2)\ \exists x\, [\alpha(x) \wedge \beta(x)] \rightarrow \alpha(\varepsilon_x\alpha/\beta).$$

Notice that the usual ε–term $\varepsilon_x\alpha$ is then $\varepsilon_x\alpha/x = x$. In the classical ε-calculus $\varepsilon_x\alpha/\beta$ may be defined by taking

$$\varepsilon_x\alpha/\beta = \varepsilon_y[[y = \varepsilon_x(\alpha \wedge \beta) \wedge \exists x\, (\alpha \wedge \beta)] \vee [y = \varepsilon_x\beta \wedge \neg\exists x\, (\alpha \wedge \beta)]].$$

But the relativized ε-scheme is not derivable in the intuitionistic ε-calculus since it can be shown to imply **SLEM**. To see this, given a formula γ define

$$\alpha(x) \equiv x = 1 \qquad \beta(x) \equiv x = 0 \vee \gamma.$$

Write a for $\varepsilon_x\alpha/\beta$. Then we certainly have $\exists x\beta(x)$, so (1) gives $\beta(a)$, i.e.

$$(3) \qquad a = 0 \vee \gamma$$

Also $\exists x\, (\alpha \wedge \beta) \leftrightarrow \gamma$, so (2) gives $\gamma \rightarrow \alpha(a)$, i.e.

$$\gamma \rightarrow a = 1,$$

whence

$$a \neq 1 \rightarrow \neg\gamma,$$

so that

$$a = 0 \rightarrow \neg\gamma.$$

And the conjunction of this with (3) gives $\gamma \vee \neg\gamma$, as claimed.

References

[1] Bell, John L. *Hilbert's epsilon operator in intuitionistic type theories*, Math. Logic Quarterly, 39, 1993.

[2] Bell, John L. *Hilbert's epsilon-operator and classical logic*, Journal of Philosophical Logic, 22, 1993.

[3] Grayson, R. J. *Heyting-valued models for intuitionistic set theory.* In Fourman, M. P., Mulvey, C. J., and Scott, D. S. (eds.) (1979) *Applications of Sheaves. Proc. L.M.S. Durham Symposium* 1977. Springer Lecture Notes in Mathematics 753, pp. 402-414.