Comparing the Smooth and Dedekind Reals in Smooth Infinitesimal Analysis

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S*mooth infinitesimal analysis*, **SIA**, is a theory formulated within higher-order intuitionistic logic and based on (at least) the following axioms:

Axioms for the continuum, or smooth real line R. These include the usual axioms for a commutative ring with unit expressed in terms of two operations + and $\bullet$, and two distinguished elements $0 \neq 1$. In addition we stipulate that **R** is a local ring, i.e., the following axiom:

$$\exists y \;\; x \bullet y = 1 \vee \exists y \, (1 - x) \bullet y = 1.$$

Axioms for the strict order relation < on R. These are:

1. $a < b$ and $b < c$ implies $a < c$.
2. $\neg(a < a)$
3. $a < b$ implies $a + c < b + c$ for any c.
4. $a < b$ and $0 < c$ implies $a \bullet c \leq b \bullet c$
5. either $0 < a$ or $a < 1$.
6. $a \neq b$[1] implies $a < b$ or $b < a$.
7. $x \neq 0$ implies $\exists y \; x \bullet y = 1$.
8. $0 < x$ implies $\exists y \;\; x = y^2$.

Arithmetical Axioms

These govern the set **N** of *Archimedean* (or smooth) *natural numbers*, and read as follows:

1. **N** is a cofinal or Archimedean subset of **R,** i.e. $\mathbf{N} \subseteq \mathbf{R}$ and $\forall x \in \mathbf{R} \; \exists n \in \mathbf{N} \; x < n$.
2. *Peano axioms*: $0 \in \mathbf{N}$

[1] Here $a \neq b$ stands for $\neg a = b$. It should be pointed out that axiom 6 is omitted in some presentations of **SIA**, e.g. those in [3] and [4].

$$\forall x \in \mathbf{R}(x \in \mathbf{N} \rightarrow x + 1 \in \mathbf{N})$$

$$\forall x \in R(x \in \mathbf{N} \rightarrow x + 1 \neq 0)$$

3. *Geometric Induction scheme.* For every formula $\alpha(x)$ involving just =, $\wedge$, $\vee$, $\top$, $\bot$, $\exists$ [2]

$$\alpha(0) \wedge \forall x{\in}\mathbf{N}(\alpha(x) \rightarrow \alpha(x + 1)) \rightarrow \forall x\alpha(x).$$

Using geometric induction it follows that

- **N** has *decidable equality*, i.e. $\forall x{\in}\mathbf{N}\forall y{\in}\mathbf{N}(x = y \vee x \neq y)$
- **N** is *linearly ordered*, i.e. $\forall x{\in}\mathbf{N}\forall y{\in}\mathbf{N}(x < y \vee x = y \vee y < x)$
- **N** satisfies *decidable induction*: for any (not necessarily geometric formula $\alpha(x)$,

$$\forall x{\in}\mathbf{N}(\alpha(x) \vee \neg\alpha(x)) \rightarrow [\alpha(0) \wedge \forall x{\in}\mathbf{N}(\alpha(x) \rightarrow \alpha(x + 1)) \rightarrow \forall x\alpha(x)].$$

The relation $\leq$ on **R** is defined by $a \leq b \Leftrightarrow \neg b < a$. The open interval (a, b) and closed interval $[a, b]$ are defined as usual, viz. $(a, b) = \{x: a < x < b\}$ and $[a, b] = \{x: a \leq x \leq b\}$; similarly for half-open, half-closed, and unbounded intervals.

Write Δ for the subset $\{x: x^2 = 0\}$ of **R**; we use the letter ε as a variable ranging over Δ. Elements of Δ are called (nilsquare) *infinitesimals* or *microquantities.* Since, clearly, $0 \in \Delta$, Δ may be regarded as an *infinitesimal neighbourhood of* 0. Δ is subject to the

Microaffineness Principle. *For any map* $g: \Delta \rightarrow \mathbf{R}$ *there exist unique* $a, b \in \mathbf{R}$ *such that, for all* ε, *we have*

$$g(\varepsilon) = a + b.\varepsilon.$$

Remark. The monoid Δ^Δ of self-maps of Δ may be regarded as acting on Δ by evaluation: for $f \in \Delta^\Delta$, $f \cdot \varepsilon = f(\varepsilon)$. The submonoid $\mathbf{R}_0$ of Δ^Δ consisting of maps vanishing at 0 may then be thought of as the space of *ratios of infinitesimals.* Now it follows from

[2] Such formulas are called *geometric*: they are preserved under [the inverse image parts) of arbitrary geometric morphisms between toposes.

the microaffineness principle that $\mathbf{R}_0$ is isomorphic to **R,** so that **R** may itself be thought as the space of ratios of infinitesimals. This was essentially the view of Euler, who regarded (real) numbers as representing the possible results of calculating the ratio 0/0. For this reason F. W. Lawvere has proposed calling **R** the space of *Euler reals.*

From these axioms it follows that the continuum in **SIA** differs in certain key respects from its counterpart in *constructive analysis* **CA***,* which is furnished with an elegant axiomatization in [2].

To begin with, the third basic property of the strict ordering relation < in **CA**, given as axiom R2(3) on p.102 of [2], and which may be written

(*) $$\neg(x < y \vee y < x) \rightarrow x = y$$

is incompatible with the axioms of **SIA**. For (*) implies

(**) $$\forall x \neg(x < 0 \vee 0 < x) \rightarrow x = 0.$$

But in **SIA** we have by Exercise 1.6 and Thm. 1.1 (i) of [1],

$$\forall x \in \Delta \neg(x < 0 \vee 0 < x) \wedge \Delta \neq \{0\},$$

which clearly contradicts (**).

Thus in **CA** the set Δ of infinitesimals would be degenerate (i.e., identical with $\{0\}$), while the nondegeneracy of Δ in **SIA** is one of its characteristic features.

Next, call a binary relation S on **R** *stable* if it satisfies

$$\forall x \forall y\ (\neg\neg xRy \rightarrow xRy).$$

In **CA**, the equality relation is stable, a fact which again follows from principle $R2(3)$ referred to above. But in **SIA** it is not stable, for, as shown in Thm. 1.1(ii) of [1], there we have $\forall x \in \Delta \neg\neg x = 0$. If = were stable, it would follow that $\forall x \in \Delta\ x = 0$, in other words, that Δ is degenerate, which is not the case in **SIA.**

Axiom 6 of **SIA**, together with the transitivity and irreflexivity of <, implies that < is stable. This may be seen as follows. Suppose $\neg\neg a < b$. Then certainly $a \neq b$, since $a = b \rightarrow \neg a < b$ by irreflexivity. Therefore $a < b$ or $b < a$. The second disjunct together with $\neg\neg a < b$ and transitivity gives $\neg\neg a < a$, which contradicts $\neg a < a$. Accordingly we are left with $a < b$. As can be deduced from assertion 8 on p. 103 of [2], the stability of < implies *Markov's principle*, which is not affirmed in **CA.**[3]

A subspace $A \subseteq$ **R** is *indecomposable* if it admits only trivial partitionings, that is, if $A = U \cup V$ and $U \cap V = \varnothing$ (under these conditions U and V are called detachable subsets of A), then $U = \varnothing$ or $V = \varnothing$. Clearly A is indecomposable iff any map $f: A \rightarrow 2 = \{0, 1\}$ is constant.

In **SIA** one also assumes the

Constancy Principle. *If* $A \subseteq$ **R** *is any closed interval on* **R,** *or* **R** *itself, and* $f: A \rightarrow$ **R** *satisfies* $f(a + \varepsilon) = f(a)$ *for all* $a \in A$ *and* $\varepsilon \in \Delta$, *then* f *is constant.*

It follows in **SIA** from the Constancy Principle that **R** *itself and each of its closed intervals is indecomposable.* From this we can deduce that in **SIA** *all intervals in* **R** *are indecomposable.* From the indecomposability of **R** it follows that the logical principle $\neg\alpha \vee \neg\neg\alpha$ is not affirmable in **SIA** (take α to be the predicate $x = 0$).

The set **Q** of (smooth) *rational numbers* is defined as usual to be the set of all fractions of the form m/n with $m, n \in$ **N**, $n \neq 0$. The fact that **N** is cofinal in **R** ensures that **Q** is dense in **R**.

Since **R** is a local ring, the set Θ of *noninvertible* elements of **R** (which of course includes all the nilpotent elements, and in particular all the microquantities) is a maximal ideal in **R**. It follows from axiom 7 that

$$\Theta = \{x \in \mathbf{R}: \neg x \neq 0\},$$

and it is not hard to show that also

$$\Theta = \{x \in \mathbf{R}: \ \forall n \in \mathbf{N} \ \ -1/(n+1) < x < 1/(n+1)\}.$$

Θ may be regarded as the space of *noninvertible infinitesimals***.**

A *Dedekind real* is a pair $(U, V) \in \mathscr{P}\mathbf{Q} \times \mathscr{P}\mathbf{Q}$ satisfying the conditions:

$$\exists x\, \exists y\, (x \in U \wedge y \in V)$$
$$U \cap V = \varnothing$$
$$\forall x\, (x \in U \leftrightarrow \exists y \in U.\ x < y)$$

[3] In versions of **SIA** that omit axiom 6 neither the stability of <, nor Markov's principle, can be derived.

$$\forall x\,(x \in V \leftrightarrow \exists y \in V.\ y < x$$

$$\forall x \forall y (x < y \rightarrow x \in U \vee y \in V).$$

Peter Johnstone shows in his book how to turn the set $\mathbf{R}_d$ of Dedekind reals into an ordered ring. (In the topos Shv(X) of sheaves over a topological space *X*, $\mathbf{R}_d$ is the sheaf of continuous real-valued functions on open subsets of *X*.) As shown by Peter Schuster, this ring is always *constructively complete*, that is, satisfies the condition: Let *A* be an inhabited subset of $\mathbf{R}_d$ that is bounded above. Then sup *A* exists if and only if for all $x, y \in \mathbf{R}_d$ with $x < y$, either y is an upper bound for *A* or there exists $a \in A$ with $x < a$. (A real number *b* is called a *supremum*, or *least upper bound*, of *A* if it is an upper bound for *A* and if for each $\varepsilon > 0$ there exists $x \in A$ with $x > b - \varepsilon$.)

Although $\mathbf{R}_d$ is constructively complete, it is not conditionally complete in the classical sense because of the failure of $\neg\alpha \vee \neg\neg\alpha$ (since, as originally shown by Johnstone, conditional completeness of $\mathbf{R}_d$ is actually equivalent to this logical law: in Shv(*X*), $\neg\alpha \vee \neg\neg\alpha$ holds iff *X* is extremally disconnected, that is, the closure of every open set is open.) But $\mathbf{R}_d$ shares some features of the constructive reals not possessed by $\mathbf{R}$, e.g.

$$\neg\neg x = y \rightarrow x = y$$

$$x \leq y \wedge y \leq x \rightarrow x = y.$$

$$x^n = 0 \rightarrow x = 0.$$

There is a natural order preserving homomorphism $\varphi: \mathbf{R} \rightarrow \mathbf{R}_d$ given by

$$\varphi(r) = (\{q \in \mathbf{Q}: q < r\}, \{q \in \mathbf{Q}: q > r\})$$

This is injective on $\mathbf{Q}$, and embeds $\mathbf{Q}$ as the rational numbers in $\mathbf{R}_d$. Moreover, $\ker \varphi = \Theta$, so φ induces an embedding of the quotient ring $\mathbf{R}/\Theta$ into $\mathbf{R}_d$. $\mathbf{R}/\Theta$ is $\mathbf{R}$ shorn of its nilpotent infinitesimals: it is both a field of fractions and an integral domain, that is, satisfies

$$\forall x (x \neq 0 \rightarrow x \text{ is invertible}) \qquad \forall x \forall y (x \cdot y = 0 \rightarrow x = 0 \rightarrow y = 0).$$

It can be shown that φ is *surjective*—so that $\mathbf{R}/\Theta \cong \mathbf{R}_d$—precisely when $\mathbf{R}$ is constructively complete in the sense above. In that event $\mathbf{R}_d$ is both a field of fractions and an integral domain, properties that the ring of Dedekind reals in a topos fails in general to possess.

In any topos with an object of natural numbers (in particular, in any model of **SIA**) the usual *open interval topology* can be defined on $\mathbf{R_d}$. Stout has shown that with this topology $\mathbf{R_d}$ is always *connected* in the sense that it cannot be partitioned into two disjoint inhabited open subsets. In **SIA** $\mathbf{R_d}$ actually inherits a stronger *indecomposability* property from **R.** In fact, if A is a detachable subset of $\mathbf{R_d}$, then $\varphi[\mathbf{R}] \subseteq A$ or $A \cap \varphi[\mathbf{R}] = \varnothing$. For suppose A detachable and let f: $\mathbf{R_d} \to 2$ be its characteristic function. Then $f \circ \varphi$: $\mathbf{R} \to 2$ must be constant since $\mathbf{R}$ is indecomposable. If $f \circ \varphi$ is constantly 1, then $\varphi[\mathbf{R}] \subseteq A$; if constantly 0, then $A \cap \varphi[\mathbf{R}] = \varnothing$. It follows that if φ is surjective then $\mathbf{R_d}$ is itself indecomposable.

Mention should be made here of $\mathbf{R_c}$, the set of *Cauchy reals*. This is defined to be the set of sequences $\boldsymbol{q} = (q_n)_{n\in\mathbf{N}}$ of rationals satisfying

$$\forall m \forall n[0 < m \leq n \to -\frac{1}{m} < q_m - q_n < \frac{1}{m}]$$

factored by the equivalence relation

$$\forall n[n > 0 \to -\frac{2}{n} \leq q_n - q'_n \leq \frac{2}{n}].$$

In his book *Topos Theory* Johnstone shows how to turn $\mathbf{R_c}$ into a commutative ring. (In the category of sheaves over a locally connected space X, $\mathbf{R_c}$ is the sheaf of locally constant real-valued functions on open subsets of X.). There is a natural injection $\mathbf{R_c} \xrightarrow{m} \mathbf{R_d}$ given by

$$m((q_n)) = \left(\left\{ q : \exists n(q < q_n - \frac{1}{n}) \right\}, \left\{ q : \exists n(q > q_n + \frac{1}{n}) \right\} \right).$$

In general, this injection is not onto, as shown by the example of sheaves on a locally connected space.

Finally, we describe how certain aspects of *nonstandard analysis* can be reproduced in **SIA.**

First, we define the set $\mathbb{N}$ of *standard* natural numbers to be the intersection of all *inductive* subsets of **N**, i.e.,

$$\mathbb{N} = \{n \in \mathbf{N}: \forall X \in \mathscr{P}\mathbf{N}\ [0 \in X \wedge \forall m \in \mathbf{N}(m \in X \rightarrow m + 1 \in X) \rightarrow n \in X]\}.$$

$\mathbb{N}$ of course satisfies *full induction*:

$$\forall X \in \mathscr{P}\mathbb{N}\ [0 \in X \wedge \forall m \in \mathbb{N}(m \in X \rightarrow m + 1 \in X) \rightarrow X = \mathbb{N}].$$

From the fact that **N** satisfies decidable induction it follows that

- $\mathbb{N}$ is a detachable subset of **N** iff $\mathbb{N}$ = **N.**

The full induction scheme for $\mathbb{N}$ may be used to show that **N** *is an end-extension of* $\mathbb{N}$, i.e.,

$$\forall x \in \mathbf{N} \forall n \in \mathbb{N}(x < n \rightarrow x \in \mathbb{N}).$$

We define the space of *infinitesimals* to be

$$\Gamma = \{x \in \mathbf{R} : \forall n \in \mathbb{N}(-1/(n+1) < x < 1(n+1))\}.$$

This contains the space Θ of noninvertible infinitesimals as well as the space of *invertible* or *Robinsonian infinitesimals*

$$\mathbb{I} = \{x \in \mathbf{R}\text{: } x \textit{ is invertible}\}.$$

Notice that noninvertible infinitesimals are strictly "smaller" than Robinsonian ones in that

$$\forall x \forall y[x \in \Theta \wedge y \in \mathbb{I} \wedge y > 0 \rightarrow x < y].$$

To assert the existence of Robinsonian infinitesimals is to assert that $\mathbb{I}$ is inhabited: this is equivalent to asserting that the set $\mathbf{N} - \mathbb{N}$ of *nonstandard integers* is inhabited, or equivalently, that the following holds:

$$(*) \qquad \exists n\in\mathbf{N}\forall m\in\mathbb{N}\ m < n.$$

When (*) is satisfied, as it is in certain models of **SIA,** we shall say that nonstandard integers, or invertible infinitesimals, are *present.* Notice that while it is perfectly consistent to assert the presence of invertible infinitesimals, i.e., that $\mathbb{I}$ be inhabited, it is inconsistent to assert the "presence" of nonzero noninvertible infinitesimals, i.e. that $\Theta - \{0\}$ be inhabited.

One can also postulate the condition

$$\forall n\in\mathbf{N}[\forall x\in\mathbf{N}-\mathbb{N}\ (x > n) \rightarrow n \in \mathbb{N}],$$

i.e. "a natural number which is smaller than all nonstandard natural numbers must be standard". This is in fact equivalent to the condition that $\mathbb{N}$ be a $\neg\neg$-stable subset of **N,** i.e. $\mathbf{N} - (\mathbf{N} - \mathbb{N}) = \mathbb{N}$. Assuming that nonstandard integers exist, this latter may be understood as asserting that as many such exist as is possible.

When nonstandard integers are present, one can establish an *overspill principle* for **N** in the following form: for any geometric formula $\alpha(x)$,

$$\forall m\in\mathbb{N}\exists n\in\mathbb{N}(m < n \wedge \alpha(n)) \rightarrow \neg\forall n[\alpha(n) \rightarrow n \in \mathbb{N}].$$

In the presence of invertible infinitesimals $\mathbf{R_d}$ is a nonstandard model of the reals without nilpotent elements. The passage via φ from **R** to $\mathbf{R_d}$ eliminates the nilpotent elements, but preserves invertible infinitesimals. When φ is onto, $\mathbf{R_d}$ is then an indecomposable nonstandard model of the reals.

Within **R** we have the subring of *accessible reals*

$$\mathbf{R_{acc}} = \{x\in\mathbf{R}\colon \exists n\in\mathbb{N}(-n < x < n)\},$$

in which $\mathbb{I}$ is an ideal. Since each open interval in **R** is indecomposable, $\mathbf{R_{acc}}$ satisfies the condition of being an inhabited set which includes, for each pair *x, y* of its members, an indecomposable subset *I* for which $\{x, y\} \subseteq I$. It follows from this that $\mathbf{R_{acc}}$ *is indecomposable.*

Within $\mathbf{R_d}$ we have the subring of *finite reals*

$$\mathbf{R_{fin}} = \{x \in \mathbf{R_d} \colon \exists n \in \mathbb{N}(-n < x < n)\}.$$

Clearly φ carries $\mathbf{R_{acc}}$ into $\mathbf{R_{fin}}$. Since $\mathbf{R_{acc}}$ is indecomposable, $\mathbf{R_{fin}}$ inherits an indecomposability property analogous to that established for $\mathbf{R_d}$, namely, if *A* is a detachable subset of $\mathbf{R_{fin}}$, then $\varphi[\mathbf{R_{acc}}] \subseteq A$ or $A \cap \varphi[\mathbf{R_{acc}}] = \varnothing$.

Finally, we observe that $\mathbf{R_{fin}}$ can only be a detachable subset of $\mathbf{R_d}$ when $\mathbf{N} = \mathbb{N}$, or equivalently, when $\mathbf{R_{acc}}$ and **R** coincide, or to put it another way, there are no invertible infinitesimals. For if $\mathbf{R_{fin}}$ is detachable in $\mathbf{R_d}$, then either $\varphi[\mathbf{R}] \subseteq \mathbf{R_{fin}}$, or $\mathbf{R_{fin}} \cap \varphi[\mathbf{R}] = \varnothing$. The latter being obviously false, it follows that $\varphi[\mathbf{R}] \subseteq \mathbf{R_{fin}}$. But then $\varphi[\mathbf{N}] \subseteq \mathbf{R_{fin}} \cap \varphi[\mathbf{N}] = \varphi[\mathbb{N}]$, whence $\mathbf{N} \subseteq \mathbb{N}$. Thus, in the presence of invertible infinitesimals, the property of being a finite Dedekind real is undecidable.

References

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