HEYTING ALGEBRAS

Heyting Algebras (named after the Dutch mathematician *Arend Heyting* (1898- 19?)) are algebraic structures that play in relation to intuitionistic logic (*q.v.*) a role analogous to that played by Boolean algebras (*q.v*) in relation to classical logic. They are most simply defined as a certain type of lattice. A *lattice* is a partially ordered set $(L \leq)$ in which every pair of elements x, y has a least upper bound, or *join*, denoted by $x \vee y$, and a greatest lower bound, or *meet*, denoted by $x \wedge y$. A *top* (*bottom*) element of a lattice L is an element, denoted by 1 (0) such that $x \leq 1$ $(0 \leq x)$ for all $x \in L$. Now a *Heyting algebra* is defined to be a lattice $(L \leq)$, possessing distinct top and bottom elements, such that, for any pair of elements $x, y \in L$, the set of $z \in L$ satisfying $z \wedge x \leq y$ has a largest element. This element, which is uniquely determined by x and y, is denoted by $x \Rightarrow y$: thus $x \Rightarrow y$ is characterized by the following condition: for all $z \in L$,

$$z \leq x \Rightarrow y \text{ if and only if } z \wedge x \leq y.$$

The binary operation on a Heyting algebra which sends each pair of elements x, y to the element $x \Rightarrow y$ is called *implication*; the operation which sends each element x to the element $-x = x \Rightarrow 0$ is called *negation* (or pseudocomplementation): we note that the latter operation satisfies

$$z \leq -x \text{ if and only if } z \wedge x = 0 \text{ if and only if } x \leq -z.$$

It is not difficult to show that any Heyting algebra is a *distributive lattice*, that is, the following equalities hold:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),\ \ x \vee (y \wedge z) = (x \vee y) \wedge (x \wedge z).$$

Heyting algebras are associated with theories in intuitionistic logic in the same way as Boolean algebras are associated with theories in classical logic. Indeed, given a consistent theory T in an intuitionistic propositional or first-order language **L**, define the equivalence relation $\approx$ on the set of formulas of **L** by $\varphi \approx \psi$ if $T \vdash \varphi \leftrightarrow \psi$. For each formula φ write $[\varphi]$ for its $\approx$-equivalence class. Now define the relation $\leq$ on the set $H(T)$ of $\approx$-equivalence classes by $[\varphi] \leq [\psi]$ if and only if $T \vdash \varphi \rightarrow \psi$. Then $\leq$ is a partial ordering of $H(T)$ and the partially ordered set $(H(T), \leq)$ is a Heyting algebra in which $[\varphi] \Rightarrow [\psi] = [\varphi \rightarrow \psi]$, with analogous equalities defining the meet and join operations, 0, and 1. $H(T)$ is called the the Heyting algebra *determined by T*. It can be shown that Heyting algebras of the form $H(T)$ are typical in the sense that, for any Heyting algebra L, there is a propositional intuitionistic theory T such that L is isomorphic to $H(T)$. This is the exact sense in which Heyting algebras are identified as the algebras of intuitionistic logic.

Topology furnishes another, and equally important, source of Heyting algebras. If X is a topological space, then the partially ordered set $(\mathbf{O}(X), \subseteq)$ is a Heyting algebra, where $\mathbf{O}(X)$ is the family of all open sets in X, and $\subseteq$ is the partial ordering of set inclusion. In $\mathbf{O}(X)$ meet and join are just set-theoretic intersection and union, while the implication operation is given by

$$U \Rightarrow V = \textit{interior of}\,(X - U) \cup V.$$

It can be shown that, for any Heyting algebra L, there is a topological space X such that L is isomorphic to a sub-Heyting algebra of $\mathbf{O}(X)$. In this sense, Heyting algebras may also be identified as the *algebras of topology.*

Bibliography

Balbes, R., and Dwinger, P., *Distributive Lattices*. Columbia, Missouri: University of Missouri Press, 1974.

Rasiowa, H., and Sikorski, R., *The Mathematics of Metamathematics.* Warszawa: PWN, 1963.