NOTES ON FORMAL LOGIC

BY

JOHN L. BELL

(COMPILED & EMBELLISHED BY ROBERT CLIFTON)

Logic is the beginning of wisdom, not the end.

Mr. Spock

On the contrary, I find nothing in logistic but shackles. It does not help us at all in the direction of conciseness, far from it; and if it requires 27 equations to establish that 1 is a number, how many will it require to demonstrate a real theorem?

Henri Poincaré

CONTENTS

I. PROPOSITIONAL LOGIC

1. Statements, Arguments and Counterexamples

An *inference* or *argument* is any list of *statements* divided into *premises* and *conclusions*. We shall confine our attention to inferences with just one conclusion, for example,

1. Either this man's dead, or my watch has stopped.

2. This man is not dead.

3. Therefore, my watch has stopped.

Here statements 1 and 2 are premises, and 3 the conclusion.

We shall assume that the constituent statements of inferences are assertive in the sense that each can be assigned either of two *truth values* – *true* (t) or *false* (f) – as the case may be. (An example of a statement that would *not* fall into this category is “Would you please go to the store”.) Granted this, it is natural to declare an inference *valid* if its conclusion is true in any case in which its premises are true. (Thus, on the basis of our usual grasp of the meaning of the terms "or" and "not", the inference above would count as valid.) A *counterexample* to an inference is a case in which all its premises are *true* but its conclusion is *false*. Thus an inference is *valid* provided it *has no counterexamples*, and *invalid* if it *has counterexamples*. As an example of an invalid inference, consider the following:

Either this man's dead, or my watch has stopped.

This man is dead.

Therefore, my watch has stopped.

This inference is invalid, because the situation in which this man is dead and my watch hasn't stopped constitutes a counterexample.

To investigate the validity of inferences we need first to consider how their constituent statements are formed, and how these are then to be assigned truth values. As the basic ingredients from which we shall fashion all such statements we shall take simple declarative sentences of the kind "It is raining", "The cranes are flying", etc. Such statements will be called *elementary* statements: we shall assume that elementary statements can be assigned truth values arbitrarily and entirely independently of one another.

From elementary statements we obtain *compound statements* by applying the *syntactical operations* "and", "or", "not", "if...then". (In this way we obtain, e.g. statements such as "It is raining and the cranes are flying", "If it is raining, the cranes are not flying", etc.) These operations are *truth functional* in the sense that the truth value of any compound statement built up from them is unambiguously determined by the truth values of its constitutive elementary statements.

We shall use capital letters A,B,C,... to denote elementary statements, and symbols

$\wedge$ for "and" (conjunction),

$\vee$ for "or" (disjunction),

$\neg$ for "not" (negation or denial),

$\rightarrow$ for "if...then" (implication).

The symbols A,B,C,... are called *statement letters*, and the symbols $\wedge$, $\vee$,$\neg$, $\rightarrow$, *logical operators*. "$A \rightarrow B$" is sometimes read "A implies B", or "B if A", or "A only if B". Also "$\vee$" has the inclusive sense of "and\or"; it may also be understood as "unless".

Using these symbols, *statements* are obtained by starting with the statement letters — which of course count as the simplest kind of statement — and applying the logical operators $\wedge$, $\vee$, $\neg$, $\rightarrow$ to these, using parentheses and brackets as necessary to eliminate ambiguity. So, for example, from the statement letters A, B, C, we obtain the compound statements $A \wedge B$, $(A \wedge B) \vee C$, $(A \to B) \vee C$, $\neg[(A \to B) \vee C]$, etc. We shall use letters p,q,r,... to denote arbitrary statements (elementary or compound).

Formally, a *statement*, or *propositional statement*, may now be defined by means of the following 'rules of formation':

1. Any statement letter is a statement.
2. If p and q are statements, so are $(p \wedge q)$, $(p \vee q)$, $(p \to q)$, $(\neg p)$.

Here p,q are called the *conjuncts* in the *conjunction* $(p \wedge q)$ and the *disjuncts* in the *disjunction* $(p \vee q)$. (Thus, for example, in the disjunction $(((\neg A) \vee B) \vee C)$ the disjuncts are $((\neg A) \vee B)$ and C.) Also p is called the *antecedent* and q the *consequent* in the *implication* $(p \to q)$.

Strictly speaking, any sequence of statement letters and logical operators that cannot be generated by repeated application of rules 1 and 2 above will not, for us, count as a statement. For example, $\neg A \to \vee B$ is obviously not allowed ("If not A, then or B" is gibberish), while $((\neg A) \vee B)$ is allowed. However, in the interests of brevity (and keeping our sanity!) we shall bend the rules slightly and feel free to omit parentheses in statements when no ambiguity is likely to result. For example, instead of $((\neg A) \vee B)$ we shall write simply $\neg A \vee B$. For obviously there's no need for the outside brackets if this statement is meant to be a complete statement and not a component of some larger statement (like one disjunct of a larger disjunction). Also, by replacing $(\neg A)$ by $\neg A$ we are agreeing to understand the "not" operator as only acting upon the statement letter A and not upon the larger statement $A \vee B$. If, instead, we wanted to symbolize the statement "Neither A nor B" then we would have to write $\neg(A \vee B)$ so that the *scope* of the $\neg$ operator covers the entire disjunction. As a general rule of thumb, insert parentheses only when it will not otherwise be clear to your readers what statement is being negated, or what statements are being disjoined, conjoined, etc. (For example, $A \wedge B \vee C$ is *not* clear, while $A \wedge (B \vee C)$ is; $A \rightarrow \neg B \vee C$ is *not* clear, $(A \rightarrow \neg B) \vee C$ is; etc.)

2. Truth Tables and Testing Validity

The rules for computing the truth values of compound statements are as follows.

▹ $A \wedge B$ is true if A and B are both true, and false if at least one of A and B is false.

▹ $A \vee B$ is true if at least one of A and B is true, and false if both A and B are false.

▹ $\neg A$ is true if A is false, and false if A is true.

▹ $A \rightarrow B$ is false when A is true and B is false, but true in all other cases.

The least intuitive of these rules is the last one. The idea here is that we want a statement of the form $p \rightarrow q$ to be false exactly when the truth values of p and q constitute a counterexample to the validity of the inference from p to q, that is, when p is true and q is false. In all other cases, $p \rightarrow q$ shall be declared true.[♣]

The above rules for computing truth values may be summed up in the form of *truth tables*.

[♣] For those to whom this seems odd, consider the following down-to-earth example to make it more palatable. Suppose that, upon leaving for work in the morning, I promise my wife "If I go to the store (p), then I will buy some milk (q)". When I arrive back from work in the evening, she asks me whether I picked up any milk, and I say No. Did I break my promise? That is: should $p \rightarrow q$ be declared false in this case; a case where, in fact, both p and q turned out to be false? Surely not: I would only have broken the promise if in fact I did go to the store but did *not* buy any milk (due to an oversight, or lack of money, or what have you). Nevertheless, it must be admitted that our definition of the truth-conditions for the so-called 'material' conditional $\rightarrow$ fails to do justice to *all* our intuitions about how the "if…then…" construction in natural language functions. For example, we are being forced to declare that "If New York is a big apple, then grass is green" is true simply on the basis of its consequent being true (which it is). A more sophisticated treatment of conditionals would involve discussing 'strict' conditionals, 'counterfactual' conditionals, etc. which are beyond the scope of these notes.)

A	B	$A \wedge B$	$A \vee B$	$\neg A$	$A \rightarrow B$
t	t	t	t	f	t
t	f	f	t	f	f
f	t	f	t	t	t
f	f	f	f	t	t

Each line under the first two (A,B) columns represents an assignment of truth values — a *(truth) valuation*. Here there are $2^2 = 4$ valuations. If we had n statement letters $A_1, \ldots, A_n$ there would be 2^n valuations.

So far we have laid down rules for forming compound statements from elementary statement letters. We also have rules for determining the truth values of any compound statement given the logical operators that occur in it and any truth valuation of its statement letters. Armed with these rules, we can now test a few inferences for validity.

It is raining and the cranes are flying.
Therefore, it is raining.

This has the form

$$\frac{A \wedge B}{A}$$

The inference is valid since, according to the truth table for $\wedge$, whenever the premise $A \wedge B$ is true, so is the conclusion A, thus showing that there are no counterexamples.

It is raining or the cranes are flying.
It isn't raining.
Therefore, the cranes are flying.

This has the form

$$\frac{\begin{array}{c} A \vee B \\ \neg A \end{array}}{B}$$

Examining the truth table for possible counterexamples we find

		premises		conclusion
A	B	A	A ∨ B	B
t	f	f	t	f
f	f	t	f	f

Notice that we only needed to examine the (two) cases in which the conclusion (B) is *false*, since counterexamples cannot arise in any other way. Since neither of *these* cases constitutes a counterexample, there are none, and the inference is, accordingly, *valid*.

If it is raining, the cranes are flying.

It is raining.

Therefore, the cranes are flying.

This has the form

$$\frac{\begin{array}{c} A \rightarrow B \\ A \end{array}}{B}$$

Examine the truth table for possible counterexamples (conclusion false):

A	B	A	A → B	B
t	f	t	f	f
f	f	f	t	f

Neither of these cases constitutes a counterexample, so the inference is *valid*.

If it is raining, the cranes are flying.

The cranes are flying.

Therefore, it is raining.

This one has the form

$$\frac{\begin{array}{c}A \rightarrow B \\ B\end{array}}{A}$$

The following line in the truth table is a counterexample (in fact the only one)

A	B	A → B	B	A
f	t	t	t	f

The inference is, accordingly, *invalid*.

If it is raining, the cranes are flying.

If the cranes are flying, the bears are restless.

Therefore, if it raining, the bears are restless.

This has the form

$$\frac{\begin{array}{c}A \rightarrow B \\ B \rightarrow C\end{array}}{A \rightarrow C}$$

Now the only possible counterexamples arise when the conclusion A → C is false. This can happen just when A is true and C is false. Therefore we need merely examine the two lines in the truth table in which this occurs. These are the following:

A	B	C	A → B	B → C	A → C
t	t	f	t	f	f
t	f	f	f	t	f

Since in neither of these lines are both premises $A \to B$ and $B \to C$ true, neither constitutes a counterexample, so there are none, and the inference is, accordingly, *valid*.

We shall use the notation

$$p_1,\ldots,p_n \models q$$

to indicate that the inference from the statements $p_1,\ldots,p_n$ to the statement q is valid. Thus the validity of the first three inferences above may be symbolized:

$$A \wedge B \models A; \quad A \vee B, \neg A \models B; \quad A, A \to B \models B.$$

We read "$p_1,\ldots,p_n \models q$" as "$p_1,\ldots,p_n$ (logically) *imply* q" or "q *follows from* $p_1,\ldots,p_n$".♦

3. Tautologies, Contradictions and Satisfiability

Sometimes conclusions are obtainable *without using premises*. For example, consider the premiseless "inference"

Therefore, it is raining or it is not raining.

$$\frac{\qquad}{A \vee \neg A}$$

This "inference" is valid because in its truth table

♦ Note that $p \models q$ is not itself a statement in our logical language, like $p \to q$, but rather a kind of 'meta-statement' or statement *about* statements—i.e. the statement that the argument from p as premise to q as conclusion is a valid argument. However, there is an obvious connection between the expressions $p \models q$ and $p \to q$, namely, the former holds exactly when the latter's truth-table has no f's.

A	$A \vee \neg A$
t	t
f	t

the conclusion $A \vee \neg A$ is *always true*: there are *no* counterexamples.

A statement which, like $A \vee \neg A$, is true in all possible cases is called (logically) *valid* or a *tautology*. An inference with a valid conclusion is *always* valid, regardless of what its premises are. We shall use the symbol t to stand for a fixed tautologous statement, which for definiteness we shall take to be the statement $A \vee \neg A$ (although any tautology would do). The symbol "t" is doing double duty: it indicates *both* a truth value *and* a particular statement. Notice that we then have

$$p \models t$$

for *any* statement p.

A set S of statements is said to be *satisfiable* or *consistent* if there is *at least one* case in which all the members of S are true, and *unsatisfiable* or *inconsistent* if not. This concept is related to that of validity in the following way.

If $p_1, ..., p_n \models q$, *then the set* $\{p_1, ..., p_n, \neg q\}$ *is unsatisfiable, and conversely*.

For the unsatisfiability of $\{p_1, ... p_n, \neg q\}$ is just the assertion that $p_1, ..., p_n, \neg q$ are never simultaneously true, which amounts to asserting that $\neg q$ is false, i.e. q is true, whenever $p_1, ..., p_n$ are. In particular, it follows that if $\{p_1, ..., p_n\}$ is *unsatisfiable*, then $p_1, ... p_n \models q$ for *any* statement q. That is, *inconsistent premises yield any conclusion whatsoever*.

A *single* unsatisfiable statement (e.g. $A \wedge \neg A$) is called a *contradiction* (or *inconsistent*). Thus a contradiction is a statement which is *always* false. Notice that contradictions are exactly the negations of tautologies. We shall use the symbol "f" to stand for a fixed contradiction, which for definiteness we take to be the statement

$A \wedge \neg A$ (although, as in the case of "t", it matters not which particular contradiction we choose). Notice that we now have, for *any* statement p,

$$f \models p \ .$$

A statement is said to be *contingent* if it is neither a tautology nor a contradiction; so, a contingent statement is one which is true in at least one case, and false in at least one case. Any statement is either tautologous, contradictory, or contingent: we shall later develop an efficient technique for deciding which

4. Exercises

<u>Truth Tables and Testing Validity</u>

A1. Use truth tables to determine whether the statements in the left column (jointly) imply the corresponding statement in the right column.

(a)	$A, B\rightarrow\neg C$	$\neg C$
(b)	$\neg(A\rightarrow B)$	$\neg B\rightarrow\neg A$
(c)	$A, [(A\vee B)\rightarrow B]$	$A\wedge B$
(d)	$A\rightarrow (\neg\neg B\rightarrow C)$	$(A\wedge\neg C)\rightarrow\neg B$
(e)	$A\rightarrow(B\vee C), A\rightarrow\neg B$	$A\rightarrow C$
(f)	$A\rightarrow\neg B, [(A\wedge C)\rightarrow B]$	$\neg[(\neg A\wedge C)\rightarrow A]$
(g)	$A\rightarrow(B\rightarrow C)$	$(A\wedge B)\rightarrow C$
(h)	$A\rightarrow(B\vee C)$	$(A\rightarrow B)\vee(A\rightarrow C)$
(i)	$\neg(A\wedge B)$	$\neg A\vee\neg B$
(j)	$\neg(A\vee B)$	$\neg A\wedge\neg B$
(k)	$(A\rightarrow B)\rightarrow B$	$A\vee B$

(l)	$A\to(A\to B)$	B
(m)	$(A\land B)\to C$, $\neg C$, A	$\neg B$
(n)	$A\to B$, $A\to\neg B$	$\neg A$
(o)	$\neg A\to A$	A

A2. Symbolize each of the following inferences and use truth tables to determine which of them are valid.

(a) Silas is either a knave or a fool. Silas is a knave; so, he's no fool.

(b) You may enter only if the Major's out. The Major is out. So you may enter.

(c) There will be a fire! For only if there's oxygen present will there be a fire. And of course there's oxygen present.

(d) If I'm right, then you're wrong. But if you're wrong, then I can't be right. Therefore you're wrong.

(e) If I'm right, then you're wrong. But if you're wrong, then I can't be right. Therefore, I can't be right.

(f) If they retreat provided we attack, then we attack. But they won't retreat. Therefore we attack.

(g) It's a duck if it walks and quacks like one. Therefore, either it's a duck if it walks like one or it's a duck if it quacks like one.

(h) You cannot serve both God and Mammon. But if you don't serve Mammon, you'll starve; if you starve, you can't serve God. Therefore, you can't serve God.

(i) If today's Friday, we must be in Toronto. Today is Friday, but we're not in Toronto. So we're in London.

(j) Computers can think only if they have emotions. If computers can have emotions then they can have desires as well. But computers can't think if they have desires. Therefore computers can't think.

A3. Knaves always lie, knights always tell the truth, and in Camelot, where everybody is one or the other (but you can't tell which by just looking), you encounter two people, one of whom says to you: "He's a knight or I'm a knave." What are they?

A4. Symbolize this argument and use a truth table (truncated, if you like) to determine whether it is valid:

> If I'm right, then you're a fool. If I'm a fool, I'm not right. If you're a fool, I am right. So one or other of us is a fool!

A5. Symbolize this set of sentences and determine (using a truth table or otherwise) whether the set is consistent (i.e. satisfiable):

> Either the witness was not intimidated, or if Flaherty committed suicide, a note was found. If the witness was not intimidated, then Flaherty did not commit suicide. If a note was found, then Flaherty committed suicide.

A6. Translate the following two arguments into logical notation (defining your symbols). Then use a truth table (truncated, if you like) to determine whether the arguments are valid. For the invalid arguments (if any), supply <u>all</u> counterexamples.

(a) You will eat and either I will eat or we shall starve. Therefore, you and I will eat or we shall starve.*(assume that: starve = not eat)*

(b) We'll win! For if they withdraw if we advance, we'll win. And we won't advance!

A7. Knaves always lie, knights always tell the truth, and in Camelot, where everybody is one or the other, you encounter two people, one of whom says to you: "He's a knight <u>and</u> I'm a knave." What are they?

A8. Politicians always lie, taxpayers always tell the truth, and in the US, where everybody is one or the other (but never both, as we all know!), you encounter two people, one of whom points to the other and grudgingly declares:

"I'm a taxpayer if and only if he is!"

What are they?

A9. Using truth tables, determine whether the following arguments are valid.

(a) $A \rightarrow (H \wedge J)$
$J \leftrightarrow H$
$\neg J$
$\therefore \neg A$

(b) $(D \leftrightarrow \neg G) \wedge G$
$(G \vee [(A \rightarrow D) \wedge A]) \rightarrow \neg D$
$\therefore G \rightarrow \neg D$

(c) $(G \leftrightarrow H) \vee (\neg G \leftrightarrow H)$
$\therefore (\neg G \leftrightarrow \neg H) \vee \neg(G \leftrightarrow H)$

A10. Consider the following argument:

> This argument is unsound, for its conclusion is false, and no sound argument has a false conclusion.

Is this argument sound? ('Sound" means "Valid + True Premises".)

Tautologies, Contradictions and Satisfiability

B1. Classify the following statements as tautologous, contradictory or contingent:

(a) $(A \rightarrow B) \vee (B \rightarrow A)$ (b) $[(A \rightarrow B) \wedge B] \rightarrow A$ (c) $[(A \wedge B) \rightarrow C] \rightarrow [(A \rightarrow C) \vee (B \rightarrow C)]$

B2. Which of the the following assertions is correct and why:

(a) There is a statement that implies every other statement.

(b) Any statement that follows from a satisfiable statement is satisfiable.

(c) Any statement implying a contingent statement is contingent.

(d) Any statement that follows from a contingent statement is contingent.

(e) Any statement that follows from a valid statement is valid.

(f) Any statement that implies a valid statement is valid.

(g) All contingent statements imply one another.

(h) No inference with a contradiction as conclusion can be valid.

(i) No statement implies its own negation.

(j) Each of the disjuncts of a valid disjunction is valid.

(k) An implication is valid precisely when the consequent follows from the antecedent.

(l) Any statement implied by its own negation is valid.

(m) Removing a premise from a valid argument cannot affect its validity.

(n) In a valid argument, the conclusion is always consistent with the premises; in a sound argument it is not. (Note: An argument is sound exactly when it is both valid and has true premises.)

B3. Determine which of the following five assertions are correct, justifying your answer.

(a) If a statement is not contingent, nor can its negation be.

(b) Every valid argument with a satisfiable set of premises has a satisfiable conclusion.

(c) If a conjunction is a tautology, so is each of its conjuncts.

(d) An invalid argument can always be made into a valid one by adding premises.

(e) The argument from p to q is valid if and only if $p \rightarrow q$ is valid.

B4. Circle the correct answer to each of the questions below.

(a) Identify the statement which is a contradiction in the following:

(i) $t \rightarrow t$ (ii) $t \rightarrow f$ (iii) $f \rightarrow t$ (iv) $f \rightarrow f$

(b) Identify the statement which is valid in the following:

(i) $A \wedge A$ (ii) $A \vee A$ (iii) $A \rightarrow A$ (iv) $(A \rightarrow A) \wedge A$

(c) Any argument with an unsatisfiable set of premises must be:

(i) valid and sound (ii) invalid and sound

(iii) valid and unsound (iv) invalid and unsound

B5. Provide a one to two sentence answer for these questions.

(a) Explain why each conjunct of a valid conjunction must itself be valid.

(b) Why is it that whenever the pair of statements $\{p, \neg c\}$ is jointly satisfiable we can't write $p \models c$?

B6. Circle the correct answer to each of the questions below.

(a) Which of the following statements is valid?

(i) $A \wedge A$ (ii) $A \vee A$

(iii) $(A \leftrightarrow A) \wedge A$ (iv) $A \leftrightarrow A$

(b) Which of the following statements is not equivalent to any of the others?

(i) $B \wedge \neg B$ (ii) $B \leftrightarrow \neg B$

(iii) $B \rightarrow \neg B$ (iv) $\neg B \leftrightarrow B$

(c) Which of the following statements is a contradiction?

(i) t (ii) $f \rightarrow t$

(iii) $t \vee f$ (iv) $t \wedge (t \rightarrow f)$

(d) Any argument that concludes with a tautology must be:

(i) valid + sound (ii) valid + <u>un</u>sound

(iii) valid (iv) sound

(e) At least one of the disjuncts of a valid disjunction <u>must</u> be:

(i) valid (ii) sound

(iii) consistent (iv) contingent

(f) The consequent of an inconsistent conditional <u>cannot</u> be:

(i) unsatisfiable (ii) satisfiable

(iii) a conjunction (iv) inconsistent

B7. Indicate whether each of the following statements is <u>true</u> or <u>false.</u>

(a) If a statement is not valid, its negation must be.

(b) If a statement fails to logically imply an other, it must imply the negation of that other.

(c) A statement that logically implies an other cannot imply the negation of that other.

(d) If a set of statements is satisfiable, so is each statement in the set.

(e) If each statement in a set is satisfiable, so is the set.

(f) You cannot make a valid argument invalid by adding more premises.

(g) You cannot make an invalid argument valid by removing premises.

(h) Sound arguments can never have f as their conclusion.

B8. Indicate whether each of the following statements is true or false.

(a) A tautologous conjunction must have a tautologous conjunct.

(b) A contradictory disjunction must have a contradictory disjunct.

(c) If neither a statement nor its negation is valid, then both must be consistent.

(d) If a conditional is unsatisfiable, its consequent must be too.

(e) A contingent statement can logically imply both a statement and the negation of that statement.

(f) No subset of a set of satisfiable statements can be unsatisfiable.

(g) Every statement logically implies at least one other statement with which it is not equivalent.

(h) You can never make an invalid argument into a sound one by dropping some of its premises.

(i) You can never make a valid argument into an unsound one by adding more premises to it.

(j) Some statements are equivalent to every statement that logically implies them.

B9. Using truth tables (where necessary), decide if the following sets of sentences are satisfiable.

(a) $\{A \rightarrow B, B \rightarrow C, A \rightarrow C\}$

(b) $\{\neg[J \vee (H \rightarrow L)], L \leftrightarrow (\neg J \vee \neg H), H \leftrightarrow (J \vee L)\}$

(c) $\{(J \rightarrow J) \rightarrow H, \neg J, \neg H\}$

(d) $\{A, B, C\}$

(e) $\{(A \wedge B) \vee (C \rightarrow B), \neg A, \neg B\}$

B10. True or False?

(a) A conjunction with one valid conjunct must itself be valid.

(b) An implication with a valid consequent must itself be valid.

(c) A disjunction with one unsatisfiable disjunct must itself be unsatisfiable.

(d) A sentence is valid iff its negation is unsatisfiable.

(e) An implication with a valid antecedent must itself be valid.

B11. Using truth tables, determine whether the following are valid.

(a) $(F \vee H) \vee (H \leftrightarrow \neg F)$ (b) $\neg A \rightarrow [(B \wedge A) \rightarrow C]$

II. EQUIVALENCE

1. Equivalence and Bi-implication

Two statements are called (logically) *equivalent* if they take the *same* truth values in all possible cases. For example, consider the truth tables for the statements $A \rightarrow B$, $\neg B \rightarrow \neg A$:

A	B	$A \rightarrow B$	$\neg B \rightarrow \neg A$
t	t	t	t
t	f	f	f
f	t	t	t
f	f	t	t

Since $A \rightarrow B$ and $\neg B \rightarrow \neg A$ have the same truth value on every line of the table, they are equivalent.

We write $p \equiv q$ to indicate that the statements p and q are equivalent. We may think of $\equiv$ as a kind of *equality* between statements. We leave it to the reader as an exercise to show that for any statements p,q the assertion that $p \equiv q$ amounts to the same thing as:

$$p \models q \text{ and } q \models p$$

In connection with $\equiv$, we can define a new logical operator" $\leftrightarrow$ " called *bi-implication* (or 'if and only if') as follows:

A	B	$A \leftrightarrow B$
t	t	t
t	f	f
f	t	f
f	f	t

Thus $A \leftrightarrow B$ has value "t" exactly when A and B have the *same* truth value. It follows from this that $p \equiv q$ holds when and only when the statement $p \leftrightarrow q$ is valid. The statements p and q are called the *components* of $p \leftrightarrow q$.

It is easy to check the following equivalences:

$$p \equiv \neg\neg p$$

$$p \vee q \equiv \neg(\neg p \wedge \neg q)$$

$$p \wedge q \equiv \neg(\neg p \vee \neg q)$$

$$p \rightarrow q \equiv \neg p \vee q$$

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

We see from these equivalences that $\rightarrow$ and $\leftrightarrow$ are in a natural sense *expressible* in terms of $\{\wedge,\vee,\neg\}$ and hence in terms both of $\{\wedge,\neg\}$ and $\{\vee,\neg\}$. The question now arises as to whether *every* possible truth function is so expressible. The answer, as we shall see, is *yes*.

2. Expressive Completeness

We begin by describing what is meant by the *disjunctive normal form* of a truth function. Let us suppose we are given a truth function H of n statement letters $A_1,\ldots,A_n$. Thus, for each valuation of $A_1,\ldots A_n$, a corresponding (truth) value of $H(A_1,\ldots,A_n)$ is obtained. (For example, $H(A_1, A_2, A_3) = (A_1 \vee A_2) \leftrightarrow A_3$ defines a truth function, since truth values for A_1, A_2, and A_3 fix a truth value for $(A_1 \vee A_2) \leftrightarrow A_3$ via the truth tables for $\vee$ and $\leftrightarrow$.) We display this situation in the form of a truth table:

A_1	A_2	...	A_n	$H(A_1,\ldots,A_n)$
t	t	...	t	*
t	t	...	f	*
f	f	...	f	*

Assume first that at least *one* of the entries in the H column is "t". For each valuation of $A_1,\ldots,A_n$ in which a "t" appears in the H column we form the conjunction $A_1^* \wedge \ldots \wedge A_n^*$ where each A_i^* is A_i if the given valuation assigns t to A_i and $\neg A_i$ if not. Notice that this conjunction is true *precisely* under the given valuation and no other. Now we form the *disjunction* of all these conjunctions arising from the "t" cases of the given truth table. The resulting statement is called the *disjunctive normal form* (*d.n.f.*) of the given truth function. Clearly, its truth table is identical to that of the given truth function.

It remains to consider the case in which the given truth function always takes the value "f". Here we may take the disjunctive normal form to be, e.g., $A_1 \wedge \neg A_1$.

Since d.n.f.s contain only the logical operators $\wedge$, $\vee$, $\neg$, it follows from all this that *every possible truth function can be expressed in terms of* $\wedge$, $\vee$, $\neg$, and so *every statement is equivalent to one whose only logical operators are these*. We sum this up by saying that the set $\{\wedge, \vee\ \neg\}$ is *expressively complete*. Moreover, since $\wedge$ is expressible in terms of $\vee$, $\neg$ and $\vee$ in terms of $\wedge$, $\neg$, we may infer that *each of the sets* $\{\wedge,\neg\}$ *and* $\{\vee\ \neg\}$ *is expressively complete*.

Before proceeding further let us determine a d.n.f. in a practical case. Suppose we are given, for instance, the truth table

A	B	C	H(A,B,C)
t	t	t	t
t	t	f	f
t	f	t	t
t	f	f	t
all remaining lines			f

The d.n.f. here is, writing $\underline{A}$ for ¬A etc. and omitting the "∧"s,

$$ABC \lor A\underline{B}C \lor A\underline{BC}$$

The question arises as to whether there are *single* logical operations (involving just two statement letters) which are expressively complete. We shall see that there are exactly *two* of these.

We define the logical operators ("Sheffer strokes") "|" —*nand* — and "↓" — *nor* — by means of the following truth tables.

A	B	A|B	A ↓ B
t	t	f	f
t	f	t	f
f	t	t	f
f	f	t	t

Clearly, $A \mid B \equiv \neg(A \land B)$ and $A{\downarrow}B \equiv \neg(A \lor B)$ (hence 'nand' is short for 'not and' and 'nor' short for 'not or'!).

First, we show that | and ↓ are each expressively complete. To do this it suffices to show that ¬ and ∨ are both expressible in terms of |, and ¬ and ∧ in terms of ↓. (Why?)

Clearly $A \mid A \equiv \neg(A \land A) \equiv \neg A$, so ¬ is expressible in terms of |. Now

$$A \mid B \equiv \neg(A \land B) \equiv \neg A \lor \neg B,$$

so

$$\neg A \,|\, \neg B \equiv \neg\neg A \vee \neg\neg B \equiv A \vee B.$$

Hence, recalling that $\neg A \equiv A \,|\, A$, we see that

$$A \vee B \equiv (A \,|\, A) \,|\, (B \,|\, B),$$

and so $\vee$ is expressible in terms of $|$.

Similarly, $\neg A \equiv A{\downarrow}A$ and $A \wedge B \equiv (A{\downarrow}A){\downarrow}(B{\downarrow}B)$. Therefore $|$ and $\downarrow$ are each expressively complete.

We next show that $|$ and $\downarrow$ are the *only* expressively complete logical operations on two statement letters.

For suppose that H(A,B) is expressively complete. If H(t,t) were t, then any statement built up using only H would take the value t when all its statement letters take value t. So $\neg A$ would not be expressible in terms of H. Therefore H(t,t) = f. Similarly, H(f,f) = t. So we obtain the partial truth table

A	B	H(A,B)
t	t	f
t	f	
f	t	
f	f	t

If the second and third entries in the last column are t,t or f,f, then H is $|$ or $\downarrow$. If they are f,t, then $H(A,B) \equiv \neg A$; and if they are t,f, then $H(A,B) \equiv \neg B$. So in both of these cases H would be expressible in terms of $\neg$. But clearly $\neg$ is not expressively complete by itself, since the truth function t is not expressible in terms of it. So H is $|$ or $\downarrow$ as claimed.

3. Arithmetical Representation of Statements and Logical Operations

Statements and logical operations can be nicely expressed within *binary arithmetic*: the arithmetic of 0 and 1.

First, we describe the rules of binary arithmetic. We suppose given the two numbers 0,1 and two operations "+" (plus) and "." (times) on them subject to the following rules (only one of them may be unfamiliar!):

$$0 + 0 = 1 + 1 = 0 \qquad 0.0 = 0.1 = 1.0 = 0$$

$$0 + 1 = 1 + 0 = 1 \qquad 1.1 = 1$$

We shall think of statements as determining *binary functions* (that is, functions taking just the values 0 and 1) as follows. Statement letters A,B,C,... will be regarded as *variables* taking values 0,1: we think of 1 as representing the truth value t and 0 as representing the truth value f. Then the operation "$\wedge$" corresponds to "." and the operation "$\neg$" to the operation "1 + " of adding 1.

Given this, how do we interpret "$\vee$" and "$\rightarrow$"? We argue as follows.

$$\begin{aligned} A \vee B &\equiv \neg(\neg A \wedge \neg B) \\ &= 1 + (1 + A).(1 + B) \\ &= 1 + 1 + A + B + A.B \\ &= 0 + A + B + A.B \\ &= A + B + A.B \end{aligned}$$

And

$$\begin{aligned} A \rightarrow B &\equiv \neg A \vee B \\ &= 1 + A + B + (1 + A).B \\ &= 1 + A + B + B + A.B \\ &= 1 + A + A.B \end{aligned}$$

In this way, any statement p gives rise to a binary function called its *binary representation* which we shall denote by the same symbol p. In that case, *tautologies* are those statements whose binary representations take only value 1, and *contradictions* those statements whose binary representations take only value 0.

When, for example, is $p \rightarrow q$ a tautology? Exactly when the corresponding binary representation $1 + p + p.q$ is constantly 1. But this is the case precisely when $0 = p + p.q = p.(1 + q)$, that is, when at least one of p and $1 + q$ is 0, in other words, if $p = 1$, then $1 + q = 0$, i.e. $q = 1$. But this means that the value of p never exceeds the value of q: we shall write this as $p \leq q$. It follows that

$$p \models q \;\Leftrightarrow\; p \rightarrow q \text{ is a tautology } \;\Leftrightarrow\; p \leq q$$

(where we have written "$\Leftrightarrow$" to indicate equivalence of assertions). That is, *in the binary representation,* $\models$ *corresponds to* $\leq$. By the same token,

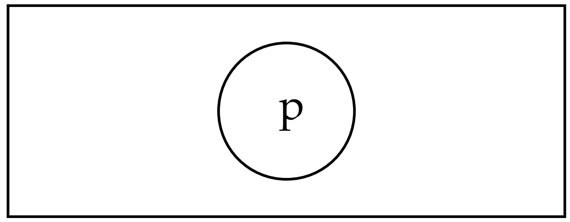
$$p \equiv q \;\Leftrightarrow\; p = q.$$

That is, *in the binary representation,* $\equiv$ *corresponds to* =.

The binary representation sheds light on expressive completeness. For example, the expressive completeness of $\{\wedge, \neg\}$ translates into the assertion that any binary function can be expressed in terms of the operations "." and "1 + ", while the expressive completeness of "|" translates into the assertion that any binary function can be expressed in terms of the single binary function $1 + x.y$.

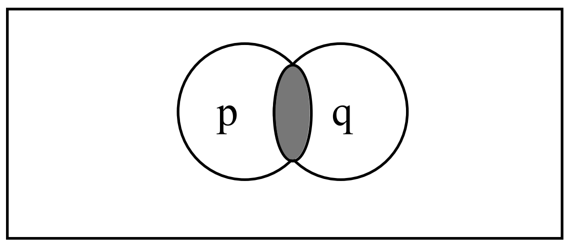
4. Venn Diagrams

Venn diagrams are a convenient method of depicting logical relationships. We represent the various truth valuations of statement letters by points in a rectangle (which itself may be thought of as a kind of "logical space"). Then, for a given statement p, the collection of valuations making p true is represented by a circle within the rectangle:

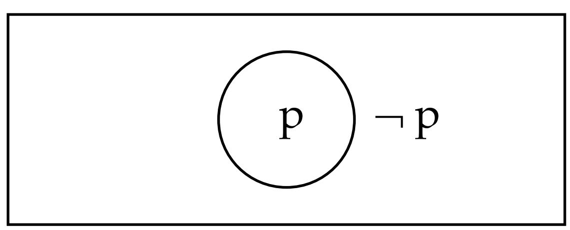


This circle is called the *region corresponding to* p, or simply *the region of* p.

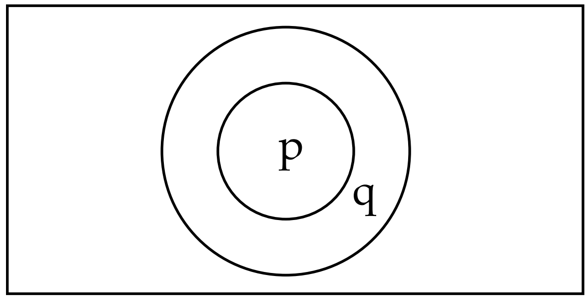
It is clear that the region corresponding to any *tautology* is the *whole rectangle*, and that corresponding to any *contradiction* is the *empty region*. For a *conjunction* $p \wedge q$ the corresponding region is the shaded portion in the figure below, that is, the *intersection*



of the regions corresponding to p and q. For a *disjunction* $p \vee q$ the corresponding region is that covered by the *union* of both circles. For a *negation* $\neg p$ the corresponding region is that lying outside the region of p: its *complement*.



The relation of *logical implication* corresponds to the relation of *inclusion* between regions: $p \models q$ is equivalent to the region of p being



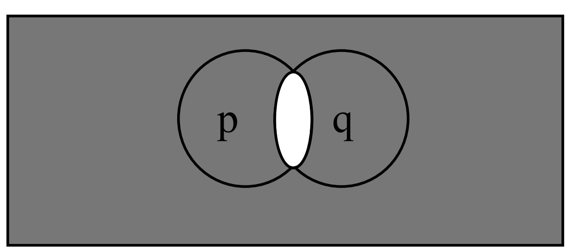
included in the region of q. Finally, the relation of *equivalence* corresponds to the relation of *identity* between regions: p ≡ q is equivalent to asserting that the regions of p and q *coincide.*

It is now straightforward to illustrate the following *rules of equivalence* by means of Venn diagrams:

1. $p \equiv \neg\neg p$
2. $p \wedge q \equiv q \wedge p \qquad p \vee q \equiv q \vee p$
3. $p \wedge (p \vee q) \equiv p \qquad p \vee (p \wedge q) \equiv p$
4. $p \wedge f \equiv f \qquad p \vee f \equiv p$
5. $p \wedge t \equiv p \qquad p \vee t \equiv t$
6. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \qquad p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
7. $\neg(p \wedge q) \equiv \neg p \vee \neg q \qquad \neg(p \vee q) \equiv \neg p \wedge \neg q.$

Here on line 1 we have the *law of double negation*, on line 2 the *commutative laws*, on line 3 the *absorptive laws*, on line 6 the *distributive laws*, and on line 7 *de Morgan's laws*.

For example, consider the Venn diagram immediately below.



Clearly the shaded region — that corresponding to $\neg(p \wedge q)$ — is the union of the region outside that of p with the region outside that of q. This latter is the region corresponding to $\neg p \vee \neg q$. This verifies the first de Morgan law. The remaining laws may be similarly verified.

5. Exercises

Equivalence

A1. Which of the following pairs of statements are equivalent?

(a)	$(A \vee B) \wedge \neg A$	$\neg A \wedge B$
(b)	$A \rightarrow (A \wedge B)$	$A \wedge B$
(c)	$A \rightarrow (A \wedge B)$	$A \rightarrow B$
(d)	$\neg(A \leftrightarrow B)$	$A \leftrightarrow \neg B$
(e)	$\neg(A \leftrightarrow B)$	$(A \wedge \neg B) \vee (\neg A \wedge B)$
(f)	$A \leftrightarrow (B \leftrightarrow C)$	$(A \leftrightarrow B) \leftrightarrow C$
(g)	$A \vee (B \wedge C)$	$(A \vee B) \wedge C$
(h)	$A \rightarrow (A \rightarrow A)$	A
(i)	$(A \rightarrow A) \rightarrow A$	A

A2. (a) Indicate which of the following statements are <u>valid</u>:

(i) $t \veebar t$

(ii) $f \leftrightarrow f$

(iii) $(t \downarrow f) \rightarrow (f \mid t)$

(iv) $\neg(p \rightarrow \neg p)$

(v) $\neg(p\leftrightarrow\neg p)$

(vi) $(p\rightarrow q)\rightarrow(q\rightarrow p)$

(vii) $(p\rightarrow q)\rightarrow(\neg q\rightarrow\neg p)$

(viii) $(\neg p\wedge\neg q)\leftrightarrow\neg(p\vee q)$

(ix) $(p\leftrightarrow(p\wedge q))\leftrightarrow(q\leftrightarrow(p\vee q))$

(b) Which of the above statements (i)-(ix) are equivalent to each other?

Expressive Completeness

B1. Find statements involving the operators $\wedge$, $\vee$, $\neg$ and the statement letters A, B, C that have the following truth tables (1), (2), (3):

A	B	C	(1)	(2)	(3)
t	t	t	t	t	f
f	t	t	t	t	t
t	f	t	t	t	f
f	f	t	f	f	f
t	t	f	f	t	t
f	t	f	f	f	t
t	f	f	f	t	f
f	f	f	t	f	t

B2. The logical operator $\underline{\vee}$ called exclusive disjunction is defined by specifying that $p\underline{\vee}q$ is true when exactly one of p, q is true, and false otherwise.

(a) Define $\underline{\vee}$ in terms of $\leftrightarrow$ and $\neg$.

(b) Show that $\underline{\vee}$ is associative, in the sense that $p\underline{\vee}(q\underline{\vee}r) \equiv (p\underline{\vee}q)\underline{\vee}r$, for any statements p, q and r.

(c) Show that $\{\underline{\vee},\wedge,t\}$ is an expressively complete set.

(d) What are the truth conditions for $p_1 \underline{\vee} p_2 \underline{\vee} \ldots \underline{\vee} p_n$? (That is: when would you regard such an expression as true and when not?)

B3. (a) Show that the pair {¬,→} is expressively complete.

(b) Show that the single truth function f(A,B,C) = (A∨B)→¬C is expressively complete. (Hint: One approach is to show that {¬, →} are both expressible in terms of the function f and invoke (a).)

B4. (a) Show that ↔ cannot be expressed in terms of → alone. (Hint: any statement containing exactly A, B, → takes value t in at least one case where A and B have opposite truth values.)

(b) Show that ∨ *can* be expressed in terms of → alone.

(c) Show that ∧ cannot be expressed in terms of → alone. (Start by showing that any statement containing just the logical operator → must take truth value t in at least two cases.)

B5. Find statements in ¬, ∨, ∧ that have the following truth functions f, g, h.

A	B	C	f(A, B, C)	g(A, B, C)	h(A, B, C)
t	t	t	t	f	t
f	t	t	t	t	f
t	f	t	f	t	f
t	t	f	f	t	f
f	f	t	f	f	f
t	f	f	f	t	t
f	t	f	f	t	t
f	f	f	t	t	t

B6. Show that the truth function h(A, B, C) determined by (A→B)→¬C is expressively complete.

B7. Find disjunctive normal forms for the following statements:

(a) ¬(A →B) ∨ (¬A ∧ C) (b) A ↔ [(B ∧ ¬A) ∨ C]

B8. (a) Explain why {f,$\rightarrow$} must be an expressively complete set.

(b) Out of the 16 possible binary logical operators one could define, exactly how many can be expressed in terms of $\rightarrow$ alone? (This one's moderately difficult!)

Binary Representations and Venn Diagrams

C1. (i) Find the binary representations of the following statements, (ii) using those representations classify each statement as valid, contradictory or contingent, and (iii) draw the Venn diagram corresponding to each statement:

(a) $A\downarrow B$ (b) $A\underline{\vee}B$ (c) $A\leftrightarrow B$ (d) $(A\rightarrow B) \vee (B\rightarrow A)$ (e) $(A\vee\neg A) \wedge (B\wedge\neg B)$

C2. Find the binary representations of the following statements and draw their Venn diagrams:

(a) $A\rightarrow(B\rightarrow B)$

(b) $(A\downarrow A)\downarrow(A\downarrow A)$

(c) $(A\leftrightarrow B)\wedge(\neg A\leftrightarrow B)$

(d) $p\rightarrow(q\rightarrow r)$

(e) $[p\rightarrow(q\rightarrow r)]\wedge(s\vee\neg s)$

(f) $(C\underline{\vee}D)|(C\underline{\vee}D)$

C3. Find the binary representations of the following statements and draw their Venn diagrams:

(a) $(A\wedge\neg A)\rightarrow(A\vee\neg A)$

(b) $\neg[(A|A)|(A|A)]$

(c) $\neg[(A\leftrightarrow B)\vee(\neg A\leftrightarrow B)]$

(d) $(p\rightarrow q)\rightarrow r$

(e) $[(p\rightarrow q)\rightarrow r]\vee(s\wedge\neg s)$

(f) $(C\vee D)\downarrow(C\vee D)$

C4. Find the disjunctive normal forms, binary representations and Venn diagrams for the following statements:

(a) $A\leftrightarrow(B\rightarrow\neg A)$

(b) $[(A\rightarrow B)\rightarrow A]\wedge\neg C$

III. TRUTH TREES

1. Introduction to Truth Trees

To test an inference for validity it suffices to conduct an exhaustive search for counterexamples. If none are found, then the inference is valid. *Truth trees* are an efficient and elegant device for unearthing counterexamples.

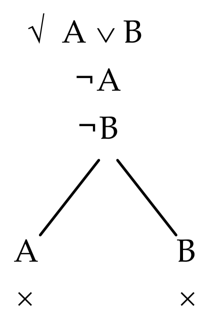
Consider, for example, the (valid) inference

$$\begin{array}{c} A \vee B \\ \neg A \\ \hline B \end{array}$$

To obtain its tree form, we start by listing its premises and the *negation* of its conclusion:

$$\begin{array}{c} A \vee B \\ \neg A \\ \hline \neg B \end{array}$$

These statements will be true in exactly the cases in which there are counterexamples to the original inference. Now we continue, generating a tree-like structure:



Here the statements A, B, ¬A, ¬B, A ∨ B occupy positions, or as we shall call them, *nodes*[1] in the tree. The statement occupying the top node is a disjunction and requires analysis: A ∨ B is true in all those cases in which A is true and all those cases in which B is true, and in no other cases. We indicate this by writing A and B at the ends of a fork at the foot of the tree. At the same time we tick the statement A∨B, using "√", to indicate

[1] We shall often identify a node in a tree with the statement occupying it.

that all its t cases have been taken into account. *Ticking a statement*[2] *is, accordingly, equivalent to erasing it*. Finally we write "×" at the foot of each path through the tree in which a statement occupies one node and its negation another. Such paths are called *closed*. In this particular tree all paths are closed; under these conditions the tree itself is said to be *closed.* And, as we shall see, the inference is then valid.

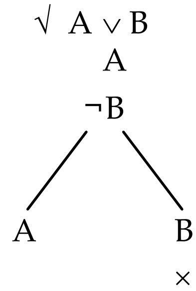
Why is this? Because the procedure was designed so that when we ticked a statement, we displayed all the possible ways in which that statement can be true. The various paths then represent all the ways in which the initial statements (i.e., the statements with which we began the tree) could possibly be true; that is, each path represents a potential counterexample to the original inference. In the case of a *closed* path, the possibility it represents does not really exist. Accordingly, if *all* paths are closed, then it is impossible for all the initial statements of the tree to be (simultaneously) true, in other words, there are no counterexamples to the original inference and so it is valid.

In contrast, observe what happens when we test an *invalid* inference, e.g.,

$$\begin{array}{c} A \vee B \\ \underline{\quad A \quad} \\ B \end{array}$$

In this case the tree looks like this:

[2] We shall often use the locution"to tick a given node" as a synonym for "to tick the statement occupying the given node".



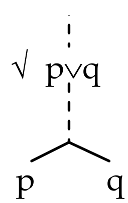
The left-hand path is not closed, that is, it is *open* and represents a genuine counterexample to the inference in question. To describe it, note which statement letters, *with or without* ¬, occupy nodes in the path. In this case they are A, ¬B, and the corresponding counterexample is that in which A is true and B is false:

A	B	A ∨ B	A	B
t	f	t	t	f

We next describe the various tree rules.

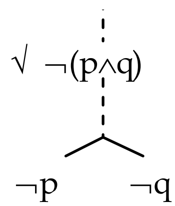
2. The Tree Rules

Disjunction



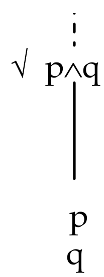
Tick a disjunction occupying a node and write the disjuncts at the end of a fork drawn at the foot of each open path containing the ticked node.

Negated conjunction



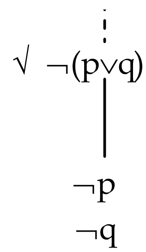
Tick a negated conjunction occupying a node and write the negations of the conjuncts at the end of a fork drawn at the foot of each open path containing the ticked node. (For: a conjunction is false exactly when some conjunct is false. Notice that, by de Morgan's law, this rule is nothing but the disjunction rule in disguise.)

Conjunction



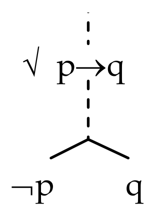
Tick a conjunction occupying a node and write the conjuncts in a column at the foot of each open path containing the ticked node. (Justification: a conjunction is true exactly when both conjuncts are true.)

<u>Negated disjunction</u>



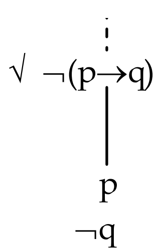
Tick a negated disjunction occupying a node and write the negations of the disjuncts in a column at the foot of each open path containing the ticked node. (Justification: a disjunction is false exactly when both disjuncts are false.)

<u>Implication</u>



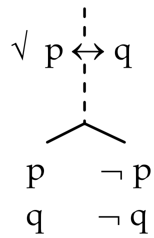
Tick an implication occupying a node and write the negation of the antecedent and the consequent at the ends of a fork drawn at the foot of each open path containing the ticked node. (For: an implication is true exactly when the negation of the antecedent is true, or the consequent is true, or both.)

<u>Negated implication</u>



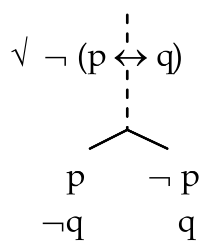
Tick a negated implication occupying a node and write the antecedent and the negation of the consequent in a column at the foot of each open path containing the ticked node. (For: an implication is false exactly when the antecedent is true and the consequent false.)

Bi-implication



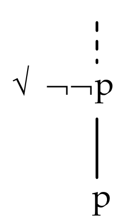
Tick a bi-implication occupying a node and draw a fork at the foot of each open path containing the ticked node. At the ends of each of these write in columns the components, and, respectively, the negations of the components, of the ticked node. (For: a bi-implication is true exactly when both components are true, or both are false.)

Negated bi-implication



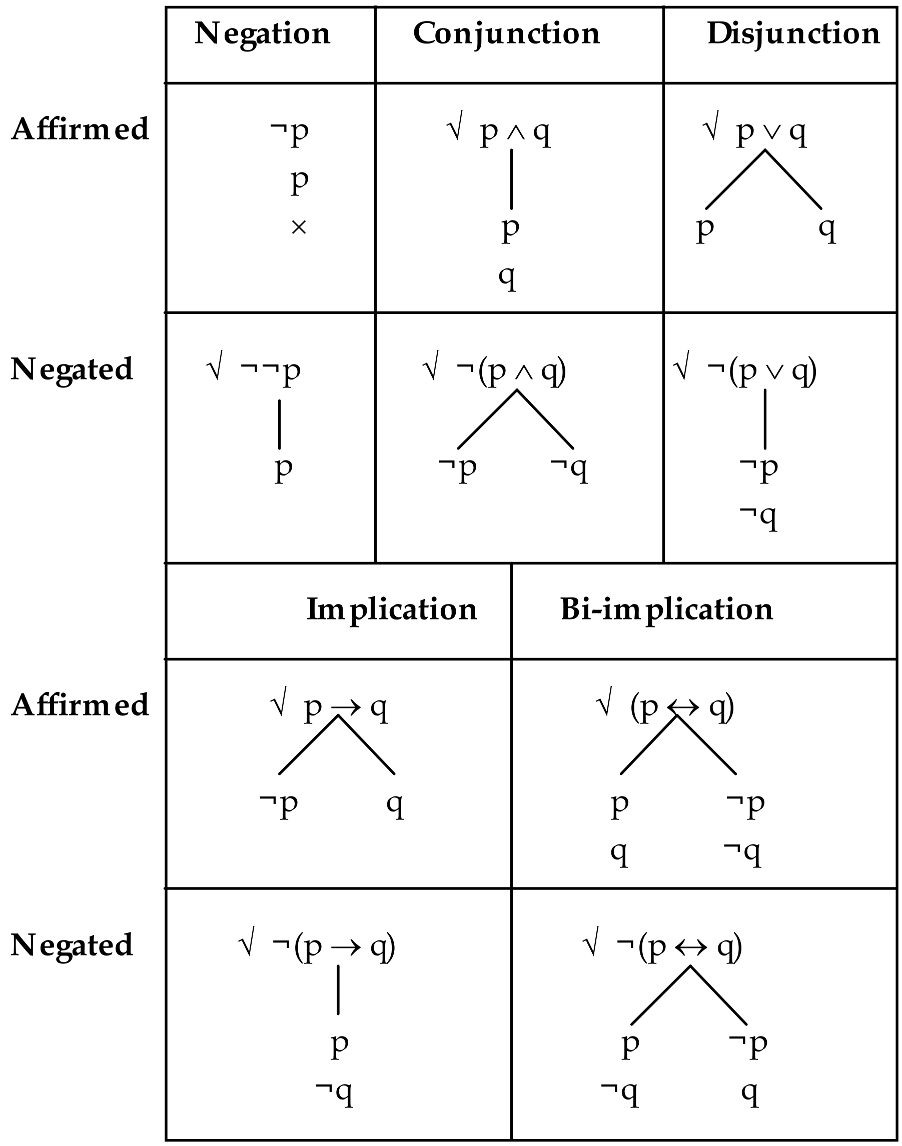
Tick a negated bi-implication occupying a node, and draw a fork at the foot of each open path containing the ticked node. At the ends of these write in columns the first component and the negation of the second, and, respectively, the negation of the first and the second. (For: a bi-implication is false exactly when one component is true and the other false.)

Double negation



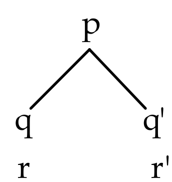
Erase double negations. (For: the negation of a statement is false exactly when the statement is true.)

We summarize these rules as follows:



When applying a tree rule of the form

p is called the *premise*, and {q,r} the *list of conclusions*, of the application. Similarly, when applying a tree rule of the form



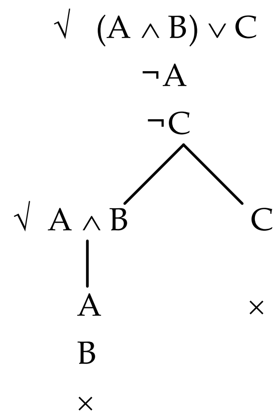
p is called the *premise*, and {q,r}, {q',r'} the *lists of conclusions*, of the application.

3. Tree Test for Validity

To test an inference for validity, write its premises and the *negation* of its conclusion in a column and apply the tree rules to all unticked lines of open paths, ticking lines to which rules are applied, until the tree is *finished*, i.e. until the only unticked nodes in any remaining open paths are statement letters and their negations. A tree obtained in this way is called a [finished] tree *associated* with the given inference. If any such tree is *closed*, i.e. if all its paths are closed, the original inference is valid.

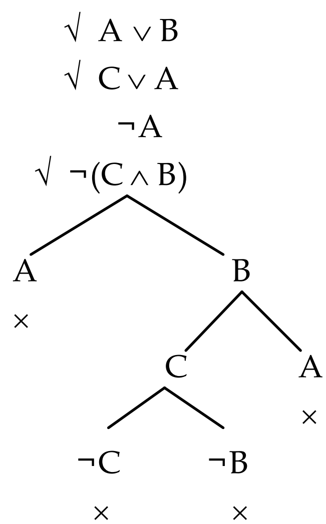
We now give some examples of the use of this test.

$$\begin{array}{c}(A \wedge B) \vee C \\ \neg A \\ \hline C\end{array}$$



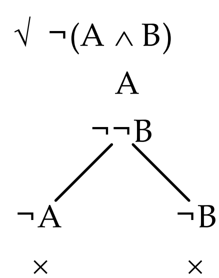
Tree closed, so inference valid.

$$\begin{array}{l}A \vee B \\ C \vee A \\ \neg A \\ \hline C \wedge B\end{array}$$



Tree closed, so inference valid.

$$\begin{array}{c}\neg(A \wedge B) \\ A \\ \hline \neg B\end{array}$$



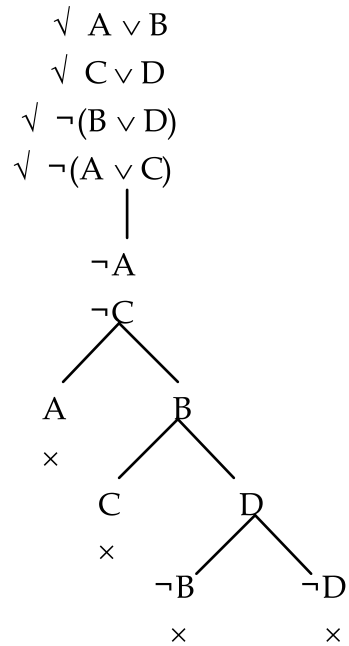
Tree closed, so inference valid.

$A \vee B$

$C \vee D$

$\underline{\neg(B \vee D)}$

$A \vee C$

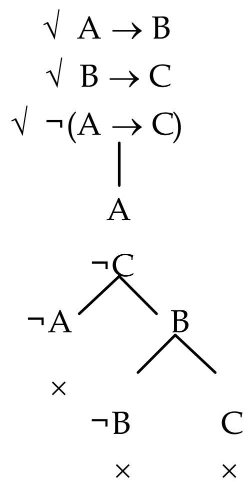


Tree closed, so inference valid.

$A \rightarrow B$

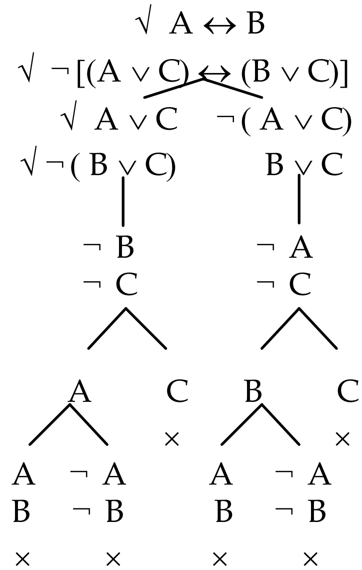
$\underline{B \rightarrow C}$

$A \rightarrow C$



Tree closed, so inference valid.

$$\frac{A \leftrightarrow B}{(A \vee C) \leftrightarrow (B \vee C)}$$



Tree closed, so inference valid.

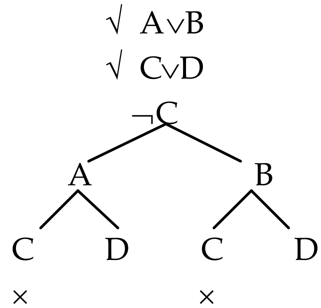
4. Further Applications of the Tree Method

A. Counterexamples from the associated tree. *Any open path remaining in a finished tree associated with an inference determines a counterexample to it (and so establishes its invalidity). And conversely, any counterexample is determined by an open path in any such tree.*

For example, here is an invalid inference:

$$\begin{array}{c} A \vee B \\ \underline{C \vee D} \\ C \end{array}$$

Consider the following finished tree associated with this inference:

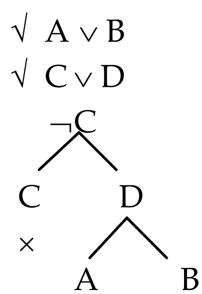


Each open path in this tree determines a counterexample to the given inference. For example, the left-hand open path, nodes of which are occupied by A, ¬C, D, but by neither B nor ¬B, determine as counterexamples all cases in which A, C, D are t, f, t respectively, regardless of the truth value of B. That is, we obtain two counterexamples A: t, B: t, C: f, D: t, and A: t, B: f, C: f, D: t. Similarly, the right-hand open path determines as counterexamples all cases in which B, C, D are t, f, t respectively, regardless of the truth value of A. In total we get the three distinct counterexamples

ABCD: ttft, tfft, ftft.

These are all the counterexamples to the given inference.

In this connection we observe that the open paths in the other finished tree associated with the above invalid inference, viz.,



of course determine exactly the same counterexamples as were obtained before.

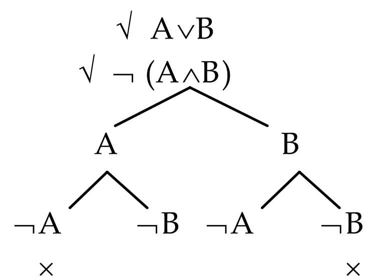
Recall that a set of statements is *satisfiable* or *consistent* if there is at least one case in which all the members of the set are true.

B. Tree test for satisfiability. *Given a set S of statements, start a tree with the members of S in a column. Then S is satisfiable precisely when there is an open path through the finished tree. Each open path determines a truth valuation that makes all the members of S true.*

We illustrate this by the following example. Consider the set of statements

$$\{A \vee B, \neg(A \wedge B)\}.$$

The relevant finished tree is



There are two open paths in which the statement letters (negated or unnegated) A, ¬B; ¬A, B respectively, occupy nodes. Thus the valuations making the given set of statements true are AB:tf or ft.

C. Tree test for logical validity. *To determine whether a given statement is logically valid, start a tree with its negation. Then the given statement is logically valid precisely when the resulting finished tree is closed.*

For example, consider the statement $(A \wedge B) \vee (\neg A \vee \neg B)$. To test for logical validity, we construct the following tree:

√ ¬ [(A∧B) ∨ (¬A∨¬B)]

√ ¬ (A∧B)

√ ¬ (¬A ∨ ¬B)

¬¬A

¬¬B

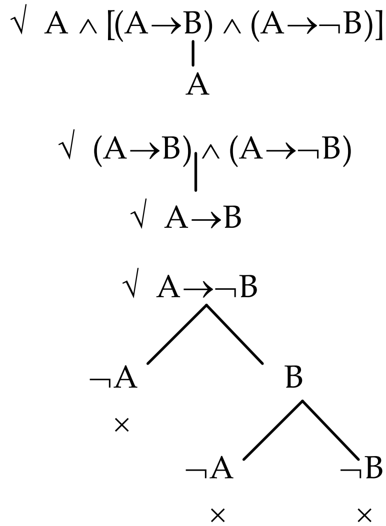
¬A ¬B

× ×

Since this (finished) tree is closed, the statement in question is logically valid.

D. Tree test for contradictions. *To test whether a given statement is a contradiction, start a tree with the (unnegated) statement. Then the statement is a* contradiction *precisely when the resulting finished tree is closed*.

For example, to test whether the statement A ∧ (A → B) ∧ (A→¬B) is a contradiction, construct the following tree:



Since this (finished) tree is closed, the given statement is a contradiction.

5. Correctness and Adequacy of the Tree Method

We conclude this chapter with some arguments designed to justify the claims we have made concerning the use of trees in establishing validity and satisfiability.

First, let us call a tree rule R *correct* if whenever the premise of R is true under a given valuation, then all the statements in *at least one* of R's lists of conclusions are also true under the valuation. And let us call R *complete* if the converse holds, that is, the premise of R is true under a given valuation whenever all the statements in at least one of R's lists of conclusions is true under the valuation.

Clearly, *all the tree rules we have introduced are correct and complete in the above senses*.

Next, we observe that the process of constructing a finished tree *always terminates*. For the tree starts with a finite number of statements, each of which has finite length (taking the *length* of a statement to be the total number of symbols in it), and it grows 'downward' by a process of choosing an unticked statement occupying a node of an open path, ticking it and adding at the foot of the path some finite number of statements, each of which is *shorter* than the ticked one. Eventually the point must be reached at which all unticked statements occupying nodes of open paths have lengths 1 or 2 (i.e., are statement letters or their negations) and the process ends.

Given a set *S* of statements, let us say that a tree *starts with* S if it has S as its initial set of statements. Now we can establish the

Correctness of the tree method. *If a set* S *of statements is satisfiable, there will be an open (complete) path through any tree that starts with* S.

To prove this, observe first that, if all the statements occupying nodes in a path P of a tree are true under a given valuation, then P is open. For if there is a valuation making all statements occupying nodes in P true, then both a statement and its negation cannot both occupy nodes in P, otherwise the (alleged) valuation would have to make both a statement and its negation true—impossible. It follows that P cannot contain both a statement and its negation, which is just to say that path P is open.

Now suppose that under some valuation V all the members of S are true. Consider the following property of a tree **T**.

(*) **T** starts with S and contains a (complete) path P such that all statements occupying nodes of P are true under V.

By the observation above, any tree satisfying (*) contains an open path.

We claim that, if **T** has property (*), so does any tree **T*** obtained from **T** by applying a tree rule. For suppose that (a) all the statements occupying nodes in a certain path P through **T** are true under V and (b) we extend **T** to **T*** by applying a tree rule to one of its statements. Clearly we may assume that this statement is in P, for if not, then P is unaffected and is a complete path of **T***. Accordingly in the transition from **T** to **T*** the path P is extended to a new path, or extended and split into two new paths, by applying some tree rule. Since any tree rule is correct, all the statements occupying nodes in the new path, or all those occupying nodes in at least one of the new paths (each of which extends the path P), are true under V. But this shows that **T*** has property (*), as claimed.

It follows that *any* tree **T** starting with S has property (*), and hence contains an open path. For any tree **T** starting with S can be 'built up' (or rather, down!) by starting first with the tree with a single path consisting of the statements in S—which has property (*) by definition—and then applying tree rules, one after another (finitely many times), until tree **T** results. By the argument of the previous paragraph, at each stage of the 'tree building' process, property (*) is preserved, therefore the end result—the tree **T**—will have that property too (and so must contain an open path, which is what we needed to show).

As an immediate consequence of this, we obtain the

Inference correctness of the tree method. *If a* finished *tree associated with an inference is* closed, *then the inference is* valid.

Now we prove the converse of the above correctness result, that is, the

Adequacy of the tree method. *If there is an* open *path through a* finished *tree starting with a given set* S *of statements, then* S *is* satisfiable.

To prove this, let **T** be a finished tree starting with S and containing an open path P. We are going to show how to define a truth valuation V on the statement letters that figure in tree **T** such that the sentences in set S all come out true under V. Consider the single statement letters that occur in path P (not negated statement letters, just the non-negated elementary statements in P). Let V be the valuation that assigns all those statement letters value t, and all the statement letters that do *not* occur in path P (i.e. that occur somewhere else in the tree **T**) the truth value f. (If there are any other statement letters left out of this assignment, let them take any truth value you want.) We claim that *all statements occupying nodes of* P *are true under* V, not just the nodes containing statement letters.

To show this first notice that all statements of lengths 1 or 2 occupying nodes of P are true under V. For those of length 1 are statement letters and are accordingly true under V by definition. And any one of length 2 is a negation ¬A of a statement letter A; since P is open, A cannot occupy a node of P, and so is false under V. Thus ¬A is true under V.

Now suppose that, if possible, some statement occupying a node of P is false under V. Let p be such a statement of *shortest* length. Then by the above the length of p must be at least 3, so a tree rule, R say, may be applied to p. Since **T** is finished, some list L of conclusions obtained by applying R to p is already part of P. But each statement in L is *shorter* than p, and so must be true under V. Since R is complete, it follows that the premise p of the specified application of R is also true under V. Therefore the falsity of p is refuted, and the claim above follows.

Given the (now established) truth of the claim that all statements occurring in path P are true under V, it follows (in particular) that the initial statements in the set S are true under V (since that set clearly lies in all paths of the tree, P included). It follows, then, that the set S is satisfiable, and hence the tree method is adequate in the sense spelled out above.

As an immediate consequence, we obtain the converse of validity correctness, that is, the

Inference adequacy of the tree method. *If an inference is valid, then any finished tree associated with it is closed.*

6. Exercises

Tree Test for Validity

A1. Use the tree method to determine whether the following arguments are valid. In the invalid cases, find all counterexamples:

(a) $(A\wedge B)\to C$
$\underline{\neg A\to D}$
$B\to(C\vee D)$

(b) $\underline{A\vee\neg(B\wedge C)}$
$(A\leftrightarrow C)\vee B$

(c) $(B\leftrightarrow\neg A)\to\neg C$
$(\neg B\wedge D)\vee(A\vee E)$
$\underline{(D\vee E)\to C}$
$A\to B$

(d) If Holmes has bungled or Watson is windy, Moriarty will escape. Thus Moriarty will escape unless Holmes bungles.

(e) Moriarty will not escape unless Holmes acts. We shall rely on Watson only if Holmes does not act. So if Holmes does not act, Moriarty will escape unless we rely on Watson.

(f) Moriarty will escape only if Holmes bungles. Holmes will not bungle if Watson's to be believed. So if Watson's to be believed, Moriarty won't escape.

A2. Use the tree method to determine which of the following inferences is valid. In the invalid cases, supply all counterexamples.

(a) $\underline{A\wedge(B\vee C)}$
$(A\wedge B)\vee(A\wedge C)$

(b) $(\neg A \vee B) \wedge (A \vee \neg B)$

$\underline{A \vee B}$

$A \wedge B$

(c) $\neg (A \wedge B) \vee C$

$\underline{A \vee D}$

$\neg B \vee C \vee D$

(d) $\underline{\neg(A \vee B) \vee C}$

$A \vee C$

A3. Use the tree method to determine which of the following inferences are valid. In the invalid cases find all counterexamples.

(i) $(A \wedge B) \rightarrow C$

$\underline{\neg A \rightarrow D}$

$B \rightarrow (C \vee D)$

(ii) $\underline{A \vee \neg(B \wedge C)}$

$(A \leftrightarrow C) \vee B$

(iii) $(B \leftrightarrow \neg A) \rightarrow \neg C$

$(\neg B \wedge D) \vee (A \vee E)$

$\underline{(D \vee E) \rightarrow C}$

$A \rightarrow B$

A4. Define the new logical operations [A, B, C] and A*B by

$$[A, B, C] = (A \rightarrow B) \rightarrow \neg C$$

$$A{*}B = A \rightarrow \neg B$$

Devise the simplest tree rules you can for these operations and their negations. Use the rules you have devised to determine which of the following inferences are valid:

(i) $\underline{[A, B, C]}$

A*B

(ii) [A, B, C]

$\underline{A{*}B}$

C*B

A5. Use the tree method to determine which of the following inferences are valid. In the invalid cases find all counterexamples.

(i)

$(A \wedge B) \rightarrow C$

$\underline{\neg A \rightarrow D}$

$B \rightarrow (C \vee D)$

(ii)

$\underline{(A \wedge B) \rightarrow C}$

$(A \rightarrow C) \vee (B \rightarrow C)$

(iii)

$\underline{A \vee \neg(B \wedge C)}$

$(A \leftrightarrow C) \vee B$

A6. Define the logical operations {A, B, C}, A*B, and A•B by

$$\{A, B, C\} = (B \rightarrow A) \wedge (\neg B \rightarrow C)$$

$$A{*}B = A \rightarrow \neg B$$

$$A \bullet B = \neg(\neg A \rightarrow B)$$

Devise the simplest tree rules you can for these operations. Use these rules to test the validity of the following inferences:

(i)

$\underline{A \bullet B}$

$A{*}B$

(ii)

$\underline{A{*}B}$

$A \bullet B$

(iii)

$\underline{\{A, B, C\}}$

$A \bullet C$

(iv)

$\{A, A \bullet B, C\}$

$\underline{A{*}C}$

B

A7. Determine which of the following arguments are valid. In the invalid cases, supply <u>all</u> counterexamples.

(a)

$A \rightarrow B$

$\neg B$

$\underline{\neg A \rightarrow C}$

$\neg C \rightarrow B$

(b)

$(\neg q \vee r) \leftrightarrow \neg p$

$q \vee \neg q$

$\underline{\neg r \leftrightarrow p}$

$\neg p \wedge (\neg q \rightarrow r)$

(c)

$\neg p \leftrightarrow (\neg q \vee r)$

$\neg q \wedge q$

$\underline{p \leftrightarrow \neg r}$

$(\neg q \rightarrow r) \wedge \neg p$

A8. Determine which of the following arguments are valid. In the invalid cases, supply <u>all</u> counterexamples.

(a)

$A \rightarrow B$

$B \rightarrow C$

$\underline{C \rightarrow D}$

$A \rightarrow D$

(b)

$(\neg q \vee r) \leftrightarrow \neg p$

$q \vee \neg q$

$\underline{\neg r \leftrightarrow p}$

$p \rightarrow p$

(c)

It's a duck if it walks
<u>and quacks like one.</u>
Either it's a duck if it walks like one
or it's a duck if it quacks like one.

A9. Translate the following arguments into logical notation (indicating what elementary sentences your symbols refer to) and then determine whether each argument is valid. If not, indicate the total number of counterexamples.

(a) If Dumb knows that he's dumb, then he's dumb. If he knows that he's dumb, then he at least knows *something*. If Dumb knows something, then he's not dumb after all! Therefore, Dumb's not dumb.

(b) Canada's economy will fail if Quebec does separate. If Canada's economy won't fail, then the market will get the jitters if Quebec does separate. The market will get the jitters even if Quebec doesn't separate. So, the market will get the jitters and Canada's economy will fail.

A10. Use the tree method to determine whether the following argument is valid; if not, supply one counterexample.

Either scientists don't know what they are talking about, or the sun will eventually burn out and Earth will become dark and cold. If scientists don't know what they are talking about, then Mars is teeming with life. If Earth becomes dark and cold, then either the human race will migrate to other planets or will die out. Mars is not teeming with life, but the human race will not die out. Therefore, the human race will migrate to other planets only if Mars is teeming with life.

A11. Use the tree method to determine whether the following argument is valid. If not, supply the exact number of counterexamples.

It will not be the case that both the Representatives and the Senators will pass the bill. If either the Representatives or the Senators pass it, the voters will be pleased; but if both pass it, the President won't be pleased. The President won't be pleased if and only if Dole rejoices. Therefore, Dole won't rejoice.

Further Applications of the Tree Method

B1. Use the tree method to determine which of the following sets of statements are satisfiable. In the latter cases, supply all the satisfying valuations.

(a) $A, B, \neg(A \wedge B)$

(b) $A, \neg B, \neg A \vee B$

(c) $A, \neg B, A \vee B$.

B2. Use the tree method to determine which of the following statements are <u>tautologies</u>.

(a) $\neg(A \wedge B) \vee A$

(b) $\neg A \vee (A \wedge B)$

(c) $\neg(A \wedge B) \vee A \vee B$

B3. Use the tree method to determine which of the following statements are <u>contradictions</u>.

(i) $\neg\big[\,[(A \to B) \to A] \to A\big]$ (ii) $\big[(A \to B) \to B\big] \wedge \neg A \wedge \neg B$

B4. In the land of knights and knaves, knights always state the truth and knaves falsehoods. Punch and Judy are two inhabitants of this land. From their assertions in each case use the tree method to deduce as much as you can about their statuses.

(i) *Punch*: Judy's a knight

Judy: We're not both knaves.

(ii) *Punch*: If Judy's a knave, we both are.

Judy: Either he's a knight, or I'm a knave.

B5. Use the tree method to determine which of the following pairs of statements are equivalent.

(i) $A \to (B \to C)$	$(A \wedge B) \to C$
(ii) $(A \leftrightarrow B) \leftrightarrow C$	$A \leftrightarrow (B \leftrightarrow C)$
(iii) $\neg(A \leftrightarrow B)$	$A \leftrightarrow \neg B$
(iv) $\neg(A \to (B \to \neg C))$	$\neg A \wedge (B \leftrightarrow C)$
(v) $\neg A \vee (B \to C)$	$A \wedge \neg B \wedge \neg C$

B6. Classify each of the following statements as tautologous, contradictory or contingent.

(a) $((A \to B) \to B) \to A$

(b) $\neg(p \wedge q) \vee p$

(c) $B \leftrightarrow (C \vee \neg C)$

(d) $(p \leftrightarrow (p \to q)) \to q$

B7. Knaves always lie, knights always tell the truth, and in Camelot, where everybody is one or the other, you encounter two people, one of whom says to you:

(i) "He's a knight and I'm a knave." What are they?

(ii) What if that person had said: "If he's a knave, then so am I"?

(iii) How about if that person had said: "I'm a knight, and, then again, I'm not; though he's a knave if I am" ?

B8. Classify each of the following statements as tautologous, contradictory or contingent.

(a) $((A \to B) \to B) \to B$

(b) $\neg(p \vee q) \wedge q$

(c) $(B\wedge\neg B)\leftrightarrow(C\vee\neg C)$

(d) $\neg[(p\to q)\leftrightarrow p)]\vee q$

B9. Knaves always lie, knights always tell the truth, and in Camelot, where everybody is one or the other, you encounter three people, Lancelot, Arthur and Merlin, who say to you:

Lancelot: Merlin's a knave.

Arthur: Either Lancelot or Merlin is a knave.

Merlin: If I'm a knave, they are too.

What are they?

B10. Circle the tautologies that occur below:

(a) $\neg\neg A \to A$ (b) $A \to (B \to A)$ (c) $A \leftrightarrow (B \vee \neg A)$

(d) $A \leftrightarrow (\neg A \wedge A)$ (e) $(A \to B) \vee (B \to A)$

B11. Circle the inconsistent sets of sentences that occur below:

(a) $B \to A$, B , $\neg A$ (b) $A \to \neg A$, $\neg A \to A$ (c) $A \leftrightarrow (B \vee \neg B)$, $\neg A$

(d) $A \to B$, $B \to C$, $A \wedge \neg C$ (e) $\neg (A \to (B \to C))$, C

B12. In the land of knights and knaves, where knaves always lie, knights always tell the truth, and everybody is either one or the other (clearly no one can be both!), you encounter two people, Dumb and Dumber, both of whom speak to you. In each case below, determine as much as you can about their <u>individual</u> identities.

(a) Dumb: 'If I'm a knave, we both are.' Dumber: 'He's a knight or I'm not.'

(b) Dumb: 'Dumber is a knight if and only if 2+2 is 4.' Dumber: 'Come on, 2+2 is *not* 4!'

B13. These puzzles concern a land populated by saints and sinners. Saints always tell the truth; sinners always lie. You are a traveler in this strange land and must try to identify those you meet as saints or sinners.

You encounter two people, Mutt and Jeff, one or both of whom speak to you. What can you deduce in each case, using the tree method, about whether they are saints or sinners?

1. *Mutt:* I'm a saint.

2. *Jeff*: Mutt is a saint.

3. *Mutt:* Jeff's a sinner.

4. *Jeff:* Either I'm a saint, or I'm not.

5. *Mutt*:I'm a saint, and, then again, I'm not.

6. *Jeff:* If Mutt is a sinner, so am I.

7. *Jeff:* Neither of us is a saint.

8. *Mutt:* We're not both saints.

9. *Mutt:* I'm a sinner if and only if Jeff's a saint.

10. *Jeff:* Mutt is a saint, and I'm a sinner.

11. *Mutt:* I'm a sinner unless Jeff's a saint.

12. *Mutt:* If either of us is a sinner, I am.

13. *Mutt*: Jeff's a sinner.

Jeff: We're not both sinners.

14. *Mutt:* I'm a saint if and only if Jeff's a sinner.

Jeff: Mutt is a sinner.

15. *Mutt:* Jeff's a saint.

Jeff: At least one of us is a sinner.

16. *Mutt:* I'm a saint if and only if Jeff is.

Jeff: Mutt is a saint.

17. *Mutt:* If I'm a sinner, we both are.

Jeff: Either he's a saint, or I'm a sinner.

18. *Mutt:* Jeff's a saint if and only if his brother is.

Jeff: Unfortunately, my brother's a sinner.

19. *Mutt:* Jeff and his brother are both saints.

Jeff: Well, I'm a saint, but my brother isn't.

At this point, you meet three curious looking people in the land of knights and knaves. What can you deduce about their status?

20. *Curly:* Larry's a sinner.

Moe: Either Curly or Larry is a sinner.

Larry: If I'm a sinner, they are too.

21. *Curly:* We're all saints.

Moe: Well, I'm a saint, but Larry's a sinner.

Larry: No, the other two are both sinners.

22. *Curly:* ThatMoe's a saint.

Moe: No,we're all sinners.

Larry: Curly, Moe, *and* their cousins are all sinners.

23. *Curly:* Well, at least we're not all of us sinners.

Moe: Curly is.

Larry: If Curly is, Moe is too.

24. *Curly:* If Moe's a saint, Larry is too.

Moe: Well, Larry's a sinner if Curly's one.

Larry: But Curly and Moe aren't both sinners.

25. *Curly:* If any of us are saints, Larry is.

Moe: But Larry's a sinner.

Larry: And I'm a sinner if and only if Moe's one.

26. *Curly:* If Moe's a sinner, Larry is too.

Moe: If Larry's a sinner, so is Curly.

Larry: If Moe's a saint, we all are.

B14. Determine which of the following sets of statements are (jointly) satisfiable, in each case describing the satisfying valuations:

(a)	(b)	(c)
$\neg A \vee B$	$\neg(\neg B \vee A)$	$\neg D \vee B$
$B \vee \neg C$	$A \vee \neg C$	$A \vee \neg B$
$\neg B \vee \neg(C \vee D)$	$\neg B \vee \neg C$	$\neg(D \wedge A)$
		D

B15. Using the tree method, determine which of the following statements are tautologies. In the non-tautologous cases, supply all the truth valuations that make the statement false.

(i) $((A \rightarrow B) \rightarrow B) \rightarrow A$

(ii) $A \rightarrow (B \rightarrow (B \rightarrow A))$

B16. Using the tree method, determine which of the following sets of statements are satisfiable. In the satisfiable cases, supply all the satisfying valuations.

(i) $A \rightarrow B$, $B \leftrightarrow C$, $(C \vee D) \leftrightarrow \neg B$

(ii) $\neg(\neg B \vee A)$, $A \vee \neg C$, $B \rightarrow \neg C$

B17. Knaves always lie, knights always tell the truth, and in Camelot, where everybody is one or the other, you encounter three people, Lancelot, Arthur and Merlin, who say to you:

Lancelot: Merlin's a knave.

Arthur: Either Lancelot or Merlin is a knave.

Merlin: If I'm a knave, they are too.

Use the tree method to determine as much as you can about each person's identity.

B18. We return for one last visit to the land of Camelot where everyone is either a knight (always speaking the truth) or a knave (always uttering falsehoods). Sir Lancelot is searching for his mistress Queen Guinevere, and happens upon King Arthur and his band of merry men. When Lancelot asks of Guinevere's whereabouts, Arthur becomes jealous and is in no mood to give Lancelot a straight answer. So he instructs Merlin to cast a spell upon his men so that each, in turn, responds to Lancelot as follows:

Sir Karl the Pauper: Guinevere is in Camelot today.

Sir Loin of Beef: Sir Karl is a knight, but Sir Rob is most certainly a knave.

Sir Rob of Cliff Town: Hey, I'm a knave if and only if Sir Loin is!

Sir Lee Fellow: Yah! If any of us are knights, Sir Rob is.

Does Arthur succeed in hiding Guinevere's present whereabouts, or do his men inadvertently disclose her location to Arthur's rival in love? Use the tree method to find out.

B19. A certain island is populated entirely by heroes and scoundrels; the former always tell the truth, the latter invariably lie.

(a) You encounter four people on the island who say to you:

Dean: If I'm a hero, so is Stan.

Stan: If I'm a hero, so is Jerry.

Jerry: If I'm a scoundrel, Ollie isn't.

Ollie: Those others are all liars!

Determine as much as you can about their individual identities.

(b) Believe it or not, I once went to the Island myself in search of buried treasure. I don't remember the details too clearly, but I do recall encountering Dean and Jerry. Dean, I remember, told me:

"Jerry is a hero and there is buried treasure on the island",

but I can't quite recall what Jerry said. All I remember is that he said exactly one of the following:

" Dean is a scoundrel and there's no buried treasure on the island"

or

"Dean is a scoundrel and there is buried treasure on the island".

Nevertheless, I do remember being able to figure out whether treasure was buried on the island and the identity of both speakers. What were their identities? Was treasure buried on the island?

(c) Suppose you were to go to the Island of Heroes and Scoundrels and wished to find out whether or not there is gold on the island. You meet Dean (not knowing his identity) and you are allowed to ask him only one question, which must be answerable by 'Yes' or 'No'. What question could you ask him that would allow you to figure out if there is buried treasure on the island? (This one's tricky — and there may be more than one question that could do the job.)

B.20 Finally, here's a toughie. On a certain island, rumoured to contain buried treasure, live three gnomes, identical in appearance, of whom it is known that one invariably tells the truth, one always lies, and the third answers "yes" or "no" at random. You arrive on the island and, encountering the three gnomes, ask them a total of two questions, each addressed to one gnome at a time, and to which the answer is a simple "yes" or "no". What questions would you ask that would allow you to figure out if there is buried treasure on the island? (*Hint:* the answer to the first question must enable you to "eliminate" the gnome who answers at random.)

IV. PREDICATE LOGIC

1. Predicates, Relations and Quantifiers

The concept of validity (which we shall call *propositional validity*) that we have employed up to now is restricted in that it does not cover a large class of arguments which are clearly logically correct. Consider, for example, the following argument:

1. All Cretans love all animals.

2. All horses are animals.

3. Epimenides is a Cretan.

∴ *4. Someone loves all horses.*

This argument, while patently not propositionally valid, is still, given the usual reading of the terms "all" and "some", logically correct. Its correctness also of course derives from our grasp of the *grammatical structure* of the statements constituting it, which involve in an essential way *predicates* —"(is a) horse", "(is an) animal", "Cretan"—and *relations* —"loves".

In order to symbolize this argument and others like it we need to enlarge our logical vocabulary. Thus, as in algebra, it is natural to introduce *variables* x,y,z,... to refer to arbitrary individuals, and then to write, for example, "Ax" for "x is an animal", "Cx" for "x is a Cretan", "Hx" for "x is a horse", "Lxy" for "x loves y", and "e" for "Epimenides". The symbols A, C and H are *predicate symbols*, L is a *relation symbol*, and e a *name*. Finally we introduce two symbols ∀ and ∃ called the *universal* and *existential quantifier*, respectively: the expression "$\forall x$" will symbolize the phrase "for all (or any) x", and "$\exists x$" the phrases "for some x", or, equivalently, "there exists x".

To put our argument in symbolic form, we first write it in the following way:

1'. For any individual x, if x is a Cretan, then for any individual y, if y is an animal, then x loves y.

2'. For any individual x, if x is a horse, then x is an animal.

3'. Unchanged.

∴ *4'. For some individual x, for all individuals y, if y is a horse, then x loves y.*

Now 1'- 4' can be symbolized directly in terms of our enlarged logical vocabulary thus:

1". $\forall x[Cx \rightarrow \forall y(Ay \rightarrow Lxy)]$

2". $\forall x(Hx \rightarrow Ax)$

3". Ce

4". $\exists x \forall y(Hy \rightarrow Lxy)$.

The logical system associated with the enlarged vocabulary of variables, predicate and relation symbols, names, and quantifiers is called *predicate* or *quantificational logic.* Statements formulated within this vocabulary will be called *predicate* or *quantificational statements*.

2. Tree Rules for Quantifiers

In order to be able to employ the tree method to test arguments within predicate logic (such as the one above) for validity we need to formulate new tree rules governing the quantifiers. In the case of $\forall$ we shall be guided by the usual meaning of generality, namely, that whenever we assert that *all* individuals under consideration have a certain property, then, given any individual, *that* individual has, or, as we shall sometimes say, *instantiates* the property. We call this the principle of *universal instantiation.* The corresponding tree rule may be formulated thus:

UI. Given a statement of the form ∀vp(v) occupying a node of an open path of a tree,

(1) if a name n appears in the path, write p(n) at its foot unless that statement already occupies a node of the path (in which case, writing p(n) once more in the path would be redundant);

(2) if no name appears in the path, choose some name n and write p(n) at its foot.

Do not tick the line ∀vp(v).

In writing p(v) we have indicated that the statement p contains an occurrence of the variable v; this done, we have written p(n) for the result of substituting "n" for "v" at each occurrence of the latter in p.[1]

Let us observe this rule in action. Consider the argument:

1. Juliet loves all who love Romeo.
2. Romeo loves himself.
∴ *3. Juliet loves herself.*

The argument may be symbolized as follows, using "r" as a name for Romeo, "j" for Juliet, and writing "L" for "loves":

1. $\forall x(Lxr \rightarrow Ljx)$
2. Lrr

∴ 3. Ljj.

[1] Strictly speaking, by "occurrence of v" here we mean *free* occurrence, that is, an occurrence of v *not* within a context of the form "∀vq(v)" or "∃vq(v)". We shall always tacitly assume that this is the kind of occurrence in question.

As usual, we start off with the premises of the argument followed by the negation of its conclusion, and then continue so as to obtain a closed tree in the following way:

1.	$\forall x(Lxr \rightarrow Ljx)$		
2.	Lrr		
3.	$\neg Ljj$		
4.	$\surd$ $Lrr \rightarrow Ljr$		(UI applied to 1)
5.	$\neg Lrr$	Ljr	(from 4)
6.	$\times$	$\surd$ $Ljr \rightarrow Ljj$	(UI applied to 1 again)
7.		$\neg Ljr$ Ljj	(from 6)
		$\times$ $\times$	

In this example we used the **UI** rule twice to obtain lines 4 and 6:

1. $\forall x(Lxr \rightarrow Ljx)$	$\forall x p(x)$
4. $Lrr \rightarrow Ljr$	$p(r)$
6. $Ljr \rightarrow Ljj$	$p(j)$

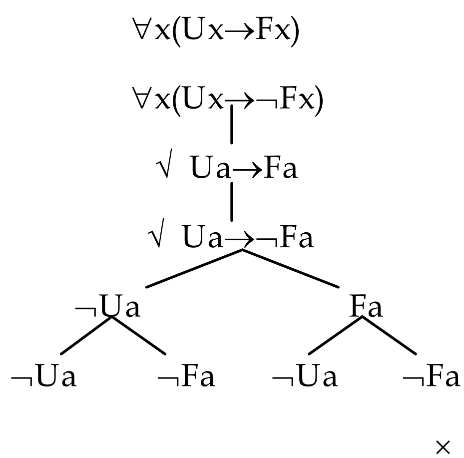
Both applications were made to the same node, 1, and in both the variable v was "x", and p(v)—that is, p(x)—the statement "$\forall x(Lxr \rightarrow Ljx)$". The two applications differed, however, in respect of the name substituted for x: in the first case it was "r" and in the second "j". In the first case we obtained p(r) by substituting "r" for "x" in p(x), and in the second p(j) by substituting "j" for "x" in p(x).

From the fact that we had to apply **UI** *twice* to the same statement 1. it should now be apparent why we do not tick a statement to which **UI** has been applied. Indeed, in this example we had to continue to apply it with every name actually appearing in the path in question before the path (and the tree) finally closed.

Let us now consider an example of an application of **UI** in which no names are initially given. Here the tree method will be used to test *satisfiability* rather than validity. Consider the conditions:

All unicorns are fleet. *No unicorns are fleet.*

Using the obvious notation, these apparently conflicting hypotheses concerning unicorns are expressible as the statements occupying the first two nodes of the following tree, which tests their joint satisfiability:



The third node here results by applying **UI** to the first node, at the same time introducing the new name a. Once this name has been introduced into the path, it must be used in any subsequent application of **UI** in that path, in particular, in the application yielding the fourth line from the second.

We note that the tree is finished since no further applications of **UI** can be made, and it has 3 open paths. Each of these open paths may be regarded as representing a possible *domain* or *universe of discourse* in which all the statements occupying lines in it are true. In general, the objects constituting the domain associated with an open path correspond to the names appearing in that path. In our example, there is only one such name—"a"—present, so that each domain of discourse has exactly one element, which we take to be named by "a". Since the statement ¬Ua occurs in each path, the statement "a is not a unicorn" holds in each domain of discourse. In the second open path the statement ¬Fa appears, so the statement "a is not fleet" holds in the associated domain of discourse. The third open path contains the statement Fa, so "a is fleet" holds in the associated domain of discourse. On the other hand, the first path contains neither Fa nor ¬Fa, so in the corresponding domain of discourse a can be fleet or not indifferently. In fact, since the object named by "a" is the *sole* individual in each domain of discourse, we see that in each of these contexts the statement ¬Ua has the stronger meaning that *nothing* is a unicorn. Thus each domain of discourse represents a "world" in which no unicorns exist, so that any assertion about all unicorns, including our two conditions above, automatically come out true, and are therefore jointly satisfiable there (contrary to what one might naively expect).

We turn now to the existential quantifier ∃. First, we note that there is a simple connection between ∃ and ∀. To see what it is, imagine that we have a domain of discourse consisting of three people, named by a, b, c, say. Consider the two statements

Someone (in our domain of discourse) is Canadian	∃xCx
Everyone (in our domain of discourse) is Canadian	∀xCx

It is clear that in our domain of discourse the statement ∃xCx is equivalent to the disjunction

$$Ca \vee Cb \vee Cc$$

and the statement ∀xCx to the conjunction

$$Ca \wedge Cb \wedge Cc.$$

Therefore the negated statement ¬∃xCx is equivalent to

$$\neg(Ca \vee Cb \vee Cc),$$

which by de Morgan's law is equivalent to

$$\neg Ca \wedge \neg Cb \wedge \neg Cc.$$

But this last statement asserts that each, and so every, individual in our domain of discourse satisfies ¬C; in other words, it is equivalent to the statement $\forall x \neg Cx$.

It is evident that the correctness of this line of reasoning is independent both of the nature of the predicate C and of the number of indviduals in the domain of discourse. Thus we may draw the general conclusion that, for any statement p(v), writing ≡ for equivalence as usual,

$$\neg\exists v p(v) \equiv \forall v \neg p(v).$$

An analogous argument shows that also

$$\neg\forall v p(v) \equiv \exists v \neg p(v).$$

Thus, in our example above, negating the statement "someone is Canadian" is equivalent to asserting "everyone is nonCanadian" and negating the statement "everyone is Canadian" to asserting "someone is nonCanadian".

All this justifies the following

RULE FOR NEGATED QUANTIFICATION

If a statement beginning with ¬∀v (or ¬∃v) occupies a node of an open path, tick it and write at the feet of all open paths containing that node the same statement with ∃v¬ in place of ¬∀v (or with ∀v¬ in place of ¬∃v) in front.	√ $\neg\forall v\ p(v)$ | $\exists v\neg\ p(v)$	√ $\neg\exists v\ p(v)$ | $\forall v\neg\ p(v)$

We now require a rule for the existential quantifier. This is the rule of *existential instantiation*:

EI Given an unticked statement of the form ∃vp(v) occupying a node of an open path, check to see whether it contains a node occupied by a statement of the form p(n). If not, choose a name n *that has not been used anywhere in the path* and write the statement p(n) at its foot. When this has been done for every open path in which the statement ∃vp(v) occupies a node, tick the node occupied by the given statement:

√ ∃vp(v)
|
p(n) (n new)

It is important to observe in applying this rule that the name n introduced *not be already present in the path*. This is imperative because we want n to name an individual *about which we assume nothing except that it satisfy* p; individuals that have already been named may have properties that conflict with this supposition. For example, consider the following (true) premises:

Someone is Canadian	∃xCx
Nixon is not Canadian	¬Cn.

Were we allowed to use the old name n instead of being forced to introduce a new one, we would be able to generate a closed tree from these premises:

√ ∃xCx
¬Cn
Cn
×

where we have (incorrectly!) applied **EI** to the first node to obtain the third. This would mean that the premises are not jointly satisfiable, in other words, that from the assertion "someone is Canadian", we would be able to infer "Nixon is Canadian". Using the same line of reasoning, we would in fact be able to infer "everyone is Canadian". Incorrectly applied, **EI** can lead to absurdities such as these.

Correctly applied, on the other hand, **EI** leads in our example to

$$\begin{array}{l} \surd\ \exists xCx \\ \quad \neg Cx \\ \quad\ Ca \end{array}$$

where a is a *new* name, denoting, as it were, an "archetypal Canadian", whose identity is not further specified.

Armed with these new rules for quantifiers, let us return to the argument with which this chapter began, and see if the associated tree closes. Here it is:

$\forall x[Cx \rightarrow \forall y\,(Ay \rightarrow Lxy)]$

$\forall x(Hx \rightarrow Ax)$

Ce

$\surd\ \neg\exists x \forall x\,(Hy \rightarrow Lxy)$

$\surd\ Ce \rightarrow \forall y\,(Ay \rightarrow Ley)$

$\neg Ce$ | $\forall y\,(Ay \rightarrow Ley)$

$\times$ | $\forall x \neg \forall y\,(Hy \rightarrow Lxy)$

$\surd\ \neg \forall y\,(Hy \rightarrow Ley)$

$\surd\ \exists y \neg\,(Hy \rightarrow Ley)$

$\surd\ \neg\,(Ha \rightarrow Lea)$

Ha

$\neg Lea$

$\surd\ Ha \rightarrow Aa$

$\neg Ha$ | Aa

$\times$ | $\surd\ Aa \rightarrow Lea$

$\neg Aa$ | Lea

$\times$ | $\times$

It does close.

3. Identity

We frequently need to assert that two names refer to the *same*, or *different*, things, as, for instance, in the (correct) argument

1. Yesterday I was home

2. Monday I was out.

$\therefore$ *3. Yesterday was not Monday.*

Writing "a" for "yesterday", "b" for "Monday", and "Hx" for "I was home on day x", we still lack a way of symbolizing statement 3. We rectify this by introducing the symbol "=" called the *identity* or *equality symbol*, which we agree is to be written in between variables or names, as in x = y, n = x or m = n. Similarly, we introduce the *diversity* or *inequality symbol* ≠, so that the statements x ≠ y, n ≠ x, m ≠ n serve as abbreviations for ¬(x = y), ¬(n = x), ¬(m = n), respectively.

Now our argument may be symbolized

$$\frac{\begin{array}{c} Ha \\ \neg Hb \end{array}}{a \neq b}$$

If we negate the conclusion of this argument so as to investigate its validity in the customary arboreal fashion, we start off with

$$\begin{array}{c} Ha \\ \neg Hb \\ a = b \end{array}$$

Now if a and b are truly identical, then any property possessed by a should be shared by b (and vice versa), so from the first and third of these statements it should be permissible to infer

$$\begin{array}{c} Hb \\ \times \end{array}$$

As a result the tree will close.

This idea leads us to introduce the following

RULE FOR IDENTITY

If an open path contains a full line of the form m = n and also a full line p in which one of the names m, n appears one or more times, write at the foot of the path a statement q obtained by replacing one or more of the occurrences of that name in p by the other name, provided that q does not already appear in that path as a full line:

$$\begin{array}{c} m = n \\ p \\ | \\ q \end{array}$$

It is also a characteristic feature of identity that every individual is *self-identical*. We formulate this idea as a tree rule by closing any path which terminates with a statement of the form n ≠ n. Thus we introduce the

RULE FOR NONIDENTITY (OR DIVERSITY)

Close any path that contains a line of the form n ≠ n:

$$\begin{array}{c} n \neq n \\ \times \end{array}$$

These rules enable us to establish the *four basic laws of identity*, viz., *substitutivity, reflexivity, symmetry, and transitivity*.

Substitutivity

$$\begin{array}{c} p(a) \\ \underline{a = b} \\ p\ (b) \end{array}$$

The validity of this inference is confirmed by the fact that the following tree closes:

$$p(a)$$
$$a = b$$
$$\neg p(b)$$
$$p(b)$$
$$\times$$

where we have used the rule for identity to obtain the fourth line from the first two.

Reflexivity

$$\frac{\qquad}{a = a}$$

The validity of this inference follows immediately from the rule for nonidentity.

Symmetry

$$\frac{a = b}{b = a}$$

The correctness of this inference follows from the closed tree

$$a = b$$
$$b \neq a$$
$$b \neq b$$
$$\times$$

in which the third statement is obtained from the first two by the rule for identity, and closure from the rule for nonidentity.

Transitivity

$$a = b$$
$$\frac{b = c}{a = c}$$

The correctness of this inference results from the fact that the following tree is closed:

$$a = b$$
$$b = c$$
$$a \neq c$$
$$a \neq b$$
$$\times$$

where the last statement arises from the second and third by the rule for identity.

4. Validity and Interpretations

The reader will observe that we have been calling an argument formulated in predicate logic *valid*, or *correct*, if the tree beginning with the argument's premises and negated conclusion closes. In consonance with this, we call a predicate statement *valid* if the tree starting with its negation closes. Further, we say that a set of predicate statements is *satisfiable* or *consistent* if any tree starting with that set of statements contains an open path. Now these are, of course, purely *formal* definitions in which, by contrast with the corresponding definitions for propositional logic, the concepts of truth and falsity do not figure. Nevertheless, with a little insight it is possible to reformulate these definitions for predicate statements in terms of truth and falsity. We give a brief outline.

The key idea we require is that of an *interpretation* of the vocabulary of predicate logic, a concept which generalizes that of a valuation of statement letters. For this purpose we shall allow our logical vocabulary to include, in addition to names and predicate symbols, relation symbols linking an arbitrary number of individuals: thus relation symbols may be binary, ternary , ... , n-ary. An *interpretation* I of our logical vocabulary now consists of:

(1) A nonempty set A called the *universe* of I;

(2) an assignment, to each name, of a definite *element* of A (which we shall call the *interpretation under* I of that name);

(3) an assignment, to each n-ary relation symbol, of a definite n-ary relation among the elements of A (which we shall call the *interpretation under* I of that relation).

Once an interpretation of our logical vocabulary has been fixed, it becomes possible to assign a definite *truth value* to each predicate statement based on that vocabulary by giving the identity symbol, the logical operators, and the quantifiers, their natural meanings. This is best conveyed by an example.

Suppose our vocabulary contains one binary relation symbol L and one name a. Consider the interpretation I whose universe is the set of natural numbers {1,2,3,...}, a is interpreted as the number 1, and L is interpreted as the "less than" relation $<$. Let us determine the truth values of the statements

$$(1)\ \exists xLax \quad (2)\ \forall xLax \quad (3)\ \forall x\exists yLxy$$

under I. To do this we use I to "translate" these statements into assertions possessing truth values as follows. The "translation" of (1) is:

"There is a number x such that $1 < x$"

which is obviously *true*. The interpretation of (2) is

"For all numbers x, $1 < x$",

which is clearly *false* since it is not the case that $1 < 1$. Finally, the interpretation of (3) is

"For any number x, there is some number y such that $x < y$",

which is obviously *true*.

This example indicates the method by which any interpretation can be made to yield a definite truth value—truth or falsity—to each predicate statement p: we call this the *truth value of* p *under* I.

Now we can define an argument **A** to be *valid* if, for any interpretation I, whenever all the premises of **A** are true under I, so is its conclusion. A statement p is *valid* if it is true under any interpretation. And a set of statements *S* is *satisfiable* if there is an interpretation under which all the statements in *S* are true.

It can be shown that the two definitions of validity, that based on *trees* and that based on *interpretations*, are equivalent. This is also the case for satisfiability.

5. Many-Sorted Logic

In English (and other languages) there are different quantifiers for different types of domain, for example, various universal quantifiers.

Domain	Places	Times	People	Things
Quantifier	Everywhere	Always	Everyone	Everything

It is convenient to introduce similar devices into our formal logical notation. The method is best illustrated by an example.

Consider the following vocabulary:

Px: x is a person
Qx: x is a politician
Tx: x is a time
Fxyz: x can fool y at (time) z

Then the statement

There is someone who can fool only himself and all politicians all of the time.

may be symbolized in our customary notation as

$$\exists x[Px \wedge \forall y[Py \rightarrow [\forall z(Tz \rightarrow Fxyz) \rightarrow (x = y \vee Qy)]]].$$

This rather involved expression may be simplified by introducing different sorts of letter to indicate individuals satisfying P ("persons") or T ("times"). Thus, if we agree to use letters x,y for persons, and letters t,u for times, the statement above assumes the simpler "many sorted" form:

$$\exists x \forall y[\forall t[Fxyt \rightarrow (x = y \vee Qy)]].$$

The advantage here is that we no longer need to employ explicit predicates to restrict the "range" of the variables. Notice that in order to transcribe this many-sorted statement back into its original "one-sorted" form we need to replace "$\exists x$" by "$\exists x(Px \wedge ...)$", "$\forall y$" by "$\forall y(Py \rightarrow ...)$" and "$\forall t$" by "$\forall z(Tz \rightarrow ...)$".

6. Functions

Relationships such as motherhood or fatherhood have the property that each individual determines a specific, indeed unique, individual (one's mother or father, respectively) with respect to which it stands in that relationship. The introduction of devices called *function symbols* into our vocabulary will enable us to give symbolic expression to this fact.

Thus consider, for example, the relation M of motherhood on the domain of discourse consisting of all persons. We introduce the *function symbol* m to stand for "mother of", so that mx is to be read "mother of (person) x". Then there are two equivalent ways of expressing the statement "y is the mother of x", viz.,

$$Mxy \text{ and } y = mx.$$

Thus, for example, if "a" names Liza Minnelli, then "ma" names Judy Garland.

Names and variables are noun-like *terms*, and function symbols may be applied to terms of this sort to yield new terms. Thus we may write, for example,

$$mmx = \text{mother of mother of } x = \text{maternal grandmother of } x.$$

Similarly, if in addition we introduce the function symbol f for "father of", then

mfx = mother of father of x = paternal grandmother of x

etc.

In general, we may introduce a function symbol in connection with a relation R precisely when R has the two following properties:

Existence: for any x, there exists y such that Rxy
Uniqueness: for any x,y,z, if Rxy and Rxz, then y = z.

When these conditions are satisfied, then for any x there is a *unique* y such that Rxy, and so we can introduce a function symbol f with the meaning that, for any x, fx denotes this uniquely determined y. Thus, for any x and y, the following conditions are equivalent:

y = fx and Rxy.

Function symbols may also be employed in trees, where such terms as fa, fma, etc. are counted as names. However, in doing this we must at the same time insist that when the **EI** rule requires us to introduce a new name, it must be a simple one, i.e. a new letter not already used. To illustrate, we establish the validity of the inference

$$\frac{\forall x\,(fa = x)}{\forall x\,(fx = a)}$$

(An example of this form of argument in English is: "Everybody's Adam's father", therefore "Adam's everybody's father".) The tree for this inference is

$\forall x\,(fa = x)$
$\surd\ \neg\forall x\,(fx = a)$
$\surd\ \exists x\,(fx \neq a)$
$fb \neq a$

fa = fb

fa = a

fb = a

×

The tree is closed and the inference valid. Notice that in the fourth line **EI** was applied to the third line, introducing a new letter "b". Also notice that both the names fb and a have been substituted in for the variable x in the first line.

7. Exercises

Trees and Translations Involving Quantifiers

A1. Using the tree method, determine which of the follwing inferences are valid.

(i) $\underline{\forall x(Px \rightarrow Qx)}$ / $\forall xPx \rightarrow \forall xQx$

(ii) $\underline{\forall xPx \rightarrow \forall xQx}$ / $\forall x(Px \rightarrow Qx)$

(iii) $\underline{\exists x(Px \wedge Qx)}$ / $\exists xPx \wedge \exists xQx$

(iv) $\underline{\exists xPx \wedge \exists xQx}$ / $\exists x (Px \wedge Qx)$

(v) $\underline{\forall x(Px \vee Qx)}$ / $\forall xPx \vee \forall xQx$

(vi) $\underline{\forall xPx \vee \forall xQx}$ / $\forall x(Px \vee Qx)$

A2. Symbolize the following arguments, and determine whether they are valid (always try to choose natural predicate letters and name letters!):

(a) All logicians are neurotic. No vegetarians are neurotic. Therefore, no vegetarians are logicians.

(b) Every Greek who loathes a Trojan is feared by all. Achilles loathes a Trojan, so, if Achilles is Greek, everybody fears him.

(c) Alma has a brother who has no brother, so she's no one's brother.

(d) I'll be home before four o'clock. Therefore there's a time before four o'clock that I'll be home before.

A3. Use the tree method to determine whether the following argument is valid:

$\underline{\forall xFx \rightarrow \forall xGx}$

$\exists x\forall y(Fx \rightarrow Gy)$

A4. Using the following translation key:

Ex = x is an epic	Sxy = x is shorter than y	Wxy = x wrote y
b = Beowulf	c = the Odyssey	d = Homer

symbolize each of the following assertions:

(i) The Odyssey is an epic.

(ii) Homer didn't write Beowulf.

(iii) The Odyssey and Beowulf are not the same length.

(iv) Homer wrote an epic which is longer than Beowulf.

(v) Of all the epics that there are, Homer wrote the longest.

A5. Symbolize the following arguments (using the given symbols), and determine whether they are valid using any reliable method:

(a) If anyone can learn physics, you can. Anyone who can learn logic can learn physics. Dr. Rob can learn logic. So *you* can learn physics! [Use — Px: x can learn physics; u: you; Lx: x can learn logic; d: Dr. Rob]

(b) No acrobats are clumsy. Therefore, if Alma is a waiter, then if all waiters are clumsy, Alma is no acrobat. [Use — Ax: x is an acrobat; Cx: x is clumsy; Wx: x is a waiter; a: Alma]

(c) All dogs are cats. Therefore, whoever loves a dog loves a cat. [Use — Dx: x is a dog; Cx: x is a cat; Lxy: x loves y]

A6. Symbolize the sentences in the following arguments and, using the tree method, determine which ones are valid.

(i) Everything has a cause. If the world has a cause, then there is a God. Hence, there is a God.

(ii) If everyone litters, the world will be dirty. Hence, if you litter, the world will be dirty.

(iii) All love all lovers. Romeo loves Juliet. Therefore, I love you.

(iv) Any barber in Seville shaves exactly those men in Seville who do not shave themselves. Hence, there is no barber in Seville.

A7. Symbolize the following (sets of) sentences and, using the tree method, determine in each case whether they are jointly satisfiable.

(i) Any reasonable person can understand logic and is fit to vote. But Joe doesn't understand logic and yet is fit to vote.

(ii) There is a barber who shaves exactly those who do not shave themselves.

(iii) Everybody loves all lovers. You love yourself, but you don't love me.

(iv) There are at least three objects in this box, and exactly one out of every two is black.

A8. Translate into logical notation:

(a) Some know all.

(b) Some know all who know them.

(c) Some know all who know themselves.

(d) All who know some know some who know all.

(e) No one who knows someone Alma knows, knows all who know Alma.

(f) All who know everyone Alma knows know some who know Alma.

A9. Using the tree method, determine which of the following arguments is valid:

(a) There is someone who is going to pay for all the breakages. Therefore, each of the breakages is going to be paid for by someone.

(b) No student in the statistics class is smarter than every student in the logic class. Hence, some student in the logic class is smarter than every student in the statistics class.

(c) Any person who is not mad can understand logic. None of Wagner's sons can understand logic. No mad persons are fit to vote. Therefore, none of Wagner's sons is fit to vote.

A10. Translate the following into logical notation (use:- **L**xy = x **likes** y, m = **me** or **I**):

(a) I like myself.

(b) Someone likes me.

(c) No one likes me.

(d) Everyone likes someone.

(e) I like myself and no one else.

(f) Someone likes everyone who likes me.

(g) I dislike anyone who dislikes me.

(h) Someone likes everyone I dislike.

(i) Someone who likes me likes everyone.

(j) Everyone who likes someone I like likes no one I dislike.

A11. Translate the following into logical notation

[Assume the domain is **persons**, and use: Txy = x **is taller than** y, r = **Rob**):

(a) Rob is taller than everyone.

(b) Everyone is taller than someone.

(c) Noone's taller than everyone.

(d) Everyone's taller than everyone *else*.

(e) Rob is taller than no more than one person.

(f) No two people are taller than eachother.

(g) If anyone's taller than Rob, Rob is.

(h) Someone is taller than everyone Rob isn't taller than.

(i) Someone taller than Rob is taller than everyone taller than Rob.

(j) Everyone taller than someone is taller than someone taller than everyone.

A12. Demonstrate the validity or invalidity of each of the following two arguments by first translating them using the given symbols, and then doing their trees.

For translation, use:

Cx = x is a **chimpanzee**	Bx = x **will get a banana**	f = **fred**
Sxy = x **can solve** y	Txy = x is **trying harder than** y	b = **barney**
Px = x is a **problem**		

(i) Not all chimpanzees are trying equally hard. No chimpanzee tries harder than himself. Therefore there are at least two chimpanzees.

(ii) Fred and Barney can solve exactly the same problems. If Fred can solve even one problem, then he will get a banana. Fred will not get a banana. Therefore Barney can't solve any of the problems, and he won't get a banana either.

A13. Using the tree method, determine which of the following pairs of statements are equivalent:

(i) $\forall x\, Px \to \forall x Qx$ $\quad\quad$ $\forall x \exists x\, (Px \to Qy)$

(ii) $\exists x Px \to \exists x Qx$ $\quad\quad$ $\exists y \forall x\, (Px \to Qy)$

A14. Define $\exists!x$ by writing $\exists!xPx$ for $\exists x[Px \wedge \forall y(Py \to y=x)]$. State, in simple language, the meaning of $\exists!xPx$.

Determine which of the following inferences are valid:

(i) $\underline{\hspace{3em}}$

$\forall x \exists! y\, (x = y)$

(ii) $\underline{\exists!x\, Px}$

$\exists x \forall y(Py \leftrightarrow y=x)$

(iii) $\underline{\exists!x(Ax \wedge Bx)}$

$\exists!xAx \wedge \exists!xBx$

(iv) $\underline{\forall x \forall y\, (x = y)}$

$\exists!x\, (x = x)$

A15. Demonstrate the validity or invalidity of each of the three arguments below by first translating them using the given symbols, and then doing their trees.

(a) Tweety bird despises cats. No cats despise Tweety bird. Sylvester is a cat. Therefore, Tweety bird despises someone who despises him. (use:- Dxy: x **despises** y; t: **Tweety bird**; Cx: x **is a cat**; s: **Sylvester**)

(b) Any good logic teaching assistant helps all and only those who don't help themselves. Hence there aren't any good logic teaching assistants! (use:- Gx = x **is a good logic teaching assistant**; Hxy = x **helps** y)

(c) I'll finish this exam before four o'clock. For any pair of times, one later than the other, there is a time in between them. So there's a time before four o'clock that I'll finish this exam before. (use:- Ex = **I'll finish this exam <u>at</u> time** x; x<y = **time** x **is earlier than time** y; f = **four o'clock**)

A16. Translate the following into logical notation (use:- Lxy: x **loves** y):

(a) Some love anyone.

(b) No one loves all who love them.

(c) Some love all who love themselves.

(d) All lovers love some who love all.

(e) Nobody who loves somebody loves somebody who loves nobody.

(f) All who love all love all lovers.

A17. Demonstrate the validity or invalidity of each of the three arguments below by first translating them using the given symbols, and then doing their trees.

(a) Ben loves cats. No cats love Ben. Whitey is a cat. Therefore, Ben loves someone who doesn't love him. (use:- Lxy: x **loves** y; b: **Ben**; Cx: x **is a cat**; w: **Whitey**)

(b) There's a set containing all and only those sets which are not members of themselves. Therefore, every set is a member of itself. (use:- Sx = x **is a set**, x ∊ y = x **is a member of** y)

(c) Everyone loves lovers. Romeo loves Juliet. So Fred loves Wilma. (assume domain = persons and use:- Lxy = x **loves** y x; r = **Romeo**; j = **Juliet**; f = **Fred**; w = **Wilma**)

Interpretations and Counterexamples

B1. Translate the following arguments into logical symbols and determine whether they are valid. If not, specify one counterexample.

(i) There's something that's tasty if it's a chocolate bar. So there's a tasty chocolate bar.

(ii) Some like it hot and some don't. Those who like it hot like Marilyn Monroe, and those who don't *don't* like her! Therefore, everybody either likes her or doesn't.

(iii) If anyone is taller than Rob, Gurpreet is. If Gurpreet is taller than Rob, anyone is. So it isn't the case that there's someone taller than Rob and someone not.

B2. Consider the domain consisting of points and straight lines in a given plane, with the following vocabulary for describing it:

Px = x is a Point Lx = x is a straight Line Oxy = (point) x lies On (line) y

Determine the truth value of each of the following statements, providing a brief justification of your answer in each case.

(a) $\forall x \forall y[(Lx \wedge Ly) \rightarrow \exists z(Pz \wedge Ozx \wedge Ozy)]$

(b) $\forall x \forall y[(Px \wedge Py \wedge x \neq y) \rightarrow \exists z \forall w[(Lw \wedge Oxw \wedge Oyw) \rightarrow z=w]]$

B3. Knaves always lie, knights always tell the truth, and in Camelot, where everybody is one or the other, you encounter some people, among them King Arthur who says to you:

"Exactly one out of every two of us is a knave"

Choose names for any other people you might need to refer to and specify an interpretation (i.e. 'case') in which Arthur is a **knight** (if indeed it's possible for him to be a knight, given what he says). Also, specify an interpretation in which Arthur is a **knave** (again, only if that's possible).

B4. Consider the following scenario involving three objects, two predicates F and G, and a relation R:

Domain = {1,2,3} F = {2,3} G = {1} R = {(1,1),(2,2),(1,3),(2,3),(3,3)}

Which of the following is true, and which is false, under this scenario?

(i) $\forall x(Fx \rightarrow Gx)$

(ii) $\exists x(Fx \rightarrow Gx)$

(iv) $\forall x \exists y \neg Ryx$

(v) $\exists y \forall x(Rxy \rightarrow Gx)$

(iii) $\exists x(Fx \vee Gx)$

B5. Consider the domain consisting of the positive whole numbers {1,2,3,...} with the following vocabulary for describing it:

Ex: x is even　　Ox: x is odd　　x<y: x is less than y

x>y: x is greater than y　　x≠y: x is unequal to y

Determine the truth value of each of the following statements, providing a brief justification of your answer in each case.

(a) $\forall x[Ox \rightarrow \exists z(Ez \wedge (x>z))]$

(b) $\exists x \forall y((x \neq y) \rightarrow (x<y))$

B6. Using the tree method, or otherwise, determine whether the following sets of statements are satisfiable. For each of the satisfiable sets, supply an interpretation in which all the statements are true.

(a) $\exists x \forall y Pxy$

$\forall x \forall y \exists z(Pxz \wedge Pxy)$

(b) $\forall x \exists y Pyx$

$\forall x \forall y(Pxy \rightarrow \neg Pyx)$

$\forall x \forall y \forall z((Pxy \wedge Pyz) \rightarrow Pxz)$

(c) $\forall x \forall y \forall z ((Pxy \wedge Pyz) \rightarrow Pxz)$

$\forall x \neg Pxx$

$\exists x \exists y(Pxy \wedge Pyx)$

B7. Symbolize the following arguments, and determine whether they are valid using any reliable method. If not, give a counterexample.

(a) A person is famous if and only everyone has heard of him or her. So, all famous people have heard of each other.

(b) Tweety bird hates cats. No cats hate Tweety bird. Sylvester is a cat. Therefore, Tweety Bird hates someone who hates him. [use:- Hxy: x **hates** y; t: **Tweety bird**; Cx: x is a **cat**; s: **Sylvester**)

(c) Logic students are taller than business students. Exactly one out of every pair of students is a logic student. So some student is taller than some other.

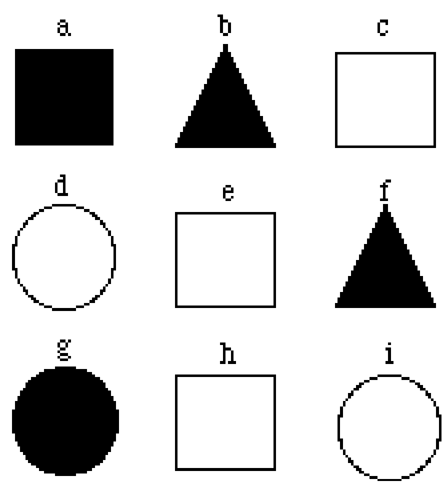
B8. Symbolize the following arguments, and determine whether they are valid using any reliable method. If not, give a counterexample.

(a) Exactly one professor lives in Talbot College. Professor Bell lives in Talbot College. Professor Bell is a toposopher. Therefore, every professor who lives in Talbot College is a toposopher. [For translation, assume domain is **professors** and use: Cx = x lives in **Talbot College**, b = **Bell**, Tx = x is a **toposopher**.]

(b) Exactly one out of every pair of balls is red. Exactly one ball is red. So exactly one ball isn't red. [Assume domain is **balls**.]

(c) Stallone can outgun everybody who can outgun anyone he can. Therefore, Stallone can outgun himself and noone else! [For translation, assume domain is **persons**.]

B9. Here is a small world:



and symbols for describing it (with any variables restricted to ranging over the above nine inhabitants):

Domain = The shapes with names a, b, c, d, e, f, g, h, i

Sx = x is a **square** Lxy = x is **directly left** of y Cxy = x is in the **same column** as y

Tx = x is a **triangle**   Axy = x is **directly above** y   Rxy = x is in the **same row** as y

Bx = x is **black**

Interpreting each of the formulae below as a statement about this world, state whether it is true or false of the world, and, if it is false, briefly state why (referring to any of the shapes above by name, if you need to):

(i) $\forall x Tx$

(ii) $\neg\exists x Lxx$

(iii) $\neg\exists x\neg(Tx\vee Sx)$

(iv) $\forall x(Tx\rightarrow Bx)$

(v) $\forall y\exists x Lxy$

(vi) $\neg\exists x\exists y(Tx\wedge Ty\wedge Cxy)$

(vii) $\exists x[Tx\wedge\forall y((Rxy\wedge x\neq y)\rightarrow\neg Ty)]$

(viii) $\forall x((Sx\wedge\exists y Axy)\rightarrow\neg Bx)$

(ix) $\forall x\forall y((Sx\wedge Sy\wedge x\neq y)\rightarrow(Cxy\rightarrow(\neg Bx\wedge\neg By)))$

(x) $\forall y\exists x(Bx\wedge Sx\wedge x\neq y)$

B10. Using the indicated key: symbolize (1) through (4); translate (5) through (8) into clear English (not just logical jargon); and say whether each of these statements (1)-(8) is true or false, briefly justifying your answer.

Key –

Domain: statements   Ixy = x logically implies y   Exy = x is logically equivalent to y

(1) Every statement implies some statement or other.

(2) Some statements are equivalent to anything that implies them.

(3) Statements with the same implications are equivalent.

(4) Some statements imply all and only what implies them.

(5) ∀x∃y¬Ixy

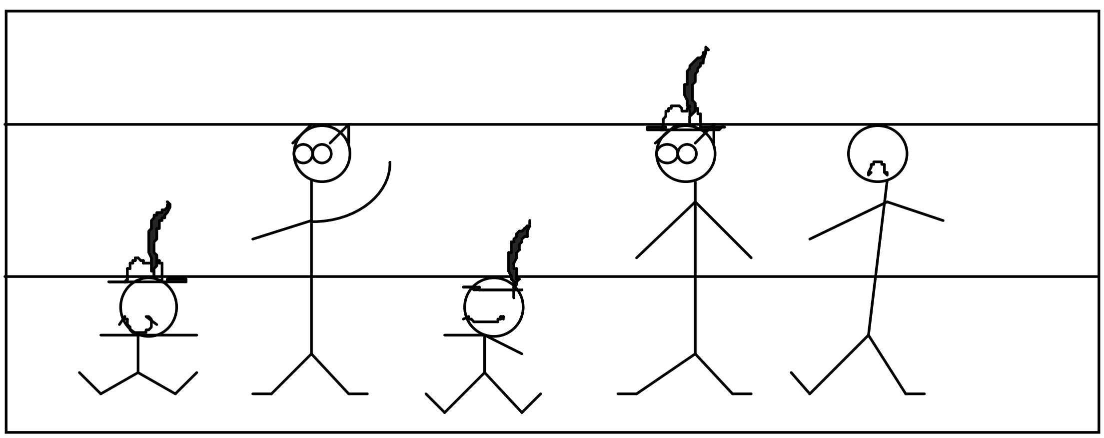
(6) ∀x∀y(Ixy∨Exy)

(7) ∃x∀y(Ixy→Exy)

(8) ∃x∀y(Ixy→∃z¬Iyz)

B11. Here is a small world:

← Left Right→



a b c d e

and a vocabulary describing it (variables restricted to the five inhabitants):

Fx : x wears a feather

Gx : x wears glasses

Hx: x wears a hat

Ixy: x is identical to y

Lxy: x is left of y

Rxy: x is right of y

Txy: x is taller than y

Which of the following quantified formulae are true and which are false of this small world? If a formula is false, describe a minimal change of the world that would make it true (e.g. take somebody's feather away, move people around, etc.. but <u>don't</u> move anybody into or out of the world).

1. $\forall x(Fx \rightarrow \exists yRxy)$
2. $\exists x\forall y\,(\neg Ixy \rightarrow Lxy)$
3. $\forall x\forall y\,[\neg Ixy \rightarrow (\neg Txy \wedge \neg Tyx)]$
4. $\forall x(Gx \rightarrow \exists y(Gy \wedge Rxy)$
5. $\forall y(\exists x\,(Fx \wedge Lyx) \rightarrow Hy)$

B12. Here is another small world:

Domain = {Mum, Pop, Junior}

F = {Pop, Junior}

G = {Mum}

R = {(Pop, Pop), (Mum, Mum), (Mum, Junior), (Pop, Junior)}

Which of the following is true, and which false, when interpreted as assertions about this world:

(a) $\forall x(Fx \leftrightarrow Gx)$

(b) $\exists x(Gx \rightarrow Fx)$

(c) $\exists x\exists y\,(Fx \wedge Gy)$

(d) $\forall x[\exists y\,Ryx \rightarrow Fx]$

(e) $\exists x[\forall yGy \vee Fx]$

(f) $\exists x\forall y\,\neg Rxy$

(g) $\forall x(\exists yRxy \rightarrow \exists z\neg Rxz)$

(h) $\forall x\forall y(Fx \vee Gy)$

(i) ∃x(Fx ∧ ¬Gx ∧ ∃yRxy)

(j) ¬∀x∀yRxy ∨ ∃x∃yRxy

(k) ∃y∀x(Fx ∧ Gy)

B13. Using any method you like, provide, for each of the following statements, an interpretation (i.e. case, scenario) which makes it true, and one which makes it false., including the domain.

(a) ∃x∀y∀z¬Rzyx

(b) ∀x∃yRxy→∃y∀xRxy

(c) ∀y∃z(Pz∧z≠y∧Qy)

B14. Here is a small world (a quack optometrist's eye chart!):

D e *f* e D **f** *D* **E** *d* **F** **e**

and symbols for describing it (with any variables restricted to ranging over the above eleven inhabitants):

Dx: x **is the letter d (or D)** Cx: x **is capitalized** Rxy: x **is right of** y

Ex: x **is the letter e (or E)** Bx: x **is bold-faced** Lxy: x **is directly left of** y

Fx: x **is the letter f (or F)** Ux: x **is underlined**

Ix: x **is italicized**

Interpreting each of the formulae below as a statement about this world, state whether it is true or false of the world:

(i) ∀xUx

(ii) ¬∃x(Bx∧Ix)

(iii) $\forall x(Fx \rightarrow Bx)$

(iv) $\forall x[Bx \rightarrow \exists y(Ryx \wedge Uy)]$

(v) $\exists x \exists y[Lxy \rightarrow (Ux \vee Uy)]$

(vi) $\forall x(\neg Cx \leftrightarrow \exists y Lyx)$

(vii) $\exists x(Ex \wedge Cx \wedge Bx \wedge \forall y[(Ey \wedge Cy \wedge By) \rightarrow y=x])$

(viii) $\exists x(Dx \wedge \forall y[(Dy \wedge Ryx) \rightarrow \neg Cy])$

B15. Here is ANOTHER quack optometrist's eye chart:

$$\underline{\mathbf{f}} \quad \mathit{h} \quad \mathbf{E} \quad \underline{\mathit{d}} \quad \mathbf{F} \quad \underline{\mathbf{E}} \quad \mathbf{D} \quad \underline{\mathrm{e}} \quad \mathit{f} \quad \mathrm{e} \quad \mathbf{d}$$

and symbols for describing it:

Dx: x **is the letter d (or D)** Cx: x **is capitalized** Rxy: x **is somewhere right of** y

Ex: x **is the letter e (or E)** Bx: x **is bold-faced** Lxy: x **is directly left of** y

Fx: x **is the letter f (or F)** Ux: x **is underlined**

Ix: x **is italicized**

Interpreting each of the formulae below as a statement about this world, state whether it is true or false of the world:

(i) $\forall x Cx$

(ii) $\neg \exists x(Ux \wedge Ix)$

(iii) $\forall x(Fx \rightarrow Bx)$

(iv) $\exists x \forall y(Lyx \rightarrow (Fy \wedge By))$

(v) $\forall x[Bx \rightarrow \exists y(Ryx \wedge Uy)]$

(vi) $\exists x\exists y[Lxy\rightarrow(Ux\vee Uy)]$

(vii) $\forall x(\neg Cx\leftrightarrow\exists yLyx)$

(viii) $\exists x(Ex\wedge Cx\wedge Bx\wedge\forall y[(Ey\wedge Cy\wedge By)\rightarrow y=x])$

(ix) $\exists x(Dx\wedge\forall y[(Dy\wedge Ryx)\rightarrow\neg Cy])$

(x) $\exists x(\exists yLxy\wedge\forall zRxz)$

Functions

C1. Symbolize each of the following, using "f" as a function symbol for "the father of" and "m" as a function symbol for "the mother of":

(i) a is b's paternal grandmother

(ii) a is a father

(iii) a is b's full sibling

(iv) a is b's grandmother

(v) a is a grandfather

(vi) a is b's first cousin.

C2. Let f and m be symbols for the functions "the **father** of" and "the **mother** of", and read xPy as "x is the **parent** of y" and Mx as "x is **male**". For each of the statements below, explain the precise relationship asserted between a and b as concisely as you can in English.

(a) $a\neq b \wedge fa=fb \wedge ma=mb$ (b) aPfb (c) $a\neq b \wedge faPb \wedge maPb \wedge Ma$

(d) $(fa=fb \leftrightarrow ma\neq mb) \wedge \neg Mb$ (e) $Mb \wedge \exists x(xPa \wedge fb=fx \wedge mb=mx \wedge b=fa)$

(f) $\neg Ma \wedge \exists x(xPa \wedge fbPx \wedge mbPx \wedge \neg bPa)$.

V. THE PROPOSITIONAL CALCULUS

In this chapter we describe a formal system—the *propositional calculus*—for *proving* propositional statements and as a result obtain a purely syntactical characterization of valid propositional inferences and tautologies. To set up the system we choose certain tautologies as *axioms* and lay down a certain *rule of inference* which will enable us to construct *deductions*.

In what follows we shall omit the logical operator $\leftrightarrow$ *in forming statements, and the rules governing* $\leftrightarrow$ *in constructing trees*. (Of course, we can always define $\leftrightarrow$ by $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ if we wish.)

1. Axioms

The *propositional calculus* (**PC**) has as *axioms* all statements of the form (1)–(10) below.

(1) $p \rightarrow (q \rightarrow p)$
(2) $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$
(3) $(p \wedge q) \rightarrow p$
(4) $(p \wedge q) \rightarrow q$
(5) $p \rightarrow (q \rightarrow p \wedge q)$
(6) $p \rightarrow (p \vee q)$
(7) $q \rightarrow (p \vee q)$
(8) $(p \rightarrow r) \rightarrow [(q \rightarrow r) \rightarrow (p \vee q \rightarrow r)]$
(9) $(p \rightarrow q) \rightarrow [(p \rightarrow \neg q) \rightarrow \neg p]$
(10) $\neg\neg p \rightarrow p$.

The sole *rule of inference* for **PC** is called *modus ponens* (Latin: "affirming mood"):

$$\textbf{MP} \qquad \frac{p,\ p \to q}{q}$$

In words, from p, $p \to q$, infer q.

2. Deductions

Let S be a set of statements. A *deduction* from S is a finite sequence $p_1,\ldots,p_n$ of statements such that, for any $i = 1,\ldots,n$, p_i is either (a) an axiom, (b) a member of S, or (c) inferrable using **MP** from earlier members of the sequence, i.e., there are numbers $j,k < i$ such that p_k is $p_j \to p_i$.

A deduction from the *empty* set of statements is called simply a *proof*. A deduction (or proof) with last statement p is called a *deduction* (or *proof*) *of* p. We write $S \vdash p$ to indicate that p is *deducible* from S, i.e. that there is a deduction of p from S. If S is empty, so that p is *provable*, i.e. there is a proof of p, we write just $\vdash p$, and call p a *theorem* of **PC**.

Example. $\vdash p \to p$.

The following is a proof of the statement $p \to p$.

1. $(p \to ((p \to p) \to p)) \to ((p \to (p \to p)) \to (p \to p))$ (Ax.2)
2. $p \to ((p \to p) \to p)$ (Ax.1)
3. $(p \to (p \to p)) \to (p \to p)$ (**MP** on 1,2)
4. $p \to (p \to p)$ (Ax.1)
5. $p \to p$ (**MP** on 3,4)

We now prove the important

Deduction Theorem *For any set* S *of statements and any statements* p,q:

$$S,p \vdash q \ \textit{ if and only if } \ S \vdash p \to q.$$

Proof. First suppose that $S \vdash p \to q$. Then there is a deduction D of $p \to q$ from S. Clearly, if we add the sequence p,q to D, the result is a deduction of q from S,p. Therefore $S,p \vdash q$.

Conversely, suppose that $S,p \vdash q$. Then there is a deduction $r_1,\ldots,r_n$ of q from S,p (so that q is r_n). *We claim that* $S \vdash p \to r_i$ *for any* $i = 1,\ldots,n$.

Suppose that the claim were false. Then there is a *least* number k such that it is *not* the case that $S \vdash p \to r_k$. There are then 4 possibilities: (1) r_k is an axiom; (2) r_k is in S; (3) r_k is p; (4) r_k is deducible using **MP** from some r_i and r_j with $i,j < k$, where r_j is $r_i \to r_k$.

We show that in each of these 4 cases we have $S \vdash p \to r_k$. This will contradict the assertion that the claim is false, and it must accordingly be true.

Case (1). r_k is an axiom. In this case the sequence of statements r_k, $r_k \to (p \to r_k)$, $p \to r_k$ is a proof of $p \to r_k$, so that $S \vdash p \to r_k$.

Case (2). r_k is in S. In this case the same sequence of statements as in case (1) is a deduction of $p \to r_k$ from S.

Case (3). r_k is p. Here we have $\vdash p \to r_k$ by our Example above, so *a fortiori*

$S \vdash p \to r_k$.

Case (4). For some $i,j < k$ r_j is $r_i \to r_k$. Since k was assumed to be the *least* number for which it is *not* the case that $S \vdash p \to r_k$, and $i,j < k$, we must have $S \vdash p \to r_i$ and

$S \vdash p \to r_j$, i.e., $S \vdash p \to (r_i \to r_k)$. By axiom 2,

$$(p \to (r_i \to r_k)) \to ((p \to r_i) \to (p \to r_k)).$$

Hence, applying **MP**,

$$S \vdash (p \to r_i) \to (p \to r_k)$$

and applying it once more,

$$S \vdash p \to r_k.$$

We have obtained a contradiction in each case, so the claim is true. In particular, taking i = n, we get $S \vdash p \to r_n$, i.e. $S \vdash p \to q$. This completes the proof.

3. Soundness

Our next result is the

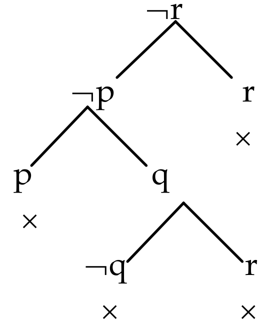
Soundness Theorem for the Propositional Calculus.

Any theorem of **PC** *is a tautology.*

Proof. Note first that, if a valuation satisfies both p and $p \to q$, then it satisfies q. Thus if both p and $p \to q$ are tautologies, so is q. In other words, **MP** leads from tautologies to tautologies.

It is also not hard to show that any axiom of **PC** is a tautology. For example, we may use the tree method to establish this for Axiom 8:

$$\surd\ \neg[(p \to r)\to((q \to r)\to(p \vee q\to r)]$$
$$\surd\ p \to r$$
$$\surd\ \neg[(q \to r)\to(p \vee q\to r)]$$
$$\surd\ q \to r$$
$$\surd\ \neg(p \vee q\to r)$$
$$\surd\ p \vee q$$
$$\neg r$$



Hence any deduction in **PC** consists entirely of tautologies, and the theorem follows.

As an immediate consequence of this, it follows that **PC** is *consistent* in the sense that for no statement p do we have both $\vdash$ p and $\vdash$ $\neg$p.

We are next going to establish a strengthened version of the Soundness Theorem by employing the Deduction theorem.

Strengthened Soundness Theorem for PC. *If* S $\vdash$ p, *then* S $\models$ p.

Proof. Suppose S $\vdash$ p, where S = $\{s_1, s_2, ..., s_n\}$. The trick is simply to apply the Deduction theorem to $s_1, s_2, ..., s_n \vdash p$ and carry each of the statements in the sequence $s_1, s_2, ..., s_n$ over to the right-hand side of the $\vdash$ sign so that the (unstrengthened) Soundness theorem, which we've already proved, can be invoked.

Thus, applying the Deduction theorem n times in succession to S $\vdash$ p yields:

$$\vdash s_1 \to (s_2 \to (\ldots (s_{n-1} \to (s_n \to p))\ldots)) \qquad (*)$$

(For example, if n=3 the first application of the Deduction theorem yields

$$s_1, s_2 \vdash s_3 \to p,$$

the next application yields

$$s_1 \vdash s_2 \to (s_3 \to p),$$

and the final application yields

$$\vdash s_1 \to (s_2 \to (s_3 \to p)).\ )$$

But by the (unstrengthened) Soundness theorem, we can infer from (*) that:

$$\models s_1 \to (s_2 \to (\ldots (s_{n-1} \to (s_n \to p))\ldots)) \qquad (**)$$

(**) simply asserts that the nested conditional above that we have concocted by applying the Deduction theorem is a tautology. But it is clear from the truth table for $\rightarrow$ that this could not be so unless there is never a case where all the statements in the set S = $\{s_1, s_2, \ldots, s_n\}$ are true and p is false. So (**) cannot be correct unless S $\models$ p, which is what we set out to prove.

Our final task will be to prove the converse of the Strengthened Soundness theorem.

4. Completeness

Let us call a set S of statements formally inconsistent if S $\vdash$ p and S $\vdash$ $\neg$p for some statement p. We now establish the following facts:

Fact A. S *is formally inconsistent if and only if* S $\vdash$ q *for all statements* q.

Fact B. S $\vdash$ p *if and only if* {S, $\neg$p} *is formally inconsistent.*

Proof of A. Clearly, if S $\vdash$ q for all statements q then S is formally inconsistent. To establish the converse, we begin by showing that $\vdash$ $\neg p \rightarrow (p \rightarrow q)$. First, note that the following sequence qualifies as a deduction of q from $\neg$p,p:

$$\neg p$$
$$p$$
$$p \rightarrow (\neg q \rightarrow p)$$
$$\neg p \rightarrow (\neg q \rightarrow \neg p)$$
$$\neg q \rightarrow p$$
$$\neg q \rightarrow \neg p$$
$$(\neg q \rightarrow p) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow \neg\neg q))$$
$$(\neg q \rightarrow \neg p) \rightarrow \neg\neg q$$
$$\neg\neg q$$

$$\neg\neg q \rightarrow q$$
$$q$$

It follows that $\neg p, p \vdash q$. Two applications of the deduction theorem now give $\vdash \neg p \rightarrow (p \rightarrow q)$ as claimed.

Now if S is formally inconsistent, we have $S \vdash p$ and $S \vdash \neg p$. Since

$$\vdash \neg p \rightarrow (p \rightarrow q),$$

two applications of **MP** yield $S \vdash q$. This proves **A**.

__Proof of B.__ If $S \vdash p$, then $S, \neg p \vdash p$ and $S, \neg p \vdash \neg p$, so $S, \neg p$ is formally inconsistent.

Conversely suppose that $S, \neg p$ is formally inconsistent. Then by Fact **A**, $S, \neg p \vdash \neg\neg p$. So by the deduction theorem $S \vdash \neg p \rightarrow \neg\neg p$. Now we have

$$p \rightarrow p, p \rightarrow \neg p \vdash \neg p$$

as the following deduction shows:

$$p \rightarrow \neg p$$
$$(p \rightarrow \neg p) \rightarrow ((p \rightarrow \neg p) \rightarrow \neg p)$$
$$p \rightarrow p$$
$$(p \rightarrow \neg p) \rightarrow \neg p$$
$$\neg p$$

Since $\vdash p \rightarrow p$, it follows that $p \rightarrow \neg p \vdash \neg p$. So, substituting $\neg p$ for p, we get $\neg p \rightarrow \neg\neg p \vdash \neg\neg p$. But $\neg\neg p \rightarrow p$ is an axiom, so an application of **MP** yields $\neg p \rightarrow \neg\neg p \vdash p$. But we have already observed that $S \vdash \neg p \rightarrow \neg\neg p$, so another application of **MP** yields $S \vdash p$ as required. This proves **B**.

We now sketch a proof of the

Theorem. *The initial set of statements of any closed tree is formally inconsistent*.

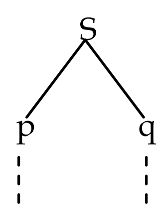
Proof (sketch). Let us define the *depth* of a tree to be the length of its longest path. Suppose that the assertion of the theorem is false. Then there is a closed tree with a *formally consistent* (i.e., not formally inconsistent) set of initial statements. Among these choose one, **T** say, of *least depth,* d say. Then **T** is a closed tree whose set S of initial statements is formally consistent. We shall derive a contradiction from this.

There are two cases to consider.

Case 1: d = 1. In this case **T** is identical with S. Since **T** is closed there must be some statement p for which both p and ¬p are in S. Clearly S is then formally inconsistent.

Case 2: d > 1. In this case, by assumption, the set of initial statements of *any* closed tree of depth < d is formally inconsistent. Now examine the statements at level 2 of **T**. We claim that however these statements were obtained, we can always conclude that S is formally inconsistent.

For example, suppose that the statements at level 2 of **T** arise by applying the ∨-rule to a statement in S of the form p ∨ q. Then **T** starts thus:



If in **T** we fuse S with p and expunge q as well as all nodes following it, we get a closed tree (recall that **T** was assumed closed) of depth < d with S,p as its set of initial statements. But then S,p is formally inconsistent. Similarly, S,q is formally inconsistent. Since p ∨ q is in S, it follows that S is formally inconsistent. For if r is any statement, we have S,p ⊢ r and S,q ⊢ r so that S ⊢ p → r and S ⊢ q → r. Two applications of **MP** and

Axiom 8 now yield S $\vdash$ $p \vee q \rightarrow r$; but since $p \vee q$ is in S, **MP** yields S $\vdash$ r. Since this holds for any statement r, S is formally inconsistent.

Similar arguments work for the other rules; in all cases we are able to conclude that S is formally inconsistent.

We have shown that assuming the theorem false leads to a contradiction. So the theorem is proved.

As a consequence of this, we finally obtain the

<u>Completeness Theorem for PC.</u> *If* S $\models$ p *then* S $\vdash$ p.

<u>Proof.</u> If S $\models$ p, then by inference adequacy any finished tree **T** associated with the inference of p from S is closed. It follows from the previous theorem that the set S, ¬p of initial statements of **T** is formally inconsistent. Hence, by fact **B** , S $\vdash$ p.

5. Exercises

<u>The Propositional Calculus</u>

Axioms:

1. $p\rightarrow(q\rightarrow p)$	6. $p\rightarrow(p\vee q)$
2. $[p\rightarrow(q\rightarrow r)]\rightarrow[(p\rightarrow q)\rightarrow(p\rightarrow r)]$	7. $q\rightarrow(p\vee q)$
3. $(p\wedge q)\rightarrow p$	8. $(p\rightarrow r)\rightarrow[(q\rightarrow r)\rightarrow((p\vee q)\rightarrow r)]$
4. $(p\wedge q)\rightarrow q$	9. $(p\rightarrow q)\rightarrow[(p\rightarrow\neg q)\rightarrow\neg p]$
5. $p\rightarrow(q\rightarrow(p\wedge q))$	10. $\neg\neg p\rightarrow p$

Rule of Inference: $p,\ldots, p\rightarrow q, \ldots,q$ (Modus Ponens)

A1. The following is a purported deduction of p from ¬¬p. Verify that it is or is not by identifying the origin of each statement in the sequence.

$\neg\neg p$, $(\neg p\rightarrow((\neg p\rightarrow\neg p)\rightarrow\neg p))$, $\neg p\rightarrow(\neg p\rightarrow\neg p)$,

$(\neg p\rightarrow((\neg p\rightarrow\neg p)\rightarrow\neg p))\rightarrow((\neg p\rightarrow(\neg p\rightarrow\neg p))\rightarrow(\neg p\rightarrow\neg p))$, $((\neg p\rightarrow(\neg p\rightarrow\neg p))\rightarrow(\neg p\rightarrow\neg p))$, $(\neg p\rightarrow\neg p)$, $(\neg p\rightarrow\neg p)\rightarrow((\neg p\rightarrow\neg\neg p)\rightarrow\neg\neg p)$, $\neg\neg p\rightarrow p$, $(\neg p\rightarrow\neg\neg p)\rightarrow\neg\neg p$, $\neg\neg p\rightarrow(\neg p\rightarrow\neg\neg p)$, $\neg p\rightarrow\neg\neg p$, $\neg\neg p$, p

Of course, there is a very simple deduction of p from ¬¬p. What is it?

A2. The sequence 1.-14. below (see over) allegedly establishes that:

$$\neg p\rightarrow\neg\neg p,\ q \vdash q\wedge p$$

Check to see whether this is so by justifying each statement below with the words "in the initial set", "modus ponens", or "axiom # so-and-so" (filling in the relevant axiom number). If a particular statement *cannot* be justified, say so!

1. $[\neg p\rightarrow((\neg p\rightarrow\neg p)\rightarrow\neg p)]\rightarrow[(\neg p\rightarrow(\neg p\rightarrow\neg p))\rightarrow(\neg p\rightarrow\neg p)]$
2. $\neg p\rightarrow((\neg p\rightarrow\neg p)\rightarrow\neg p)$
3. q
4. $(\neg p\rightarrow(\neg p\rightarrow\neg p))\rightarrow(\neg p\rightarrow\neg p)$
5. $\neg p\rightarrow(\neg p\rightarrow\neg p)$
6. $q\rightarrow(p\rightarrow(q\wedge p))$
7. $\neg p\rightarrow\neg p$
8. $\neg p\rightarrow\neg\neg p$
9. $(\neg p\rightarrow\neg p)\rightarrow[(\neg p\rightarrow\neg\neg p)\rightarrow\neg\neg p]$
10. $(\neg p\rightarrow\neg\neg p)\rightarrow\neg\neg p$
11. $\neg\neg p$
12. $\neg\neg p\rightarrow p$
13. p
14. $q\wedge p$

A3. There are two sequences of statements below (set aside in two separate columns), each purporting to be a deduction from the set of statements S = {p, q→r}. Identify the origin of each statement in each sequence, and thus discern whether or not these sequences really are deductions from S.

p	$q \to r$
$q \to r$	p
$p \to [(q \to r) \to (p \wedge (q \to r))]$	$p \to (r \to p)$
$(q \to r) \to (p \wedge (q \to r))$	$r \to p$
$p \wedge (q \to r)$	$q \to p$

A4. By the Completeness and Strengthened Soundness theorems for the propositional calculus, each concept on the left below corresponds to one on the right and vice-versa. Match them up.

tautology	deduction
unsatisfiable	theorem
valid argument	proof
valid argument without premises	formally inconsistent

A5. State the theorems below first in symbols and then in your own words.

(i) (Strengthened) Soundness Theorem

(ii) Completeness Theorem

(iii) What is the point of introducing the propositional calculus and proving these theorems?

A6. Indicate whether each of the following statements is true or false.

(a) A statement deducible from its negation cannot be a theorem.

(b) Consistency of the propositional calculus follows from the completeness theorem.

(c) If $p \vdash (q \to \neg p)$, then the pair $\{p,q\}$ is formally inconsistent.

(d) The conclusion of a proof cannot be formally inconsistent.

(e) A theorem cannot be deduced from a formally inconsistent set of statements.

(f) Assuming every tautology is a theorem, completeness of the propositional calculus follows from the deduction theorem.

(g) a $\vdash$ c implies a,$\neg$b $\vdash$ b $\rightarrow$ c for all statements a, b and c.

(h) The propositional calculus would not be sound unless it employed the modus ponens rule.

For the last two questions, call a set of statements maximally consistent if and only if it is formally consistent but not a subset of any other formally consistent set of statements.

(i) No maximally consistent set of statements can contain all theorems.

(j) Every maximally consistent set of statements must contain either p or $\neg$p, for any statement p.

A7. Why does p $\vdash$ r imply p $\vdash$ (q$\rightarrow$r) for any statements p, q and r?

A8. (a) Assuming the Completeness and Strengthened Soundness theorems, prove the Deduction theorem.

(b) Use the Deduction theorem to show directly (i.e. without explicitly constructing a deduction sequence) that $\vdash$ (p$\rightarrow$p) and that $\neg\neg$p $\vdash$ p.

A9. (i) For any statements p, q and set of statements S, S $\vdash$ p and p $\vdash$ q implies S $\vdash$ q. Why?

(ii) Let S_1, S_2, ...,S_n be n *sets* of statements, and let S be the set of statements $\{p_1,p_2,...,p_n\}$. Show that if $S_i \vdash p_i$ for all i=1 to n and S $\vdash$ p for some statement p, then $S_1 \cup S_2 \cup ... \cup S_n \vdash$ p.

(iii) Show that if S $\vdash$ p and S $\vdash$ q, then S $\vdash$ $(p \wedge q)$. (You are not allowed to assume the completeness theorem!)

A10. (a) By relying on the Completeness and Strengthened Soundness theorems, prove that a set S of statements is formally inconsistent if and only if S $\vdash$ p for all statements p.

(b) Without relying on Completeness and Soundness, show that S $\vdash$ $\neg p$ implies that the set $T=S\cup\{p\}$ is formally inconsistent.

APPENDIX A

LOGIC AND THE FOUNDATIONS OF MATHEMATICS

Mathematics, the traditional science of form and quantity, and logic, the traditional science of reasoning, are among the oldest of human intellectual endeavours. However, it is only in the last century or so that the connections between the two have been explicitly recognized and systematically developed, leading to the enrichment of both. In this appendix, we shall take a look at how and why this came about, and describe the effect that logical analysis has had on the foundations of mathematics.

Medieval scholars divided learning into two categories. The first category was the *quadrivium*, or 'fourfold way to knowledge', comprising the mathematical arts arithmetic, geometry, astronomy and music. The second was the *trivium*, or 'threefold way of eloquence', comprising the verbal arts grammar, rhetoric and logic. Logic was thus regarded by the schoolmen as an essentially linguistic discipline, having little or no bearing on mathematics. In any case, the official scholastic view was that logic had been perfected by Aristotle, so that any further contributions to the discipline could be no more than embellishments on the Aristotelian edifice. Although in the seventeeth century Leibniz had expressed the desirability of transforming logic into a universal scientific language, little progress was made in this respect until the middle of the nineteenth century when the English mathematicians George Boole (1815-1864) and Augustus de Morgan (1806-1871) and the American philosopher Charles Sanders Peirce (1839-1914) took the first steps in extending logic beyond its Aristotelian horizons. Original as these contributions were, however, the major impetus behind the transformation of logic was furnished by the later appearance of difficulties in the foundations of mathematics.

Towards the end of the nineteenth century, the German mathematician Georg Cantor (1845-1918), in the course of his mathematical research, had come to reject the received idea that the actual infinite is an inadmissible concept in mathematics, and proceeded to build a mathematical theory of infinite totatilities—the so-called *set theory*. Cantor held that there is no difference in principle between finite and infinite sets, as is revealed in his 1895 definition of the set concept:

> *By a 'set' we understand every collection to a whole of definite, well-differentiated objects of our intuition or thought.*

The problem, of course, is to determine exactly which collections consititute legitimate 'wholes'. Traditionally, only *finite* collections were admitted to be 'wholes' in this sense, since, it was held, infinite collections fail to accord with the time-honoured dictum that a whole must always be greater than any of its (proper) parts.

The curious way in which the principle that a whole must always exceed its parts is violated by infinite collections is strikingly conveyed by a fable attributed to the German mathematician David Hilbert (1862-1943). In Hilbert's tale, he finds himself the manager of a vast hotel, so vast, indeed, that it has an infinite number of rooms. Thus the hotel has a first, second, ..., nth,... room, *ad infinitum*. At the height of the tourist season, Hilbert's hotel is full: each room is occupied. (Note that we are tacitly assuming the existence of an infinite collection of occupants.) Now a newcomer seeking accommodations shows up. 'Alas', says Hilbert, 'I have not a room to spare.' But the newcomer is desperate. At that point an idea occurs to Hilbert. He telephones the occupant of each room and tells him to move to the next one; thus the occupant of room 1 moves to room 2, that of room 2 to room 3, etc. This leaves the original occupants housed (one to a room, as before), only now room 1 is vacant, and the relieved newcomer duly takes possession. In this way we see that the whole set of rooms is in some sense no larger than the part obtained by removing the first room.

The fable does not end here, however. Hilbert is about to switch on the 'no vacancy' sign when a vast assembly of tourists desirous of accommodation descends on the hotel. A quick tally reveals that the assembly is *infinite*, filling Hilbert with dismay (and the reader, no doubt, with incredulity). But now another idea occurs to him. He tells each occupant that he is to move to the room assigned double the number of his present one: thus the occupant of room number 1 proceeds to room number 2, that of room number 2 to room number 4, etc. This again leaves all the original occupants housed, only now each of the infinite set of rooms carrying odd numbers is vacant. Thus each newcomer can be accommodated: the first in room 1, the second in room 3, the third in room 5, etc. Clearly this procedure can be repeated indefinitely, enabling an infinite number of infinite assemblies of tourists to be accommodated.

Hilbert's tale shows that infinite sets are intriguing *paradoxical* but *not*, be it noted, that they are *contradictory*. (Indeed, if, for example, the physical universe happens to contain infinitely many stars—which is at least possible—then it can serve as 'Hilbert's hotel', with the stars playing the role of 'rooms'.) Set theory as originally formulated *does* contain contradictions; however, they arise not from the admission of infinite totalities *per se*, but are rather the result of countenancing totalitites consisting of *all* entities of a given kind. This is best illustrated by the infamous *Russell paradox*, discovered by Bertrand Russell (1872-1970) in 1901.

Russell's paradox arises in the following way. It starts with the truism that any set is either a member of itself or not. For instance, the set of all men is *not* a member of itself (since it is not a man), while the set of all possible sets *is* a member of itself (since it is, presumably, a set). Now consider the set consisting precisely of all those sets which are *not* members of themselves: call this set 'R' (for Russell, of course). Is R a member of itself or not? Suppose it is. Then it must satisfy the defining condition for inclusion in R, i.e., it must *not* be a member of itself. Conversely, suppose R isn't a member of itself. Then it *fails* to satisfy the defining condition for inclusion in R, i.e., it *must* be a member of itself. We have thus arrived at the unsettling, indeed contradictory, conclusion that R is a member of itself if and only if it isn't. We also note that whether R is finite or infinite is immaterial to this argument.

Russell's paradox has a purely linguistic counterpart due to Kurt Grelling (1886-1941) which is, perhaps, even more unsettling since it appears to strike at the very fabric of language. Call an (English) adjective *autological* if it is true of itself and *heterological* if not. For instance, the adjectives 'polysyllabic', 'English' are autological while 'palindromic', 'French' are not. Now, mischievously, we enquire as to whether the adjective 'heterological' is autological or not. A moment's thought should reveal that 'heterological' is autological if and only if it isn't autological.

Another principle of set theory which occasioned much dispute when first formulated is the so-called *axiom of choice*. In its simplest form, the axiom asserts that, if we are given any collection S of sets, each of which has at least one member, then there is a set M containing exactly one member from each set in S. We can imagine the set M as being formed by choosing one member from each set in S. When there are only finitely many sets in S, or if S is infinite and we have a definite rule for choosing a member from each set in S, no difficulty is encountered. The problem arises when S contains infinitely many sets, and we have no rule for choosing a member from each: how can we justify the procedure of making infinitely many arbitrary choices (and forming a set from the result)? The difficulty is well illustrated by a Russellian anecdote. A millionaire possesses an infinite number of pairs of shoes, and a corresponding number of pairs of socks. In a fit of eccentricity, he requests his valet to select one shoe from each pair. When the valet asks for instructions as to how to perform the selection, the millionaire suggests that the left shoe be chosen from each pair. The following day the millionaire proposes that the valet select one *sock* from each pair. When asked as to how this operation should be carried out, the millionaire is at a loss for a reply, since, unlike shoes, there is no intrinsic way of distinguishing one sock of a pair from the other. In other words, the selection of the socks must be truly arbitrary.

One curious consequence of the axiom of choice is the so-called *paradoxical decomposition of the sphere*, formulated in 1924 by the Polish mathematicians Stefan Banach (1892-1945) and Alfred Tarski (1902-1983). In its most striking form, this states that a solid sphere can be cut into finitely many (later shown to be reducible to 5) pieces which can themselves be rearranged to form *two* solid spheres of the same size as the original. (Of course, the use of the phrase 'can be cut' is metaphorical, but this does not detract from the wierdness of the result.) Note that this assertion, like the situation obtaining at Hilbert's Hotel, and *unlike* Russell's paradox, is counter-intuitive but not contradictory.

The perplexities surrounding the foundations of set theory are collectively designated by historians of mathematics as the 'third crisis' in the foundations of mathematics. (For the record, the first two were the Pythagorean discovery of irrational numbers c. 450 B.C. and the scandalously shaky state of the foundations of the calculus throughout the seventeenth and eighteenth centuries.) Attempts to resolve this crisis took several different forms, but they all required the deployment of subtle kinds of *logical analysis* of mathematical concepts and reasoning, thereby occasioning the rise of *mathematical logic* as a powerful new discipline.

Three principal tendencies were to emerge concerning the foundations of mathematics and the resolution of the 'crisis': *Logicism*, *Intuitionism*, and *Formalism.*

Logicism, associated with (among others) Gottlob Frege (1848-1925), Russell and Alfred North Whitehead (1861-1947), pivots on an attempt to reduce mathematics (i.e. set theory) to pure logic, the former's inconsistencies being dissolved (it was hoped) by proper formulation in terms of unimpeachable logical concepts. In Russell and Whitehead's approach, Cantor's set theory is replaced by the *theory of types*, in which a logical distinction is drawn between a set and its members, the former being, so to speak, of higher 'type' than the latter. The paradoxes are avoided by scrupulous observance of the so-called *vicious circle principle*, according to which no totality can contain members defined in terms of itself. (For example, the totality of all sets violates the vicious circle principle since it contains itself as a member and is therefore excluded as a legitimate totality.) The logicist programme culminated in the publication in 1910 of the monumental, and formidably recondite, *Principia Mathematica* of Russell and Whitehead. That the term 'recondite' is apposite here may be deduced from the following extract from a review of the work which appeared in a 1911 number of the *Spectator*:

> *It is easy to picture the dismay of the innocent person who out of curiosity looks into the later part of the book. He would come upon whole pages without a single word of English below the headline; he would see, instead, scattered in wild profusion, disconnected Greek and Roman letters of every size interspersed with brackets and dots and inverted commas, with arrows and exclamation marks standing on their heads, and with even more fantastic signs for which he would with difficulty so much as find names.*

The complexity of Russell and Whitehead's theory, allied with doubts that, despite their Herculean efforts, they had truly succeeded in reducing mathematics to pure logic, resulted in a certain lack of enthusiasm on the part of mathematicians for the logicist programme.

Intuitionism, the creation of the Dutch mathematician L. E. J. Brouwer (1882-1966), hinges on the conviction that a mathematical concept is admissible only if it is adequately grounded in *intuition*. For Brouwer, the source of mathematical ideas is ultimately to be found, not in the external world, but in our own intuitive awareness. (In particular, Brouwer shares with Kant the view that the natural numbers arise from our intuitive grasp of temporal succession.) As a result, for intuitionism a mathematical object may be said to exist only if it can be constructed (in thought) in some definite way, the construction constituting the proof of existence. Moreover, an infinite totality must never be treated as if it were a completed whole, but rather as if it were continually 'growing' in time. These principles, if accepted, have profound implications for logic and mathematics. For example, the first implies that the classical logical law of the excluded middle is no longer generally valid, that is, we can no longer assert, of any given proposition A, that either A or its negation *not* A holds. According to the intuitionist view, in order to be able to correctly assert this disjunction, we must actually be in possession of a proof of A or a proof of *not* A: however, there are many mathematical assertions—for example the statement that every even number is the sum of two primes—for which we possess neither. As for the second principle, it necessitates the outright rejection of Cantor's theory and the reconstruction of mathematics along entirely different lines.

The radical nature of the intuitionist program and the fact that it apparently entailed sacrifice of large parts of classical mathematics caused it to appear less than attractive to the majority of mathematicians.

Formalism, the brainchild of David Hilbert, had as its aim the provision of a new foundation for mathematics by reducing it, not to logic, but to the manipulation of formal symbols. Hilbert believed that the only completely reliable parts of mathematics are those which involve nothing more than mechanical reasoning about surveyable domains of concrete objects, in particular, mathematical symbols considered as marks on paper. Propositions referring only to such parts of mathematics he regarded as *real* propositions while all other mathematical propositions are to be considered *ideal* statements, rather like ideal points or lines 'at infinity' in projective geometry. (For example, '2+2=4' is a real proposition, while 'there exists an infinite set' is an ideal one.) Thus Hilbert's real propositions correspond to the 'verifiable statements' of the logical positivists, while the ideal statements are, strictly speaking, meaningless. The central objective of Hilbert's *formalist program* was to show by strictly concrete and unimpeachable means that the classical use of ideal propositions—in particular, those of Cantor's set theory (suitably adjusted to avoid the contradictions already mentioned)—would never lead to falsehoods among the real propositions. In short, the aim was to prove classical mathematics *consistent*. Once this was achieved, mathematicians would be able to roam freely within 'Cantor's paradise' (as Hilbert called it) without fear of a sudden descent into the inferno of contradiction.

Hilbert intended to establish the consistency of classical mathematics by setting it out as a purely formal system of symbols, devoid of meaning, and then showing that no proof in the system leads to a false assertion, e.g., 0=1. This, in turn, was to be done by replacing each (ideal) proof of a real proposition by a real concrete proof. Since, clearly, there is no concrete proof of the real (false) proposition 0=1, this leads to the conclusion that classical mathematics is consistent.

However, in 1931 the Austrian logician Kurt Gödel (1906-1976) shattered Hilbert's program by demonstrating, through his famous Incompleteness Theorem, that there would always be real propositions provable by ideal means which are not provable by concrete means. He achieved this by means of an ingenious modification of the ancient Liar paradox (attributed to the Greek philosophers Epimenides and Eubulides). To obtain the Liar paradox in its most transparent form, one considers the sentence 'This sentence is false'. Calling this sentence A, an instant's thought reveals that A is true if and only if A is false; in short, A asserts its own falsehood. Now Gödel showed that, if in A one replaces the term "false" by the phrase "not concretely provable", then the resulting statement B is *true* (i.e. provable by ideal means) but *not* concretely provable. This is so because B actually *asserts* its own concrete unprovability. Gödel showed, moreover, that the *consistency of arithmetic* cannot be proved by concrete means. Thus, the soundness of even such an apparently concrete and perspicuous part of mathematics as arithmetic must in some sense remain an article of faith. (See the next appendix for further discussion of Gödel's theorems.)

Although Logicism, Intuitionism and Formalism are unacceptable as complete accounts of the foundations of mathematics each one embodies an important partial truth concerning the nature of mathematics: Logicism, that mathematical truth is intimately connected with logical truth; Intuitionism, that mathematical activity proceeds by the performance of mental constructions; and Formalism, that the results of these constructions are presented symbolically.

In practice, most mathematicians regard *set theory* as constituting an adequate foundation for their work. This became possible when, in the first few decades of this century, set theory was axiomatized in such a way as to avoid the evident contradictions by suitably restricting the formation rules for sets. (Any residual doubts concerning the acceptability of the axiom of choice were dispelled in 1938 when Gödel established its consistency with respect to the remaining axioms of set theory.) Mathematicians find set theory acceptable not solely for the pragmatic reason that it enables mathematics to be done but also because it accords with the unspoken belief of many of them that mathematical objects actually exist in some sense and mathematical theorems express truths about these objects. This is a version of *Realism*, also termed, somewhat inaccurately, *Platonism*.

APPENDIX B

HILBERT'S PROGRAM AND GODEL'S THEOREMS

In response to the inconsistencies which had made their appearance in set theory (see Appendix A), Hilbert proposed to set up a precise formal language for mathematics and then prove that the resulting formalization is *consistent*, i.e. does not lead to contradiction. To ensure that the consistency proof convinced the maximum possible number of mathematicians, it was supposed to involve only elementary mathematical notions, and ideally should not transcend school arithmetic.

In 1931 Gödel showed that Hilbert's program could not be carried out *even for arithmetic!* He showed, in fact, that if arithmetic *is* consistent, then any consistency proof *must* transcend arithmetic itself. And, a fortiori, this applies also to any branch of mathematics (such as set theory) which is stronger than arithmetic.

We shall sketch proofs of Gödel's result, obtaining along the way an important theorem of Tarski on the undefinability of mathematical truth.

First we set up a precise formal language $\mathcal{L}$ for arithmetic as follows. $\mathcal{L}$ has the following symbols:

Numerical variables x_1, x_2, x_3, ...

Numerical constants $\underline{0}$, $\underline{1}$, $\underline{2}$, ...

Numerical function symbols +, ×

Equality symbol =

Logical operators $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$

Punctuation symbols.

An *expression* of $\mathcal{L}$ is a finite sequence (string) of symbols of $\mathcal{L}$.

Terms or noun-like expressions of $\mathcal{L}$ are obtained as follows.

(i) Any variable or constant symbol is a term.

(ii) If t,u are terms, so are $t \times u$, $t + u$.

(iii) An expression is a term if and only if it follows that it is one from finitely many applications of (i) and (ii).

Examples of terms: $x_1 + \underline{2}$, $\underline{6}$, $x_1 \times x_5 + \underline{27}$.

Formulas (or grammatical assertions) of $\mathcal{L}$ are obtained as follows.

(a) If t,u are terms, then $t = u$ is a formula.

(b) If A and B are formulas, so are $\neg A$, $A \wedge B$, $A \vee B$, $A \rightarrow B$, $A \leftrightarrow B$, $\forall xA$, $\exists xA$ for any variable x.

(c) An expression is a formula if and only if it follows from finitely many applications of (a) and (b) that it is one.

Examples of formulas: $x_1 + \underline{2} = \underline{3}$, $\exists x_1[x_1 + \underline{2} = \underline{3}]$, $\forall x_1[x_1 + \underline{2} = \underline{3}]$.

Formulas assume *truth values* (truth or falsehood) when their constituents are interpreted arithmetically in the obvious way. We observe that, e.g. the first formula in the examples above is true or false depending on the *value* assigned to the variable x_1, while the second or third is simply true or false *independently* of the value assigned to the variables. A formula of this latter kind in which each variable x occurring in it is governed by a quantifier $\forall x$ or $\exists x$ is called a *sentence*.

We next assign *code numbers* to the expressions of $\mathcal{L}$ as follows. Suppose that the symbols of $\mathcal{L}$, excluding variables and constants, are k in number. To these symbols we assign, in some arbitrary but fixed manner, the *label* 0, ..., k-1. Then to each variable x_n we assign the *label* $k + 2n$ and to each numeral (i.e. constant) $\underline{n}$ the *label* $k + 2n + 1$. Thus each symbol s has been assigned a label which we shall denote by s*. Now each expression $s_1 \ldots s_n$ is assigned the *code number*

$$2^{s_1^*} \cdot 3^{s_2^*} \cdots p_n^{s_n^*}$$

where p_n is the n^{th} prime number. It follows, by the fact that every natural number has a unique prime factorization, that distinct expressions are assigned distinct code numbers by this procedure. The expression with code number n will be denoted by A_n.

A property $P(a_1,\ldots,a_n)$ of natural numbers is called *expressible* in $\mathcal{L}$ if there is a formula $A(x_1,\ldots,x_n)$ of $\mathcal{L}$ such that, for all numbers $m_1,\ldots,m_n$,

$$P(m_1,\ldots,m_n) \text{ holds } \Leftrightarrow A(\underline{m}_1,\ldots,\underline{m}_n) \text{ is true.}$$

A property of *expressions* of $\mathcal{L}$ is called *expressible* in $\mathcal{L}$ if the corresponding property of their code numbers is expressible. It can be shown without much difficulty that the property of being a (code number of a) formula of $\mathcal{L}$ is expressible in $\mathcal{L}$.

But what about the property of being a *true sentence* of $\mathcal{L}$? We shall see that this property is *not* expressible!

Since the assignment of code numbers to expressions of $\mathcal{L}$ is evidently effective, we should be able to compute, for a given expression $A_m(x_1)$ with code number m, and any number n, the code number of the expression $A_m(\underline{n})$ obtained by substituting $\underline{n}$ for x_1 in $A_m(x_1)$. Thus it is plausible to assume (and, indeed, provable) that this is expressible in $\mathcal{L}$ in the sense that there is a formula

$$s(x_1, x_2) = x_3$$

for some term s of $\mathcal{L}$ such that, for any numbers m, n, and p,

$$(s\,(\underline{m},\underline{n}) = \underline{p}\,) \text{ is true} \Leftrightarrow p \text{ is the code number of the expression } A_m(\underline{n}).$$

Now let S be any collection of sentences of $\mathcal{L}$: S that may be thought of as a set of *true arithmetical* statements in $\mathcal{L}$.

We can now prove:

Theorem 1. *Suppose that* S *satisfies the following conditions:*

(1) *Each member of* S *is true.*

(2) *The property "*n *is the code number of a member of* S*" is expressible in* $\mathcal{L}$.

Then there is a true sentence G *of* $\mathcal{L}$ *such that* $G \notin S$ (*and* $\neg G \notin S$).

Proof. By (2) there is a formula $T(x_1)$ of $\mathcal{L}$ such that, for all n,

$$T(\underline{n}) \text{ is true} \Leftrightarrow n \text{ is the code number of a sentence of } S$$

$$\Leftrightarrow A_n \in S.$$

Write $B(x_1)$ for $\neg T\,(s(x_1, x_1))$ and let m be the code number of B. Then

$$B \text{ is } A_m\,.$$

Next, let p be the natural number such that

$$\underline{p} = s(\underline{m}, \underline{m})$$

is a true sentence. Then, by definition, p is the code number of the sentence $A_m(\underline{m})$.

Now write G for $A_m(\underline{m})$. Then p is the code number of G, so that

$$G \text{ is } A_p.$$

So we have

$$\begin{aligned} G \text{ is true} &\Leftrightarrow A_m(\underline{m}) \text{ is true} \\ &\Leftrightarrow B(\underline{m}) \text{ is true} \\ &\Leftrightarrow \neg\, T(s(\underline{m}, \underline{m})) \text{ is true} \\ &\Leftrightarrow T(\underline{p}) \text{ is false} \\ &\Leftrightarrow A_p \notin S \\ &\Leftrightarrow G \notin S. \end{aligned}$$

(Thus G asserts "I am not in S".) It follows that G is true, for

$$G \text{ is false} \Rightarrow G \in S \Rightarrow G \text{ is true}.$$

So $G \notin S$, and since G is true, $\neg\, G \notin S$ also. This completes the proof of the theorem.

By taking S to be the collection of all *true* sentences in this theorem, we immediately obtain:

Tarski's Theorem in the Undefinability of Truth. *The property of being a true sentence of $\mathcal{L}$ is not expressible in $\mathcal{L}$.*

Notice that the appropriate sentence G in Tarski's theorem asserts "I am not in the set of true sentences," i.e., "I am false." So Tarski's theorem is closely connected with the well-known *Liar Paradox*.

Now one can introduce the notion of a *proof* from S and that of a formula *provable* from S in such a way that:

- if each member of S is true, so is each sentence provable from S,
- if the property of being a member of S is expressible in $\mathcal{L}$, so is the property of being a formula provable from S.

One now obtains immediately from Theorem 1

Godel's 1st Incompleteness Theorem (weak form). *Let S be a set of true sentences of* $\mathcal{L}$ *such that the property of being a member of* S *is expressible in* $\mathcal{L}$. *Then* S *is* *incomplete*, i.e. *there is a (true) sentence* G *of* S *such that neither* G *nor* ¬ G *is provable from* S.

Here the sentence G asserts "I am *unprovable*."

By refining the argument it is possible to strengthen this result as follows. Let us call S *consistent* if no formula of the form A ∧ ¬ A is provable from S. Let $P(x_1,\ldots,x_n)$ be a property expressible by a formula $A(x_1,\ldots,x_n)$. P is said to be *S-definite* if, for any natural numbers $m_1, \ldots, m_n$

$$P(m_1,\ldots,m_n) \text{ holds} \Leftrightarrow A(\underline{m}_1,\ldots,\underline{m}_n) \text{ is provable from S}$$

$$P(m_1,\ldots,m_n) \text{ fails} \Leftrightarrow \neg A(\underline{m}_1,\ldots,\underline{m}_n) \text{ is provable from S.}$$

Let $Q(m_1,m_2,m_3)$ be the *substitution property* of natural numbers, i.e.

$$Q(m_1,m_2,m_3) \text{ holds} \Leftrightarrow s(\underline{m}_1, \underline{m}_2) = \underline{m}_3 \text{ is true.}$$

Then one can prove

Godel's 1st Incompleteness Theorem (strong form). *Suppose that* S *is consistent, the property of being a member of* S *is expressible and the property* Q *and that of being an* S-*provable formula are both* S-*definite*. *Then* S *is incomplete*.

Sketch of proof. As before, we take G to be a sentence of $\mathcal{L}$ which asserts its own unprovability from S. For any A, let us write S ⊢ A for "A is provable from S".

Suppose that S $\vdash$ G. Then because of the assumptions on S,

$$S \vdash \text{“G is provable from S”}.$$

But the assertion “G is provable from S” is just ¬ G, so we get

$$S \vdash \neg G,$$

contradicting the supposed consistency of S. Therefore, not (S $\vdash$ G).

Now suppose that S $\vdash$ ¬ G. Then since S is consistent, we get not (S $\vdash$ G) and by the assumptions on S it follows that

$$S \vdash \text{“G is unprovable from S”}.$$

But G *is* the assertion “G is unprovable from S,” so we get

$$S \vdash G,$$

again contradicting the consistency of S. Hence not (S $\vdash$ ¬ G). This completes the proof.

In the proof of this last theorem we showed that

$$\text{S is consistent} \Rightarrow \text{not } (S \vdash G). \qquad (*)$$

Now the assertion on the left-hand side of (*) can be expressed as a sentence Con_S of $\mathcal{L}$ as follows. Let n_o be the code number of the sentence $\underline{0} = \underline{1}$ and let $P(x_1)$ be the formula of $\mathcal{L}$ expressing “x_1 is the code number of a sentence provable from S”. Then Con_S may be taken to be the sentence $\neg P(\underline{n_o})$.

The implication (*) can now be *formalized* in S yielding a proof in S of the implication

$$Con_S \rightarrow G$$

(observing that G is essentially the formalization in S of the assertion not $(S \vdash G)$).

Now suppose that

$$S \vdash \mathrm{Con}_S .$$

Then since, as we have seen,

$$S \vdash \mathrm{Con}_S \rightarrow G,$$

it follows that $S \vdash G$. But, by the 1st incompleteness theorem, if S is consistent, then not $(S \vdash G)$. Accordingly, we have

<u>Godel's Second Incompleteness Theorem</u>. *Under the same conditions as the strong form of the first incompleteness theorem, the sentence* Con_S *formalizing* "S is consistent" *is not provable from* S.

<u>Conclusion</u>: if we classify as *theological* any domain of thought which rests on faith, i.e. on unproved assumptions, then mathematics is a branch of theology which contains a rigorous demonstration that it must be so classified!

APPENDIX C

SOLUTIONS TO EXERCISES

CHAPTER I

Truth Tables and Testing Validity

A1. (a) invalid; (c), (e), (g), (i), (k), (m), (o) all valid

A2.

(a) $K \vee F$	(c) $F \to O$	(e) $R \to W$	(g) $(W \wedge Q) \to D$	(i) $F \to T$
K	O	$W \to \neg R$	$\therefore (W \to D) \vee (Q \to D)$	$F \wedge \neg T$
$\therefore \neg F$	$\therefore F$	$\therefore \neg R$		$\therefore L$
invalid	invalid	valid	valid	valid!

A3. Both knights

A4. With 'I'm right' = IR, 'You're a fool' = YF, and 'I'm a fool' = IF, the argument symbolized is:

$$IR \to YF$$
$$IF \to \neg IR$$
$$\underline{YF \to IR}$$
$$IF \vee YF$$

To check for counterexamples, and hence validity, we only need look at the cases in the truth table where the conclusion comes out false. That means we only look at cases where both IF and YF are false (otherwise $IF \vee YF$ comes out true!), and that can happen in two ways according to what the truth value of IR is. So the truncated truth table for this argument looks like:

IF	YF	IR		IR→YF	IF→¬IR	YF→IR		IF ∨ YF
f	f	t	|	f	t	t	|	f
f	f	f	|	t	t	t	|	f

The second row exhibits a case where the premises are all true but conclusion false, so the argument is *invalid*.

A5. With 'The witness was not intimidated' = W, 'Flaherty committed suicide' = F, and 'A note was found' = N, the set of sentences is {W ∨ (F→N), W→¬F, N→F}. Fairly quickly one can see that W = true, F = false, and N = false is a satisfying truth valuation!

A6. (a) Y ∧ [I ∨ (¬Y ∧ ¬I)], therefore (Y ∧ I) ∨ (¬Y ∧ ¬I). valid

A7. Both knaves.

A8. I = can't tell! He = taxpayer for sure.

A9. (a) valid; (c) valid

A10. NO. The argument's certainly valid, and its second premise is true. But supposing premise one is true (i.e. 'for its conclusion is false') leads to a contradiction: for since we would then have a sound argument, the conclusion would have to be true, contradicting premise one's truth (and also the argument's soundness!). So premise one can't be true! So in fact the conclusion *is* true, and therefore the argument is unsound (as its conclusion claims!).

Tautologies, Contradictions and Satisfiability

B1. (a), (c) tautologous

B2. (a) correct; (c) incorrect; (e) correct; (g) incorrect;

(i), (k) correct; (m) incorrect

B3. (a) Correct. If a statement is not contingent it is either a tautology or contradiction, therefore its negation is either a tautology or contradiction, hence its negation can't be contingent.

(c) Correct. For a conjunction to be a tautology, it must come out true under all possible valuations; and since it's a conjunction, that means each of its conjuncts has to come out true under all possible valuations, otherwise the entire conjunction will come out false under some valuation. Hence all the conjuncts must also be tautologies.

(e) Correct. If p→q is valid that means it's a tautology, which means there is never any case where it comes out true. By the truth-conditions for '→', that means there can never be any case where p comes out true and q false (otherwise p→q would be false). But if there is never any case where p is true and q false, the argument from p as premise to q as conclusion faces no counterexamples, and so must be valid. Conversely, suppose the argument from p to q is a valid one. Then there is never any case where p is true and q is false (otherwise we'd have a counterexample!). But since p→q is false only in such a case, there is never any case where p→q comes out false, which means that it is a tautology, i.e. valid.

B4. (a) (ii); (c) (iii)

B5. (a) If there *were* a case where a conjunct comes out false, the conjunction would have to come out false in that case too, and so couldn't be valid!

B6. (a) (iv); (c) (iv); (e) (iii)

B7. (a) false; (c) false; (e) false; (g) true

B8. (a) true; (c) true; (e) false;

(g) false - tautologies only imply tautologies!; (i) false - just add a contradiction!

B9. (a) satisfiable; (c) unsatisfiable; (e) satisfiable

B10. (a) false; (c) false; (e) false.

B11. (a) Not valid

CHAPTER II

<u>Equivalence</u>

A1. (a), (c), (e) equivalent; (g) <u>in</u>equivalent; (i) equivalent

A2. (a) (ii), (iii), (v), (vii)-(ix) are all valid, the rest invalid.

(b) Just the tautologies, which are of course all equivalent to each other.

<u>Expressive Completeness</u>

B1. (1) $\equiv ABC \vee \underline{A}BC \vee A\underline{B}C \vee \underline{ABC}$ (3) $\equiv \underline{A}BC \vee AB\underline{C} \vee \underline{A}B\underline{C} \vee \underline{ABC}$

B2. (a) $p \vee q \equiv \neg(p \leftrightarrow q)$

(c) Use the facts: $\{\wedge, \neg\}$ is expressively complete and $\neg A \equiv A \vee t$

B3. (a) Just use the fact that $p \vee q \equiv \neg p \to q$ and that $\{\neg, \vee\}$ are expressively complete.

B4. (a) To prove the hint: show that if p and q are statements in the letters A and B taking value t in at least one case where A and B have opposite truth values, then $p \to q$ has exactly the same property (i.e. it *too* takes value t in a case where A and B have opposite truth values). Using the hint, *any* statement using *just* A, B and $\to$ takes value t in a case where A and B have opposite values. But in such a case, $A \leftrightarrow B$ is false! So you could *never* express it using just A, B and $\to$.

(c) Use: if p(A, B), q(A, B) both are true in at least two cases, $p(A, B) \to q(A, B)$ has the same property. Then, since $A \wedge B$ only takes value t in one case, you're done!

B5. f: $ABC \vee \underline{A}BC \vee \underline{ABC}$ h: $ABC \vee A\underline{BC} \vee \underline{A}B\underline{C} \vee \underline{ABC}$

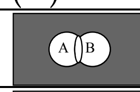
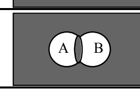
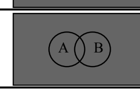
B6. $h(A, A, A) \equiv \neg A$, $h(A, h(A, A, A), h(C, C, C)) \equiv A \vee C$

B7. (a) $A\underline{B}C \vee A\underline{BC} \vee \underline{A}BC \vee \underline{AB}C$

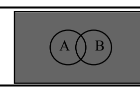
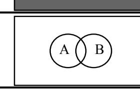
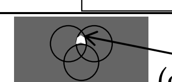
B8. (a) Since the set $\{\neg, \vee\}$ is expressively complete and those logical operators can be expressed in terms of $\{f, \rightarrow\}$ as: $\neg A \equiv A \rightarrow f$, $A \vee B \equiv (A \rightarrow B) \rightarrow B$

Binary Representations and Venn Diagrams

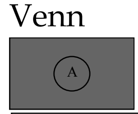
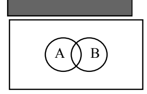
C1.

	(i)	(ii)	(iii)
(a)	$1 + A + B + A \cdot B$	contingent	
(c)	$1 + A + B$	contingent	
(e)	1	tautology	

C2.

	Binary	Venn
(a)	1	
(c)	0	
(e)	$1 + pq + pqr$	 (only the indicated space is empty)

C3.

	Binary	Venn
(a)	1	
(c)	0	
(e)	$r+p+pr+pq+pqr$	....shade everything except the region inside of q and outside

note: this = 0 exactly   of r!
when q=1 and r=0 so....

C4. (a) $A\underline{B}$; A+AB; shade only what's inside circle A and outside circle B.

CHAPTER III

Tree Test for Validity

A1. (a) valid (c) invalid; counterexample: A=t, B=C=D=E=f

(e) ¬E ∨ A, R → ¬A ∴ ¬A →(E ∨ R) invalid, counterexample: A=E=R=f

A2. (a) valid; (c) valid

A3. (i) valid (iii) invalid c.examples: A true, all other statement letters false

A4. [A, B, C]   ¬[A, B, C]   A * B   ¬(A * B)   (i) valid

[A, B, C] branches to: A, ¬B | ¬C

¬[A, B, C] branches to: ¬A, C | B, C

A * B branches to: ¬A | ¬B

¬(A * B) continues to: A, B

A5. (i) valid; (iii) invalid, c. examples: A=t, B=C=f

A6. Rules:

{A, B, C} branches to: A, B | A, C | ¬B, C

A*B branches to: ¬A | ¬B

A•B continues to: ¬A, ¬B

(i) valid; (iii) invalid

A7. (a) valid; (c) valid

A8. (a) valid; (c) valid

A9. (a) K → D   Invalid, one counterexample

$K \rightarrow S$ [Note that the argument fails simply because Dumb could

$\underline{S \rightarrow \neg D}$ be so dumb that he doesn't even know it!]

$\neg D$

A10.

S = scientists don't know what they are talking about

B = the sun will eventually burn out

E = Earth will become dark and cold

M = Mars is teeming with life

H = the human race will migrate to other planets

D = the human race will die out

Argument has H=E=B=true, S=M=D=false for a counterexample

A11. Invalid, exactly 4 counterexamples

<u>Further Applications of the Tree Method</u>

B1. (a) <u>un</u>satisfiable; (c) satisfiable, A=t, B=f

B2. (a) tautology; (c) tautology

B3. (i) contradiction

B4. (i) P & J are both Knights or both Knaves.

B5. (i), (iii) equivalent; (v) <u>in</u>equivalent

B6. (a) contingent; (c) contingent

B7. (i) Both knaves; (iii) I=knave, He=can't tell! (this one's tricky)

B8. (a) contingent; (c) contradiction

B9. Lancelot is a knave, Arthur a knight and Merlin a knight.

B10. (a), (b) and (e) are the tautologies, the others aren't.

B11. They're <u>all</u> inconsistent!

B12. (a) Both knights.

B13.

1. No information derivable.

3. M is a saint if and only if J is a sinner.

5. M is a saint.

7. M is a saint and J a sinner.

9. J is a sinner.

11. M and J are both saints.

13. M is a sinner and J a saint.

15. Inconsistent.

17. M and J are both saints.

19. M is a sinner and J and his brother are not both saints.

21. C is a sinner and the others aren't both saints.

23. C and L are saints and M a sinner.

25. M is a saint and the others are sinners.

B14. (a) Satisfiable in 4 ways:

A	B	C	D
f	t	f	f
f	f	f	f

f	f	f	t
t	t	f	f

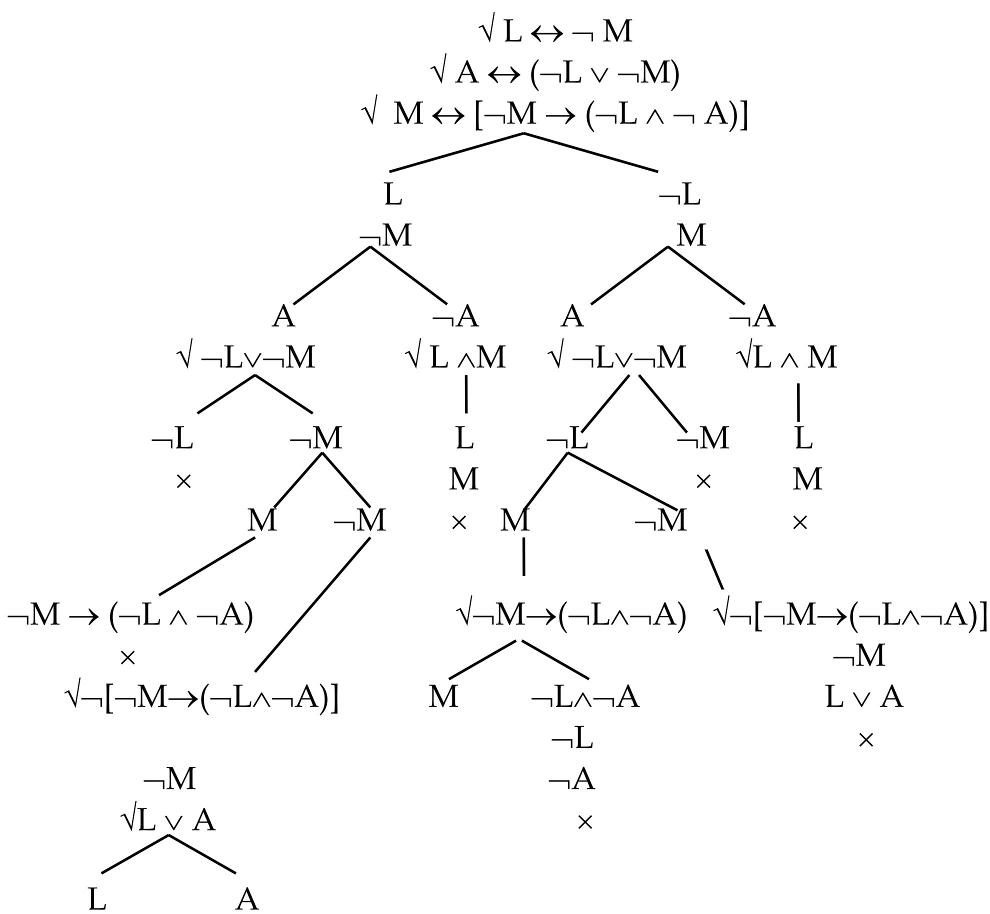
(c) Unsatisfiable

B15. (i) Non-tautologous; A=f, B=t only way to make false.

B16. (i) Satisfiable in one way:

A	B	C	D
f	f	f	t

B17. L=Lancelot a knight
A=Arthur a knight
M= Merlin a knight



From 3 open branches we see: A is true and M and L have opposite truth values.

B18. Guinevere *is* in Camelot today - Lancelot is *not* deceived!

B19. (a) Dean, Jerry, Stan are heroes, and Ollie is a scoundrel.

(c) One question you could ask is "Is it the case that you're a knight if and only if there is no buried treasure on the island?" It is easy to verify (with the tree method) that if Dean answers Yes, there's no gold on the island; and if he answers No, there is!

CHAPTER IV

Trees and Translations Involving Quantifiers

A1. (i) valid; (iii) valid; (v) invalid

A2. (a) $\forall x(Lx \rightarrow Nx), \neg\exists x(Vx \wedge Nx) \therefore \neg\exists x(Vx \wedge Lx)$, valid.

(c) $\exists x(Bxa \wedge \neg\exists yByx) \therefore \neg\exists Bax$, invalid.

A3. valid

A4. (i) Ec; (iii) $Sbc \vee Scb$; (v) $\exists x(Ex \wedge Wdx \wedge \forall y(Ey \rightarrow Syx))$

A5. (a) $\exists xPx \rightarrow Pu, \forall x(Lx \rightarrow Px), Ld \therefore Pu$; valid.

(c) $\forall x(Dx \rightarrow Cx) \therefore \forall x(\exists y(Dy \wedge Lxy) \rightarrow \exists z(Cz \wedge Lxz))$; valid.

A6. (i) $\forall xCx$ | $\underline{Cw \rightarrow G}$ | G — valid

(iii) $\forall w[\exists xLwx \rightarrow \forall zLzw]$ | $\underline{Lrj}$ | Liy — valid

A7. (i) $\{\forall x[Rx \rightarrow (Lx \wedge Vx)], \neg Lj \wedge Vj\}$ satisfiable

(iii) $\{\forall w[\exists xLwx \rightarrow \forall zLzw], Lyy, \neg Lym\}$ unsatisfiable

A8. (a) $\exists x\forall y\, Kxy$

(c) $\exists x\forall y[Kyy \rightarrow Kxy]$

(e) $\neg\exists x[\exists y(Kxy \wedge Kay) \wedge \forall y(Kya \rightarrow Kxy)]$

A9. (a) $\underline{\exists x[Hx \wedge \forall y[By \rightarrow Pxy]]}$

$\forall y[By \rightarrow \exists x(Hx \wedge Pxy)]$

valid

(c) $\forall x(Sx \rightarrow Ux)$

$\neg\exists x(Wx \wedge Ux)$

$\underline{\neg\exists x(\neg Sx \wedge Vx)}$

$\neg\exists x(Wx \wedge Vx)$

valid

A10. (a) Lmm; (c) $\neg\exists x Lxm$; (e) $Lmm \wedge \forall x(Lmx \rightarrow x=m)$; (g) $\forall x(\neg Lxm \rightarrow \neg Lmx)$;

(i) $\exists x(Lxm \wedge \forall y Lxy)$

A11. (a)$\forall x Trx$; (c) $\neg\exists x\forall y Txy$; (e) $\forall x\forall y((Trx \wedge Try) \rightarrow x=y)$; (g) $\exists x Txr \rightarrow Trr$;

(i) $\exists x(Txr \wedge \forall y(Tyr \rightarrow Txy))$

A12. (i) $\neg\forall x\forall y ((Cx \wedge Cy \wedge x \neq y) \rightarrow (\neg Txy \wedge \neg Tyx))$

$\underline{\neg\exists x(Cx \wedge Txx)}$

$\exists x\exists y (Cx \wedge Cy \wedge x \neq y)$ Valid.

A13. (i) equivalent

A14. $\exists! x Px$ means there's exactly one thing with property P. (i) valid; (iii) <u>in</u>valid

A15. (a) $\forall x(Cx \rightarrow Dtx)$

$\neg\exists x(Cx \wedge Dxt)$

$\underline{\quad Cs \quad}$

$\therefore \exists x(Dtx \wedge Dxt)$ NOT Valid.

(c) $\exists x(x < f \wedge Ex)$

$\underline{\forall x\forall y(x<y \rightarrow \exists z(x <z \wedge z <y))}$

$\therefore \exists x(x <f \wedge \exists y(y < x \wedge Ey))$

Valid.

A16. (a) $\exists x\forall y Lxy$; (c) $\exists x\forall y(Lyy \rightarrow Lxy)$; (e) $\neg\exists x(\exists y Lxy \wedge \exists z(Lxz \wedge \neg\exists w Lzw))$

A17. (a) $\forall x(Cx \rightarrow Lbx)$

$\neg\exists x(Cx \wedge Lxb)$

$\underline{\quad Cw \quad}$

$\therefore \exists x(Lbx \wedge \neg Lxb)$

Valid.

(c) $\forall x\forall y (\exists zLyz \rightarrow Lxy)$

$\underline{\quad Lrj \quad}$

$\therefore$ Lfw

Valid.

Interpretations and Counterexamples

B1. (i) $\underline{\exists x (Cx \rightarrow Tx)}$

$\exists x (Tx \wedge Cx)$ Not valid. Counterex: Domain = {1}, C= { }, T = { }

(iii) $\exists x\, Txr \rightarrow Tgr$

$\underline{Tgr \rightarrow \forall y\, Tyr}$

$\neg(\exists x\, Txr \wedge \exists x\, \neg Txr)$ Valid. No counterex.

B2. (a) False, because parallel lines don't intersect.

B3. For the knight case, you can take:

Domain={Arthur, Lancelot}, Knaves={Lancelot}, Knights={Arthur}

For the knave case, you can take:

Domain={Arthur, Lancelot}, Knaves={Lancelot, Arthur}, Knights={ }

B4. (i) false; (iii) true; (v) true

B5. (a) False, because the number 1 is odd but there are *no* positive even numbers less than it!

B6. (a) With Domain = {1} and Pxy taken to mean x=y, trivially satisfiable!

(c) <u>un</u>satisfiable

B7. (a) $\forall x(Fx \leftrightarrow \forall yHyx) \therefore \forall x\forall y((Fx \wedge Fy) \rightarrow Hxy)$, valid.

(c) $\forall x \forall y((Lx \wedge By) \rightarrow Txy)$, $\forall x \forall y(x \neq y \rightarrow (Lx \leftrightarrow \neg Ly))$, $\therefore$ $\exists x \exists y(x \neq y \wedge Txy)$ not valid. Counterex: Domain = {a}, L={ }, B={ }, T={ }

(always shoot for a simple counterexample by trial and error first!!!)

B8. (a) $\exists x(Sx \wedge \forall y(Cy \rightarrow y=x))$, Cb, Tb $\therefore$ $\forall x(Cx \rightarrow Tx)$, valid.

(c) $\forall x(\forall y(Osy \rightarrow Oxy) \rightarrow Osx)$ $\therefore$ $Oss \wedge \forall x(Osx \rightarrow x=s)$, not valid,

counterex: Domain ={s,a}, O ={(s,s), (a,s), (s,a)}

B9. (i) False, because shape a is not a triangle.

(iii) False, because g is neither a triangle nor square.

(v) False, because g has nothing left of it.

(vii) True ; (ix) True

B10. (1) $\forall x \exists y Ixy$, which is true because every statement implies <u>itself</u>!

(3) $\forall x \forall y(\forall z(Ixz \leftrightarrow Iyz) \rightarrow Exy)$, which is true because if x and y have the same implications, then since x implies itself, y must imply x too, and also (by the same argument) x must imply y, which means x and y must be equivalent.

(5) This says: 'Every statement fails to imply some statement', which is equivalent to saying: 'No statement implies all statements'. But contradictions do! So the stated claim is false.

(7) This says: 'Some statements are equivalent to anything they imply'— Yes, that's right: just consider any tautology!

B11. 1. False. To make it True, remove a's feather.

3. False. To make it True, increase a's and c's heights to the second line.

5. False. To make it True, give d's hat to b and remove d's feather.

B12. (a) false; (c) true; (e) true; (g) true; (i) true; (k) false

B13. (a) Case where true: Domain = {1}, R = { };

Case where false: Domain = {1}, R= {(1,1,1)}

*Note: These are obviously not the only possible answers!

(c) Case where true: Domain = {1,2}, P = {1,2}, Q = {1,2};

Case where false: Domain = {1}, P= { }, Q = { }

B14. (i) False; (iii) True; (v) True; (vii) True

B15. (i) false (iii) false (v) false (vii) false (ix) true

Functions

C1. (i) a=mfb; (iii) $fa=fb \wedge ma=mb \wedge a\neq b$; (v) $\exists x\,(a=fmx \vee a=ffx)$

C2. (a) a and b are siblings

(c) a is b's brother; (e) b is a's father

CHAPTER V

The Propositional Calculus

A1. Justification for the sequence: in initial set, Axiom 1, Axiom 1, Axiom 2, Modus Ponens, Modus Ponens, Axiom 9, Axiom 10, Modus Ponens, Axiom 1, Modus Ponens, in initial set, Modus Ponens.

So, Yes $\Rightarrow$ a legit deduction of $\neg\neg p \vdash p$. Simple deduction is: $\underline{\neg\neg p}$, $\underline{\neg\neg p \rightarrow} p$, $\underline{p}$!

In initial set Ax.10 MP

A2. 1. axiom 2; 2. axiom 1; 3. in initial set; 4. modus ponens; 5. axiom 1; 6. axiom 5; 7. modus ponens; 8. in initial set; 9. axiom 9; 10. modus ponens; 11. modus ponens; 12. axiom 10; 13 modus ponens; 14. You ain't foolin' me!

A3. For first sequence, justification is: in S, in S, Axiom 5, MP, MP; so the sequence is indeed a legitimate deduction from S.

A4. <u>tautology</u> goes with <u>theorem</u>

<u>unsatisfiable</u> goes with <u>formally inconsistent</u>

<u>valid argument</u> goes with <u>deduction</u>

<u>valid argument, no premises</u> goes with <u>proof</u>

A5. (i) $S \vdash p \Rightarrow S \models p$. Every deduction in the propositional calculus generates a valid argument!

(iii) To show that logic (at least propositional logic without quantifiers) can be completely captured through formal rules of symbol manipulation without any reference to the external (and potentially problematic) notion of 'truth' or 'meaning'.

A6. (a) false; (c) true; (e) false; (g) true; (i) false

A7. Because if $p \vdash r$, then clearly $p,q \vdash r$ for any q, from which it follows by the deduction theorem that $p \vdash (q \to r)$.

A8. (a) The Deduction theorem says: $S, p \vdash q$ if and only if $S \vdash p \to q$.

Assuming completeness and soundness, all we have to show is:

$$S, p \models q \text{ iff } S \models p \to q \quad (*)$$

because we could then argue as follows:

$$S, p \vdash q \Leftrightarrow S, p \models q \Leftrightarrow S \models p \to q \Leftrightarrow S \vdash p \to q.$$

Argument for (*). Assuming S, p $\models$ q, it follows that {S, p, ¬q} is unsatisfiable. So if all statements in S are true, p → q can't be false; because if it were, both p and ¬q would have to be true as well, contradicting the unsatisfiability of {S, p, ¬q}. So if all statements in S are true, so is p → q, which is just to say that S $\models$ p → q. Now we need to argue the other way around. If S $\models$ p → q, then there's never a case where all the statements in S are true and p → q is false, i.e. p true, ¬q true. So {S, p, ¬q} is unsatisfiable, which implies S, p $\models$ q.

A9. (i) If S $\vdash$ p→q then there is a legitimate deduction sequence of form S, …, p. Similarly, if p $\vdash$ q then a legitimate sequence p, …, q exists. So now just concatenate these 2 sequences to yield the following legitimate sequence: S, …, p, p, …, q which justifies S $\vdash$ q !

(iii) First note that p,q $\vdash$ p ∧ q which is justified by the sequence: p, q, p → (q → (p ∧ q)), q→ (p ∧ q), p ∧ q. Now suppose S $\vdash$ p, i.e. S, …, p is a legitimate deduction sequence, and S $\vdash$ q, i.e., S, …,q is legitimate. Since p,q, …, p ∧ q is legitimate (as we've just shown), concatenating these three sequences gives S, …, p, S, …, q, p, q,…, p ∧ q, which *also* is a legitimate deduction sequence, thus establishing S $\vdash$ (p∧q).

A10. (a) By completeness and (strengthened) soundness, all you have to argue is that:
∃q: S $\models$ q and S $\models$ ¬q if and only if ∀p: S $\models$ p

But that's clear: if the left-hand side is true, then S can't be satisfiable, which means that the right-hand side is true. (The argument from right to left is trivial.)