Measurable Cardinals

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Let κ be an infinite cardinal. A κ-complete nonprincipal ultrafilter, or, for short, a κ- *ultrafilter* on a set A is a (nonempty) family U of subsets of A satisfying (i) $S \subseteq U$ & $|S|$[1] $< \kappa \Rightarrow \bigcap S \in U$ (κ-*completeness*) (ii) $X \in U$ & $X \subseteq Y \subseteq A \Rightarrow Y \in U$, (iii) $\forall X \subseteq A\, [X \in U \text{ or } A - X \in U]$ (iv) $\{a\} \notin U$ for any $a \in A$. Notice that, if these conditions are satisfied, then every member of U has cardinality $\geq \kappa$, U contains the complement of every subset of A of cardinality $< \kappa$, and, for any $S \subseteq \mathbf{P}A$[2], if $|S| < \kappa$ and $\bigcup S \in U$, then $S \cap U \neq \varnothing$.

The cardinal κ is said to be *measurable* if $\kappa > \aleph_0$ and there exists a κ-ultrafilter on any set of cardinality κ, or, equivalently, on κ itself. Henceforth, we fix a measurable cardinal κ and a κ-ultrafilter U on κ.

Now let $\mathfrak{V} = \langle V, \in \rangle$ be the universe of sets and write V^κ for the collection of all functions with domain κ. Define the equivalence relation $\sim$ on V^κ by stipulating, for $f, g \in V^\kappa$, that

$$f \sim g \iff \{\xi < \kappa \colon f(\xi) = g(\xi)\} \in U.$$

For each $f \in V^\kappa$ let σ_f be the least rank of an element $g \in V^\kappa$ for which $f \sim g$ and define

$$f/U = \{g \in V^\kappa \colon \mathrm{rank}(g) = \sigma_f \;\&\; f \sim g\}.$$

Evidently $f/U = g/U \iff f \sim g$. Now define

$$V^\kappa/U = \{f/U \colon f \in V^\kappa\}.$$

And define the relation E on V^κ/U by

$$(f/U)E(g/U) \iff \{\xi < \kappa \colon f(\xi) \in g(\xi)\} \in U.$$

(Observe that

$$(f/U)E(g/U) \Rightarrow f/U \in \{h/U \colon h \in (\bigcup \mathrm{range}(g))^\kappa\}$$

so that $\{f/U \colon (f/U)E(g/U)\}$ is a *set*.)

The structure

$$\mathfrak{V}^\kappa/U = \langle V^\kappa/U, E \rangle$$

is then the *ultrapower* of $\mathfrak{V}$ over U. This being the case, we have

Łoś's Theorem for $\mathfrak{V}^\kappa/U$. Let $\varphi(v_0,\ldots, v_n)$ be a formula of the language of set theory and let $f_0,\ldots, f_n \in V^\kappa$. Then we have

$$\mathfrak{V}^\kappa/U \vDash \varphi(f_0/U,\ldots, f_n/U) \iff \{\xi < \kappa \colon \varphi(f_0(\xi)\ldots, f_n(\xi))\} \in U. \;\blacksquare$$

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[1] We write $|X|$ for the cardinality of a set X.

[2] For any set X, $\mathbf{P}X$ denotes the power set of X.

For each $x \in V$, let $\hat{x}$ be the map on κ with constant value κ, and define d: $V \to V^\kappa / U$ by

$$d(x) = \hat{x} / U.$$

Putting $\widehat{x_i}$ for f_i in Łoś's theorem, we get

$$\mathfrak{V}^\kappa / U \vDash \varphi(d(x_0),\ldots, d(x_n)) \Leftrightarrow \varphi(x_0,\ldots, x_n).$$

Thus d is an elementary embedding of $\mathfrak{V}$ into $\mathfrak{V}^\kappa / U$. In particular

$$\mathfrak{V}^\kappa / U \vDash \boldsymbol{ZFC}.$$

Next, we show that E is well-founded on V^κ / U. For suppose that $f_0, f_1,\ldots$ is a sequence of members of V^κ such that $(f_{n+1}/U)E(f_n/U)$ for all $n \in \omega$. Then

$$X_n = \{\xi < \kappa\colon f_{n+1}(\xi) \in f_n(\xi)\} \in U$$

for all n, so that $\bigcap_{n\in\omega} X_n \in U$ by κ-completeness of U. In particular $U \neq \emptyset$, so we may choose $\xi_0 \in \bigcap_{n\in\omega} X_n$. In that case $f_{n+1}(\xi_0) \in f_n(\xi_0)$ for all n, contradicting the well-foundedness of $\in$. Therefore is well-founded.

Accordingly $\mathfrak{V}^\kappa / U$ is isomorphic to a (unique) transitive $\in$-structure

$$\mathfrak{M} = \langle M, \in\rangle,$$

where the isomorphism e is given by

$$e(f/U) = \{e(g/U)\colon (g/U)E(f/U)\}.$$

Since d is an elementary embedding of $\mathfrak{V}$ into $\mathfrak{V}^\kappa / U$, the composite

$$j = e \circ d$$

is an elementary embedding of $\mathfrak{V}$ into $\mathfrak{M}$.

$$\begin{tikzcd}
\mathfrak{V} \arrow[r, "d"] \arrow[dr, "j"'] & \mathfrak{V}^\kappa/U \arrow[d, "e"] \\
 & \mathfrak{M}
\end{tikzcd}$$

We now note the following fact: for $\alpha < \kappa$, we have

(*) $$(f/U)Ed(\alpha) \Leftrightarrow \exists\beta{<}\alpha[f/U = d(\beta)].$$

For

$$\begin{aligned}
(f/U)Ed(\alpha) &\Leftrightarrow \{\xi < \kappa\colon f(\xi) \in \alpha\} \in U \\
&\Leftrightarrow \bigcup_{\beta<\alpha}\{\xi : f(\xi) = \beta\} \in U \\
&\Leftrightarrow (\exists\beta{<}\alpha)\{\xi : f(\xi) = \beta\} \in U \\
&\Leftrightarrow \exists\beta{<}\alpha[f/U = d(\beta)].
\end{aligned}$$

Lemma 1. $j(\xi) = \xi$ for all $\xi < \kappa$.

Proof. By induction on ξ. Suppose $\xi < \kappa$ and $(\forall\eta{<}\xi)\ j(\eta) = \eta$. Then

$j(\xi) = e(d(\xi)) = \{e(f/U)\colon (f/U)Ed(\xi)\} = \{ e(d(\eta))\colon \eta < \xi\}$ (by (*)) $= \{j(\eta)\colon \eta < \xi\} = \xi$. ■

Now let ORD be the class of all ordinals.

Lemma 2.

(i) $j|\text{ORD}$ is an order preserving map of ORD into itself.

(ii) $\alpha \leq j\alpha$ for all α.

(iii) $\text{ORD} \subseteq M$.

Proof. (i) If $\alpha \in \text{ORD}$, then since j is elementary,

$$\mathfrak{M} \vDash j\alpha \textit{ is an ordinal},$$

so $j\alpha$ really is an ordinal. Similarly, $\alpha < \beta \Rightarrow \mathfrak{M} \vDash j\alpha < j\beta \Rightarrow j\alpha < j\beta$.

(ii) follows immediately from **(i)**.

(iii) We have $\alpha \leq j\alpha \in M$, so $\alpha \in M$ by transitivity of M. ■

Lemma 3. $\kappa < j\kappa$.

Proof. Let id be the identity map on κ. By Łoś's theorem, we have

$$\mathfrak{V}^{\kappa}/U \vDash \mathrm{id}/U \textit{ is an ordinal},$$

whence

$$\mathfrak{M} \vDash e(\mathrm{id}/U) \textit{ is an ordinal},$$

And therefore $\alpha = e(\mathrm{id}/U)$ actually is an ordinal. If $\xi < \kappa$, then

$$\{\eta < \kappa\colon \hat{\xi}(\eta) < \mathrm{id}(\eta)\} \in U,$$

so that $d(\xi)E(\mathrm{id}/U)$, whence $\xi = j\xi < \alpha$. Hence $\kappa \leq \alpha$. On the other hand, we have

$$\{\eta < \kappa\colon \mathrm{id}(\eta) < \hat{\kappa}(\eta)\} \in U,$$

so that by Łoś's theorem $(\mathrm{id}/U)Ed(\kappa)$, whence $\alpha < j\kappa$. The result follows. ■

We have therefore proved

Theorem 1. Let $\kappa > \aleph_0$ be a measurable cardinal. Then there is an elementary embedding j of the universe of sets $\mathfrak{V}$ into a transitive $\in$-structure $\mathfrak{M} = \langle M, \in\rangle$ (containing ORD) such that $j\kappa > \kappa$ and $j\xi = \xi$ for all $\xi < \kappa$. ■

The converse of this result also holds.

Theorem 2. Suppose that $\kappa > \aleph_0$ and there is an elementary embedding j of $\mathfrak{V}$ into a transitive $\in$-structure $\mathfrak{M} = \langle M, \in\rangle$ such that $j\kappa > \kappa$ and $j\xi = \xi$ for all $\xi < \kappa$. Then κ is measurable. In fact, the set

$$U_j = \{x \subseteq \kappa\colon \kappa \in jx\}$$

is a κ-ultrafilter on κ.

Proof. (i) *U_j is "ultra".* If $x \subseteq \kappa$, then since j is elementary and M is transitive, we have: $j(\kappa - x) = j\kappa - jx$. Thus since $\kappa \in j$, either $\kappa \in jx$ or $\kappa \in j(\kappa - x)$. Hence $x \in U_j$ or $\kappa - x \in U_j$.

(ii) $x \in U_j$ & $x \subseteq y \subseteq \kappa \Rightarrow y \in U_j$. Similar to **(i)**.

(iii) *U_j is nonprincipal.* If $\xi < \kappa$, then

$$\forall\eta<\kappa[\eta \in \{\xi\} \Leftrightarrow \eta = \xi],$$

so

$$\forall\eta<j\kappa[\eta \in j\{\xi\} \Leftrightarrow \eta = j\xi = \xi].$$

Hence $j\{\xi\} = \{\xi\}$ and consequently $\{\xi\} \notin U_j$.

(iv) *U_j is κ-complete.* Suppose $\alpha < \kappa$ and $g\colon \alpha \to U_j$. We want to show that $\bigcap_{\xi<\alpha} g(\xi) \in U_j$. For this it suffices to establish

$$(*) \qquad j\Big(\bigcap_{\xi<\alpha} g(\xi)\Big) = \bigcap_{\xi<\alpha} j(g(\xi)).$$

Writing $b = \bigcap_{\xi<\alpha} g(\xi)$, we have

$$\forall\eta[\eta \in b \Leftrightarrow (\forall\xi<\alpha)\eta \in g(\xi)],$$

so

$$\forall\eta[\eta \in jb \Leftrightarrow (\forall\xi<\alpha)\eta \in (jg)(\xi)].$$

Since $j\xi = \xi$, we have $(j(g)(\xi) = (jg)(j\xi) = j(g(\xi))$, so that

$$\forall\eta[\eta \in jb \Leftrightarrow (\forall\xi<\alpha)\eta \in j(g(\xi))],$$

which is (*). ■

Theorem 3. If κ is measurable, it is inaccessible.

Proof. Suppose κ measurable, and fix j, $\mathfrak{M}$ to satisfy the conditions of Theorem 1.

Were κ singular, there would exist $\alpha < \kappa$ and a map $f : \alpha \to \kappa$ with range(f) cofinal in κ. In that case

$$\mathfrak{M} \models jf\colon \alpha \to j\kappa,$$

and since $\forall\xi<\kappa\exists\eta<\alpha(\xi \leq f(\eta))$,

$$\mathfrak{M} \models \forall\xi<j\kappa\exists\eta<\alpha(\xi \leq (jf)(\eta)).$$

Since $\kappa < j\kappa$, there is accordingly $\eta < \alpha$ for which $\kappa \leq (jf)(\eta)$. But $f(\eta) < \kappa$, so

$$(jf)(\eta) = (jf)(j\eta) = j(f(\eta)) = f(\eta) < \kappa.$$

Contradiction. So κ is regular.

Suppose now that $\alpha < \kappa$ and $\kappa \leq 2^\alpha$. Let f be an injection of κ into **P**α; then jf is an injection of $j\kappa$ into **P**α. We show that:

(i) $x = jx$ for all $x \subseteq \alpha$. For

$$\xi \in x \Rightarrow \xi = j\xi \in jx$$

and

$$\xi \in jx \Rightarrow \xi \in jx \subseteq j\alpha = \alpha \Rightarrow j\xi = \xi \in jx \Rightarrow \xi \in x.$$

(ii) range(jf) ⊆ range(f). For if $x \in$ range (jf), then $x \subseteq \alpha$ and $x = jx$.

Also $\mathfrak{M} \vDash \exists\xi[\langle\xi, jx\rangle \in jf]$, so that $\exists\xi[\langle\xi, x\rangle \in f]$, whence $\exists\xi[f\xi = x]$, and so $x \in$ range (f).

Finally, if $\xi < \kappa$, then, using **(i),**

$$(jf)(\xi) = (jf)(j\xi) = j(f(\xi)) = f(\xi).$$

From this it follows, using the injectivity of jf, that $(jf)(\kappa) \notin$ range(f), contradicting **(ii).** ■

A κ-ultrafilter U on a measurable cardinal κ is *normal* if whenever $f \in \kappa^\kappa$ satisfies then there is $\alpha < \kappa$ for which

$$\{\xi<\kappa: f(\xi) = \alpha\} \in U.$$

Łoś's theorem immediately yields

Lemma 4. U is normal iff $e(\mathrm{id}/U) = \kappa$. ■

Lemma 5. If κ is measurable, there is a normal ultrafilter on κ.

Proof. Let j, $\mathfrak{M}$ satisfy the conditions of Thm. 2 We show that U_j, as defined in the proof of that theorem, is a normal ultrafilter on κ.

Suppose $f \in \kappa^\kappa$ and

$$A = \{\xi<\kappa: f(\xi) < \xi\} \in U_j.$$

Then $A \in jA$. Also, we have

$$\forall\xi<\kappa[\xi \in A \Leftrightarrow f(\xi) < \xi\},$$

and so

$$\forall\xi<j\kappa[\xi \in jA \Leftrightarrow (jf)(\xi) < \xi\}.$$

Since $\kappa < j\kappa$ and $\kappa \in jA$, it follows that $(jf)(\kappa) < \kappa$. Hence there is $\beta < \kappa$ for which

(*) $$(jf)(\kappa) = \beta.$$

Putting $B = \{\xi<\kappa: f(\xi) = \beta\}$, then, as before, since $j\beta = \beta$, we have

$$\forall\xi<j\kappa[\xi \in jB \Leftrightarrow (jf)(\xi) = \beta\};$$

hence, by (*), $\kappa \in jB$, so $B \in U_j$. ■

Lemma 6. Let U be a normal ultrafilter on the measurable cardinal κ, and let $\mathfrak{M}$ be the transitive $\in$-structure isomorphic to $\mathfrak{V}^\kappa/U$. Then, for any formula $\varphi(v_0)$ of the language of set theory,

$$\mathfrak{M} \vDash \varphi[\kappa] \;\Leftrightarrow\; \{\xi<\kappa: \varphi(\xi)\} \in U.$$

Proof. By Lemma 4 and Łoś's theorem, we have

$$\begin{aligned}
&\mathfrak{M} \vDash \varphi[\kappa] \\
&\Leftrightarrow \mathfrak{M} \vDash \varphi[e(\mathrm{id}/U)] \\
&\Leftrightarrow \mathfrak{V}^\kappa/U \vDash \varphi[\mathrm{id}/U] \\
&\Leftrightarrow \{\xi<\kappa: \varphi(\mathrm{id}(\xi))\} \in U \\
&\Leftrightarrow \{\xi<\kappa: \varphi(\xi)\} \in U.
\end{aligned}$$ ■

Theorem 4. If κ is measurable and U is a normal ultrafilter on κ, then

$$\{\xi<\kappa\colon \xi \text{ is inaccessible}\} \in U.$$

Proof. Write $In(\xi)$ for "ξ is inaccessible". We know that $In(\kappa)$ by Theorem 3, so $\mathfrak{M} \models In[\kappa\}$. The result now follows from Lemma 6. ■

Lemma 7. Let κ be measurable, U a normal ultrafilter on κ, and let $\mathfrak{M} = \langle M, \in\rangle$ be the transitive $\in$- structure isomorphic to $\mathfrak{V}^\kappa/U$. Then

$$\mathbf{P}\kappa \in M.$$

Proof. It is enough to show that $\mathbf{P}\kappa \subseteq M$. For then, since $\mathfrak{M} \models$ ***ZFC,*** we have $\mathbf{P}^{(\mathfrak{M})}\kappa \in M$ and $\mathbf{P}^{(\mathfrak{M})}\kappa = \mathbf{P}\kappa \cap M = \mathbf{P}\kappa$.

As before, we let $d : \mathfrak{V}^\kappa \prec \mathfrak{V}^\kappa/U$ and e: $\mathfrak{V}^\kappa/U \cong \mathfrak{M}$. Let $a \in \mathbf{P}\kappa$; we show that $a \in M$. To do this, we define $f \in V^\kappa$ by

$$f(\xi) = a \cap \xi$$

for $\xi < \kappa$, and prove that

$$a = e(f/U) \in M.$$

To do *this* it suffices to show:

(*) $$(g/U)E(f/U) \Leftrightarrow \exists\alpha\in a[g/U = d(\alpha)].$$

One direction is easy: indeed if $\alpha \in a$, then

$$\{\xi<\kappa\colon \hat{\alpha}(\xi) \in f(\xi)\} \in U,$$

whence

$$d(\alpha) = \hat{\alpha}/U \;\; E \;\; f/U.$$

Conversely, suppose that $(g/U)E(f/U)$. Then

(§) $$\{\xi\colon g(\xi) \in a \cap \xi\} \in U,$$

so that

$$\{\xi\colon g(\xi) < \xi\} \in U.$$

Since U is normal, there is $\alpha < \kappa$ such that

(§§) $$\{\xi\colon g(\xi) = \alpha\} \in U,$$

whence

$$g/U = d(\alpha).$$

But

$$\{\xi\colon g(\xi) \in a\} \cap \{\xi\colon g(\xi) = \alpha\} \subseteq \{\xi\colon \alpha \in a\},$$

and, by (§) and (§§), the intersection on the left of "$\subseteq$" is a member of U. So therefore is the set on the right, which means that $\alpha \in a$. This proves (*), and the Lemma. ■

Theorem 5. Let κ be measurable, and let U be a normal ultrafilter on κ. If

$$\{\lambda<\kappa\colon 2^\lambda = \lambda^+\} \in U,$$

then

$$2^\kappa = \kappa^+.$$

Thus, if $2^\kappa > \kappa^+$, then

$$|\{\lambda<\kappa\colon 2^\lambda > \lambda^+\}| = \kappa.$$

Proof. Let $\mathfrak{M} = \langle M, \in\rangle$ be the transitive $\in$-structure isomorphic to $\mathfrak{V}^\kappa/U$. Suppose now that

$$\{\lambda<\kappa\colon 2^\lambda = \lambda^+\} \in U.$$

Then by Lemma 6,

$$\mathfrak{M} \vDash 2^\kappa = \kappa^+.$$

That is,

$$\mathfrak{M} \vDash \exists f\,[f : \kappa^+ \xrightarrow{onto} \mathbf{P}\kappa],$$

and hence

(*) $$\exists f\,[f : (\kappa^+)^{(\mathfrak{M})} \xrightarrow{onto} \mathbf{P}^{(\mathfrak{M})}\kappa],$$

Now, by Lemma 7, $\mathbf{P}\kappa \in M$, so $\mathbf{P}^{(\mathfrak{M})}\kappa = \mathbf{P}\kappa$. Therefore, by (*)

(**) $$|\mathbf{P}\kappa| \le |(\kappa^+)^{(\mathfrak{M})}|.$$

But clearly $(\kappa^+)^{(\mathfrak{M})} \le \kappa^+$, so $|(\kappa^+)^{(\mathfrak{M})}| \le \kappa^+$, and hence, by (**), $|\mathbf{P}\kappa| \le \kappa^+$. Therefore $2^\kappa = \kappa^+$. ■

Theorem 6. (Scott) If there is a measurable cardinal, then $V \neq L$.

Proof. Suppose that a measurable cardinal exists and let κ_0 be the least one. Let U be a κ-ultrafilter on κ, let $\mathfrak{M} = \langle M, \in\rangle$ be the transitive $\in$-structure isomorphic to $\mathfrak{V}^\kappa/U$, and let $j : \mathfrak{M} \to \mathfrak{V}^\kappa/U$ be the associted elementary embedding. If $V = L$, then then $M = V$ since $\mathfrak{M}$ is a transitive model of ***ZFC*** containing ORD. (Accordingly j is an elementary self-embedding of $\mathfrak{V}$.) Let $\varphi(x)$ be the formula expressing: *x is the least measurable cardinal.* Then we have $\varphi(\kappa_0)$, so that

$$\mathfrak{M} \vDash \varphi[j\kappa_0].$$

Since $M = V$, it follows that $\varphi(j\kappa_0)$. Therefore $j\kappa_0$ would itself be the least measurable cardinal, contradicting the fact (Lemma 3) that $\kappa_0 < j\kappa_0$. ■

Corollary. If κ is measurable and U is a κ-ultrafilter on κ, then $U \notin L$.

Proof. Suppose $U \in L$. Then, since $\kappa \in L$,

$$\langle L, \in\rangle \vDash \kappa \textit{ is measurable.}$$

But, by Theorem 6,

$$\textbf{\textit{ZFC}} \vdash \exists\kappa(\kappa \textit{ is measurable}) \to V \neq L.$$

Since $\langle L, \in\rangle \vDash$ ***ZFC***, it follows that

$$\langle L, \in\rangle \vDash V \neq L,$$

a contradiction. ■

For each set X write $\mathbf{Fin}(X)$ for the collection of all finite subsets of X, and, for each $n \in \omega$, $X^{[n]}$ for the collection of all n-element subsets of X.

A cardinal κ is a *Ramsey* cardinal if for each set I with $|I| < \kappa$ and each $f: \mathbf{Fin}(\kappa) \to I$ there is a subset $Z \subseteq \kappa$ such that $|Z| = \kappa$ and $|f[Z^{[n]}]| = 1$ for all $n \in \omega$. Under these conditions Z is said to be *homogeneous* for f. Equivalently κ is Ramsey if for each partition $\{C_i: i \in I\}$ of $\mathbf{Fin}(\kappa)$ with $|I| < \kappa$ there is a subset $Z \subseteq \kappa$ with $|Z| = \kappa$ and a sequence $i_1, i_2, \ldots \in I$ such that $Z^{[n]} \subseteq C_{i_n}$ for $I = 1,2,\ldots$.

Theorem 8. Each measurable cardinal is Ramsey.

Proof. Let $\kappa > \aleph_0$ be measurable and let U be a κ-ultrafilter over κ. Now suppose $|I| < \kappa$, $f: \mathbf{Fin}(\kappa) \to I$, and $X \in \kappa^{[n]}$. Define

$$A_i = \{y: f(X \cup \{y\}) = i\}$$

for each $i \in I$. Then $\{A_i: i \in I\}$ forms a partition of κ and consequently exactly one of the A_i is in U. We put

$$f^*(X) = \text{that } i \text{ for which } A_i \in U.$$

Thus $f^*: \mathbf{Fin}(\kappa) \to I$. We now define $f_0 = f, f_1 = f_0^*, \ldots, f_{n+1} = f_n^*, \ldots$. Let

$$S = (Z \subseteq \kappa: \forall n \forall X[X \subseteq Z \wedge |X| = n \Rightarrow \forall m[f_m(X) = f_{m+n}(\varnothing)]]\}.$$

It is easy to see that S is closed under unions of chains, so it has a maximal member Z_0. It is clear that each member of S is homogeneous for f, so to prove the Theorem it suffices to show that $|Z_0| = \kappa$.

Suppose on the contrary that $|Z_0| < \kappa$. We will show that Z_0 can be extended to a larger member of S, contradicting the former's maximality.

Let $X \in \mathbf{Fin}(\kappa)$ and $|X| = n - 1$. Then for $m \in \omega$ we have

$$f_m^*(X) = f_{m+1}(X) = f_{m+n}(\varnothing),$$

and so, by definition of f_m^*,

$$D(X, m) = \{y: f_m(X \cup \{y\}) = f_{m+n}(\varnothing)\} \in U.$$

Hence

$$D(X) = \bigcap_{m\in\omega} D(X,m) \in U,$$

and, since $|Z_0| < \kappa$,

$$D_{Z_0} = \bigcap_{X\in\mathbf{Fin}(\kappa)} D(X) \in U.$$

Clearly, if $y \in D_{Z_0}$, $X \subseteq Z_0$ and $|X| = n - 1$, we have for all $m \in \omega$

$$f_m(X \cup \{y\}) = f_{m+n}(\varnothing),$$

so that $Z_0 \cup \{y\} \in S$ for all $y \in D_{Z_0}$. Since $D_{Z_0} \in U$, $|D_{Z_0}| = \kappa$ and so there is $y_0 \in D_{Z_0} - Z_0$. Hence $Z_0 \cup \{y_0\} \in S$ is the required proper extension of Z_0. ■

Now let $\mathfrak{A}$ be a structure and let $\mathscr{L}_{\mathfrak{A}}$ be the language for $\mathfrak{A}$. Let $\mu(\mathfrak{A})$ be the number of symbols in $\mathscr{L}_{\mathfrak{A}}$. We shall say that $\mathfrak{A}$ is *good* if whenever $\mathfrak{B}$ is a substructure of $\mathfrak{A}$ closed under the $\mathfrak{A}$-denotations of all $\mathscr{L}$-terms, then $\mathfrak{B} \prec \mathfrak{A}$.

Fact. Each structure $\mathfrak{A}$ has a good expansion $\mathfrak{B}$ for which $\mu(\mathfrak{A}) = \mu(\mathfrak{B})$. ($\mathfrak{B}$ may be obtained by expanding $\mathfrak{A}$ to a Skolem structure.)

Lemma 8. Suppose that κ is measurable . Let $\mathfrak{A} = \langle A, R, \ldots\rangle$ be a structure with $\mu(\mathfrak{A}) = \aleph_0$, $R \subseteq A$, $|A| = \kappa$ and $|R| < \kappa$. Then $\mathfrak{A}$ has an elementary substructure $\mathfrak{B} = \langle B, S, \ldots\rangle$ with $|B| = \kappa$ and $|S| \leq \aleph_0$.

Proof. We may assume without loss of generality that $\mathfrak{A}$ is good. Let $|R| = \beta < \kappa$. Choose a linear ordering $<$ of A and an element $a \in A - R$. Let T be the set of all $\mathscr{L}$– terms, and let

$$I = (R \cup \{a\})^T.$$

Since $|T| = \aleph_0$, $|R| = \beta$, and κ is strongly inaccessible, we have $|I| = \beta^{\aleph_0} < \kappa$.

Now define a partition $\{C_i \colon i \in I\}$ of **Fin**(A) as follows. Call a term $t \in T$ an *n-term* if all its free variables are among $v_0,\ldots,v_n$, so enabling t to be written as $t(v_0,\ldots,v_n)$. Let

$$x = \{a_0,\ldots,a_{n-1}\} \in \mathbf{Fin}(A)$$

with $a_0 <\ldots < a_{n-1}$. Then we put $x \in C_i$ where $i \in I$ is given by:

$i(t) = a$ if t is not an n-term *or* t is an n-term and $t^{\mathfrak{A}}(a_0,\ldots,a_{n-1}) \notin R$
$i(t) = u$ if t is an n- term and $t^{\mathfrak{A}}(a_0,\ldots,a_{n-1}) \notin R$.

Since κ is a Ramsey cardinal, there is a subset $X \subseteq A$ of cardinality κ and elements $i_1, i_2,\ldots \in I$ such that

$$X^{[n]} \subseteq C_{i_n}\text{, } n = 1, 2, \ldots$$

Let B be the closure of X under the denotations in $\mathfrak{A}$ of all $\mathscr{L}$–terms, and let $\mathfrak{B} = \langle B, S, \ldots\rangle$ be the restriction of $\mathfrak{A}$ to $\mathfrak{B}$. Then $|B| = \kappa$ and since $\mathfrak{A}$ is good, $\mathfrak{B} \prec \mathfrak{A}$. It remains to show that $|S| \leq \aleph_0$.

Now each $b \in S$ is of the form

$$b = t^{\mathfrak{A}}(x_0,\ldots,x_{n-1})$$

for some $t \in T$ and some $x_0 <\ldots< x_{n-1} \in X$. Since

$$\{x_0,\ldots,x_{n-1}\} \in X^{[n]} \subseteq C_{i_n},$$

we have $i_n(t) = b$. Hence

$$b \in \bigcup_{n\in\omega} range(i_n)$$

and so

$$S \subseteq \bigcup_{n\in\omega} range(i_n).$$

Thus $|S| \leq \aleph_0.|T| = \aleph_0$. ■

We conclude with

Theorem 8. (Gaifman-Rowbottom). If a measurable cardinal exists, then $\mathbf{P}\omega \cap L$ is countable, in particular, there are only countably many constructible real numbers.

Proof. Suppose given a measurable cardinal κ. Let $R = \mathbf{P}\omega \cap L$; then $R \subseteq L_\kappa$. Consider the structure $\mathfrak{A} = \langle L_\kappa, \in, R\rangle$. We have $|L_\kappa| = \kappa$, and $|R| \leq 2^{\aleph_0} < \kappa$. Hence, by Lemma 8, there is a structure $\mathfrak{B} = \langle B, \in, S\rangle$ such that $\mathfrak{B} \prec \mathfrak{A}$, $|B| = \kappa$ and $|S| \leq \aleph_0$. Then $\langle B, \in\rangle \equiv \langle L_\kappa, \in\rangle$, and so there is a unique ξ for which there is an isomorphism f of $\langle B, \in\rangle$ onto $\langle L_\kappa, \in\rangle$. Now we have

$$|\xi| = |L_\xi| = |B| = \kappa,$$

whence $\xi \geq \kappa$. But every ordinal of $\langle B, \in\rangle$ is an ordinal of $\langle L_\kappa, \in\rangle$ (the two structures being isomorphic), so the ordinals of the former have order type $\leq \kappa$. It follows that $\xi \leq \kappa$, so that $\xi = \kappa$, whence

$$f : \langle B, \in\rangle \cong \langle L_\kappa, \in\rangle.$$

Now let P be a unary predicate symbol; take $P^{\mathfrak{A}} = R$ and $P^{\mathfrak{B}} = S$. Since the formula $x \in \omega$ is absolute, we have

$$\mathfrak{A} \vDash \forall x[P(x) \leftrightarrow \forall y(y \in x \rightarrow y \in \omega)],$$

whence

$$\mathfrak{B} \vDash \forall x[P(x) \leftrightarrow \forall y(y \in x \rightarrow y \in \omega)].$$

Therefore, if $b \in b$, we have

$$\begin{aligned} b \in S &\Leftrightarrow \mathfrak{B} \vDash \forall y(y \in b \rightarrow y \in \omega)] \\ &\Leftrightarrow \langle L_\kappa, \in\rangle \vDash \forall y(y \in f(b) \rightarrow y \in \omega)] \\ &\Leftrightarrow f(b) \in R. \end{aligned}$$

Accordingly f carries S onto R, whence

$$|\mathbf{P}\omega \cap L| = |R| \leq |S| \leq \aleph_0$$

as required. ■