CHAPTER 10

THE CONTINUOUS AND THE DISCRETE

THE RELATIONSHIP BETWEEN THE IDEAS of *continuity* and *discreteness* has played no less important a role in the development of mathematics than it has in science and philosophy. Continuous entities are characterized by the fact that they can be *divided indefinitely* without altering their essential nature. So, for instance, the water in a bucket may be continually halved and yet remain water[1]. Discrete entities, on the other hand, typically cannot be divided without effecting a change in their nature: half a wheel is plainly no longer a wheel. Thus we have two contrasting properties: on the one hand, the property of being indivisible, separate or discrete, and, on the other, the property of being indefinitely divisible and continuous although not actually divided into parts.

Now one and the same object can, in a sense, possess both of these properties. For example, if the wheel is regarded simply as a piece of matter, it remains so on being divided in half. In other words, the wheel regarded as a wheel is discrete, but regarded as a piece of matter, it is continuous. From examples such as these we see that continuity and discreteness are complementary attributes originating through the mind's ability to perform acts of abstraction, the one arising by abstracting an object's divisibility and the other its self-identity.

In mathematics the concept of whole number provides an embodiment of the concept of pure discreteness, that is, of the idea of a collection of separate individual objects, all of whose properties—apart from their distinctness—have been refined away. The basic mathematical representation of the idea of continuity, on the other hand, is the geometric figure, and more particularly the straight line. Continuity and discreteness are united in the process of measurement, in which the continuous is expressed in terms of separate units, that is, numbers. But these separate units are unequal to the task of measuring in general, making necessary the introduction of fractional parts of the individual unit. In this way fractions issue from the interaction between the continuous and the discrete.

A most striking example of this interaction—amounting, one might say, to a collision—is the Pythagorean discovery of incommensurable magnitudes. Here the

[1]For the purposes of argument we are ignoring the atomic nature of matter which has been established by modern physics.

realm of continuous geometric magnitudes resisted the Pythagorean attempt to reduce it to the discrete form of pure number. The "proportions" invented by Eudoxus to resolve the problem were in essence an extension of the idea of number—i.e., of the discrete—adequate to the task of expressing the relations between continuous magnitudes. Many centuries later, in the modern era, the concept of real number finally crystallized as a complete solution to the problem of representing continuous magnitudes as numbers.

The opposition between the continuous and the discrete also arises in physicists' account of the nature of *matter,* or *substance*. An early instance of this opposition—perhaps the first—in physics occurred in Greece in the third century B.C. with the emergence of two rival physical theories, each of which became the basis of a fully elaborated physical doctrine. One is the *atomic theory,* due to *Leucippus* (fl. 450 B.C.) and *Democritus* (b. *c*.470 B.C.) The other—the *continuum theory*—is the creation of the *Stoic* school of philosophy and is associated with the names of *Zeno of Cition* (fl. 250 B.C.) and *Chrysippus* (280–206 B.C.).

The continuum of Stoic philosophy is an infinitely divisible continuous substance which was conceived as providing the ultimate foundation for all natural phenomena. In particular the Stoics held that *space* is everywhere occupied by a continuous invisible substance which they called *pneuma* (Greek: "breath"). This pneuma—which was regarded as a kind of synthesis of air and fire, two of the four basic elements, the others being earth and water—was conceived as being an elastic medium through which impulses are transmitted by wave motion. All physical occurrences were viewed as being linked through tensile forces in the pneuma, and matter itself was held to derive its qualities form the "binding" properties of the pneuma it contains.

The atomists, on the other hand, asserted that material form results from the arrangement of the atoms—the ultimate building blocks—to be found in all matter, that the sole form of motion is the motion of individual atoms, and that physical change can only occur through the mutual impact of atoms.

A major problem encountered by the Stoic philosophers was that of the nature of *mixture*, and, in particular, the problem of explaining how the pneuma mixes with material substances so as to "bind them together". The atomists, with their granular conception of matter, did not encounter any difficulty here, since they could regard the mixture of two substances as an amalgam of their constituent atoms into a kind of lattice or mosaic. But the Stoics, who regarded matter as continuous, had difficulty with the notion of mixture. For in order to mix fully two continuous substances, they would either have to interpenetrate in some mysterious way, or, failing that, they would each have to be subjected to an infinite division into infinitesimally small elements which would then have to be arranged, like finite atoms, into some kind of discrete pattern.

This controversy over the nature of mixture shows that the problem of continuity is intimately connected with the problems of infinite divisibility and of the infinitesimally small. The mixing of particles of finite size, no matter how small they may be, presents no difficulties. But this is no longer the case when we are dealing with a continuum, whose parts can be divided *ad infinitum*. Thus the Stoic philosophers were confronted with what was at bottom a *mathematical* problem.

In fact, the problem of infinite divisibility had already been posed in a dramatic but subtle way more than a century before the rise of the Stoic school, by *Zeno of Elea* (fl. 450 B.C.), a pupil of the philosopher *Parmenides* (fl. 500 B.C.), who taught that the universe was a static unchanging unity. Zeno's arguments take the form of *paradoxes* which are collectively designed to discredit the belief in motion, and so in any notion of change. We consider, in modern formulation, three of these paradoxes, which are, perhaps, the most famous illustrations of the opposition between the continuous and the discrete. The first two of these, both of which rest on the assumption that space and time are continuous, purport to show that under these conditions continuous motion engenders, *per impossibile,* an actual infinity.

The first paradox, that of the *Dichotomy*, goes as follows. Before a body in motion can reach a given point, it must first traverse half of the distance; before it can traverse half of the distance, it must traverse one quarter; and so on *ad infinitum*. So, for a body to pass from one point to another, it must traverse an infinite number of divisions. But an infinite number of divisions cannot be traversed in a finite time, and so the goal cannot be reached.

The second paradox, that of *Achilles and the Tortoise*, is the best known. Achilles and a tortoise run a race, with the latter enjoying a head start. Zeno asserts that no matter how fleet of foot Achilles may be, he will never overtake the tortoise. For, while Achilles traverses the distance from his starting-point to that of the tortoise, the tortoise advances a certain distance, and while Achilles traverses this distance, the tortoise makes a further advance, and so on *ad infinitum.* Consequently Achilles will run *ad infinitum* without overtaking the tortoise.

This second paradox is formulated in terms of two bodies, but it has a variant involving, like the *Dichotomy,* just one. To reach a given point, a body in motion must first traverse half of the distance, then half of what remains, half of this latter, and so on *ad infinitum*, and again the goal can never be reached. This version of the *Achilles* exhibits a pleasing symmetry with the *Dichotomy*. For the former purports to show that a motion, once started, can never stop; the latter, that a motion, once stopped, can never have started.

The third paradox, that of the *Arrow*, rests on the assumption of the *discreteness of time*. Here we consider an arrow flying through the air. Since time has been assumed discrete we may "freeze" the arrow's motion at an indivisible instant of time. For it to move during this instant, time would have to pass, but this would mean that the instant contains still smaller units of time, contradicting the indivisibility of the instant. So at this instant of time the arrow is at rest; since the instant chosen was arbitrary, the arrow is at rest at any instant. In other words, it is always at rest, and so motion does not occur.

Let us examine these paradoxes more closely. In the case of the *Dichotomy*, we may simplify the presentation by assuming that the body is to traverse a unit spatial interval—a mile, say—in unit time—a minute, say. To accomplish this, the body must first traverse half the interval in half the time, before this one-quarter of the interval in one-quarter of the time, etc. In general, for every subinterval of length $1/2^n$ (n = 1, 2, 3,…), the body must first traverse half thereof, i.e. the subinterval of length $1/2^{n+1}$. In that case both the total distance traversed by the body and the time taken is given by the convergent series

$$\sum_{n=0}^{\infty}(1/2^n - 1/2^{n+1}) = \sum_{n=0}^{\infty} 1/2^{n+1} = 1,$$

as expected. So, *contra* Zeno. the infinite number of divisions is indeed traversed in a finite time.

More troubling, however, is the fact that these divisions, of lengths $\ldots 1/2^n, \ldots, 1/2^3, 1/2^2, 1/2$ constitute an infinite *regression* which, like the negative integers, *has no first term.* Zeno seems to be inviting us to draw the conclusion that it cannot be supplied with one, so that the motion could never get started. However, from a strictly mathematical standpoint, there is nothing to prevent us from placing 0 before all the members of this sequence, just as it could be placed, in principle at least, before all the negative integers. Then the sequence of correlations $\ldots(1/2^n, 1/2^n), \ldots, (1/2, 1/2) \ldots,$ (1, 1) between the time and the body's position is simply preceded by the correlation (0, 0), where the motion begins. There is no contradiction here.

In the case of the *Achilles*, let us suppose that the tortoise has a start of 1000 feet and that Achilles runs ten times as quickly. Then Achilles must traverse an infinite number of distances—1000 feet, 100 feet, 10 feet, etc.—and the tortoise likewise must traverse an infinite number of distances—100 feet, 10 feet, 1 foot, etc.—before they reach the same point simultaneously. The distance of this point in feet from the starting points of the two contestants is given, in the case of Achilles, by the convergent series

$$\sum_{n=0}^{\infty} 1000 \cdot 10^{-n} = 1111\tfrac{1}{9},$$

and in the case of the tortoise

$$\sum_{n=0}^{\infty} 100 \cdot 10^{-n} = 111\tfrac{1}{9}.$$

And, assuming that Achilles runs 10 feet per second, the time taken for him to overtake the tortoise is given, in seconds, by the convergent series

$$\sum_{n=0}^{\infty} 1000 \cdot 10^{-n} = 1111\tfrac{1}{9},$$

so that, again *contra* Zeno, Achilles overtakes the tortoise in a finite time.

Although the use of convergent series does confirm what we take to be the evident fact that Achilles will, in the end, overtake the tortoise, a nagging issue remains. For consider the fact that, at each moment of the race, the tortoise is somewhere, and, equally, Achilles is somewhere, and neither is ever twice in the same place. This means that there is a biunique correspondence between the positions occupied by the tortoise and those occupied by Achilles, so that these must have the same number. But when Achilles catches up with the tortoise, the positions occupied by the latter are only *part* of those occupied by Achilles. This would be a contradiction if one were to insist, as did Euclid in his *Elements,* that the whole invariably has more terms than any of its parts. In fact, it is precisely this principle which, in the nineteenth century, came to be repudiated for infinite sets[2] such as the ones encountered in Zeno's paradoxes. Once this principle is abandoned, no contradiction remains.

The paradox of the *Arrow* can be resolved by developing a theory of *velocity*, based on the differential calculus. By definition, (average) velocity is the ratio of distance travelled to time taken. It will be seen at once that in this definition *two* distinct points in space and *two* distinct points in time are required. Velocity at a point is then defined as the *limit* of the average velocity over smaller and smaller spatiotemporal intervals around the point. According to this definition, a body may have a nonzero "velocity" at each point, but at each instant of time will not "appear to be moving".

Although, as we have seen, Zeno's paradoxes can be resolved from a strictly *mathematical* standpoint, they present difficulties for understanding the nature of *actual* motion which have persisted to the present day.

The Stoic philosophers, as well as mathematicians such as Eudoxus, grasped fully the idea of passage to the limit or convergence of purely *spatial* quantities. But the notion of convergence of points or intervals of *time* eluded them, because getting at this notion involves the idea of a *functional correspondence* between time and space, a conception which never received adequate formulation in ancient Greek science. Nevertheless, the Stoics made a bold attempt to overcome the difficulties involved in the analysis of motion. Chrysippus, for instance, perceived the intimate connection between time and motion, as is revealed in his "definition" of time, namely, as "the interval of movement with reference to which the measure of speed and slowness is reckoned." He also held that the present moment, the *now*, is given by an infinite sequence of nested time intervals shrinking toward the mathematical "now", a strikingly modern conception.

The problem of the continuum arises also in connection with the *method of exhaustion*. We are told by Archimedes that, using his principle of convergence, Eudoxus successfully proved that the volume of a cone is one third that of the circumscribed cylinder. Archimedes also claims that Democritus originally discovered the result, but was unable to prove it rigorously. The obstacle was that he could see no way of actually building the cone from circular segments, each one of which would differ slightly in area from the two flanking it (the method he had apparently used in discovering the result). The atomist Democritus, with his belief in ultimate finite units, would presumably have understood this "slightly" as entailing a *discrete* difference between the areas of these circular segments, which would produce, not a smooth cone, but instead a ziggurat-like figure with a surface consisting of a series of tiny steps. If, on the other hand, this "slightly" were to be taken to mean "continuously", or "infinitesimally", then the difference between the areas of the segments would seem as a result to be nonexistent, and one would end up, not with a cone, but a cylinder. Eudoxus later surmounted this difficulty by taking the *limit* of the volumes in a manner essentially similar to the method employed in the integral calculus. This concept of limit is in fact completely in accord with the Stoic conception of the continuum.

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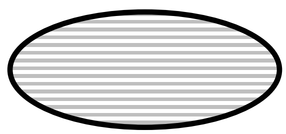
2 See the following chapter for a discussion of infinite sets.

The opposition between the continuous and the discrete resurfaced with renewed vigour in the seventeenth century with the emergence of the differential and integral calculus. Here the controversy centred on the concept of *infinitesimal*. According to one school of thought, the infinitesimal was to be regarded as a real, infinitely small, indivisible element of a continuum, similar to the atoms of Democritus, except that now their number was considered to be infinite. Calculation of areas and volumes, i.e., integration, was thought of as summation of an infinite number of these infinitesimal elements. An area, for example, was taken to be the "sum of the lines of which it is formed", as indicated in the diagram below. Thus the continuous was once again reduced to the discrete, but, with the intrusion of the concept of the infinite, in a subtler and more complex way than before.



Infinitesimals enjoyed a considerable vogue among seventeenth and eighteenth century mathematicians. In the guise of the charmingly named "linelets" and "timelets", they played an essential role in *Isaac Barrow's*[3] (1630–1677) "method for finding tangents by calculation", which appears in his *Lectiones Geometricae* of 1670. As "evanescent quantities" they were instrumental (although later abandoned) in Newton's development of the calculus, and, as "inassignable quantities", in Leibniz's. The *Marquis de l'Hospital* (1661–1704), who in 1696 published the first treatise on the differential calculus (entitled *Analyse des Infiniments Petits pour l'Intelligence des Lignes Courbes*), invokes the concept in postulating that "a curved line may be regarded as being made up of infinitely small straight line segments," and that "one can take as equal two quantities differing by an infinitely small quantity."

However, the conception of infinitesimals as real entities suffered from a certain vagueness and even, on occasion, logical inconsistency. Memorably derided by the philosopher *George Berkeley* (1685–1753) as "ghosts of departed quantities" (and in the twentieth century roundly condemned by Bertrand Russell as "unnecessary, erroneous, and self-contradictory"), this conception of infinitesimal was gradually displaced by the idea—originally suggested by Newton—of the infinitesimal as a *continuous variable* which becomes arbitrarily small. By the start of the nineteenth century, when the rigorous theory of limits was in the process of being created, this conception of infinitesimal had been accepted by the majority of mathematicians. A line, for instance, was now understood as consisting not of "points" or "indivisibles", but as the domain of values of a continuous variable, in which separate points are to be

[3]Barrow is remembered not only for his own outstanding mathematical achievements but also for being the teacher of Newton.

considered as locations. At this stage, then, the discrete had given way to the continuous.

But the development of mathematical analysis in the later part of the nineteenth century led mathematicians to demand still greater precision in the theory of continuous variables, and above all in fixing the concept of *real number* as the value of an arbitrary such variable. As a result, in the eighteen seventies a theory was formulated—independently by Dedekind, *Karl Weierstrass* (1815–1897), and *Georg Cantor* (1845–1918)—in which a line is represented as a set of points, and the domain of values of a continuous variable by a set of real numbers. In this scheme of things there was no place for the concept of infinitesimal, which accordingly disappeared for a time. Thus, once again, the continuous was reduced to separate discrete points and the properties of a continuum derived from the structure of its underlying point set. This reduction, underpinned by the development of *set theory*, has led to immense progress in mathematics, and has met with almost universal acceptance by mathematicians.

A new phase in the long contest between the continuous and the discrete has opened in the past few decades with the refounding of the concept of infinitesimal on a solid basis. This has been achieved in two essentially different ways.

First, in the nineteen sixties *Abraham Robinson* (1918–1974), using methods of mathematical logic, created *nonstandard analysis,* an extension of mathematical analysis embracing both "infinitely large" and infinitesimal numbers in which the usual laws of the arithmetic of real numbers continue to hold, an idea which in essence goes back to Leibniz. Here by an infinitely large number is meant one which exceeds every positive integer; the reciprocal of any one of these is infinitesimal in the sense that, while being nonzero, it is smaller than every positive fraction $^1/_n$. Much of the usefulness of nonstandard analysis stems from the fact that within it every statement of ordinary analysis involving limits has a succinct and highly intuitive translation into the language of infinitesimals. For instance, if we call two numbers x and y *infinitesimally close* when $x - y$ is infinitesimal, we find that the statement

$$\ell \textit{ is the limit of } f(x) \textit{ as } x \to a$$

is equivalent in meaning to the statement

$$f(a + \varepsilon) \textit{ is infinitesimally close to } \ell \textit{ for all infinitesimal } \varepsilon \neq 0,$$

and the statement

$$f \textit{ is continuous at } a$$

to the statement

$$f(a + \varepsilon) \textit{ is infinitesimally close to } f(a) \textit{ for all infinitesimal } \varepsilon \neq 0.$$

Nonstandard analysis is presently in a state of rapid development, and has found many applications.

The second development in the refounding of the concept of infinitesimal has been the emergence in the nineteen seventies of *smooth infinitesimal analysis*. Founded on the methods of category theory, this is a rigorous framework of mathematical analysis in which every function between spaces is smooth (i.e., differentiable arbitrarily many times, and so in particular continuous) and in which the use of limits in defining the basic notions of the calculus is replaced by *nilpotent infinitesimals*, that is, of quantities so small (but not actually zero) that some power—most usefully, the square—vanishes[4]. Smooth infinitesimal analysis provides an image of the world in which the continuous is an autonomous notion, not explicable in terms of the discrete.

Thus we see that the opposition between the continuous and the discrete has not ceased to stimulate the development of mathematics.

[4] We have already touched on this in the previous chapter: a fuller account will be found in Appendix 3.