APPENDIX 3

THE CALCULUS IN SMOOTH INFINITESIMAL ANALYSIS

In the usual development of the calculus, for any differentiable function f on the real line $\mathbb{R}$, $y = f(x)$, it follows from Taylor's theorem that the increment $\delta y = f(x + \delta x) - f(x)$ in y attendant upon an increment δx in x is determined by an equation of the form

$$\delta y = f'(x)\delta x + A(\delta x)^2, \qquad (1)$$

where $f'(x)$ is the derivative of $f(x)$ and A is a quantity whose value depends on both x and δx. Now if it were possible to take δx so *small* (but not demonstrably identical with 0) that $(\delta x)^2 = 0$ then (1) would assume the simple form

$$f(x + \delta x) - f(x) = \delta y = f'(x)\,\delta x. \qquad (2)$$

We shall call a quantity having the property that its square is zero a *nilsquare infinitesimal* or simply an *infinitesimal.* In *smooth infinitesimal analysis* "enough" infinitesimals are present to ensure that equation (2) holds *nontrivially* for *arbitrary* functions $f: \mathbb{R} \rightarrow \mathbb{R}$. (Of course (2) holds trivially in standard mathematical analysis because there 0 is the sole infinitesimal in this sense.) The meaning of the term "nontrivial" here may be explicated in following way. If we replace δx by the letter ε standing for an arbitrary infinitesimal, (2) assumes the form

$$f(x + \varepsilon) - f(x) = \varepsilon f'(x). \qquad (3)$$

Ideally, we want the validity of this equation to be independent of ε , that is, given x, for it to hold for *all* infinitesimal ε. In that case the derivative $f'(x)$ may be *defined* as the unique quantity H such that the equation

$$f(x + \varepsilon) - f(x) = \varepsilon H$$

holds for all infinitesimal ε.

Setting $x = 0$ in this equation, we get in particular

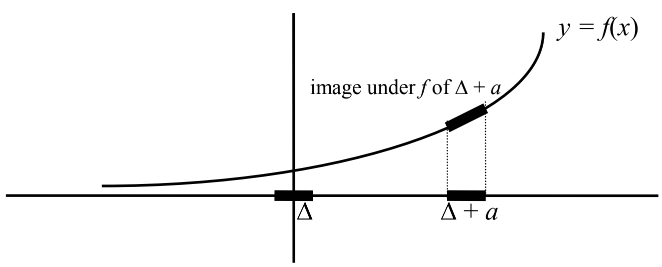
$$f(\varepsilon) = f(0) + H\varepsilon, \qquad (4)$$

for all ε. *It is equation* (4) *that is taken as axiomatic in smooth infinitesimal analysis.* Let us write Δ for the set of infinitesimals, that is,

$$\Delta = \{x: x \in \mathbb{R} \wedge x^2 = 0\}.$$

Then it is postulated that, for any f: $\Delta \to \mathbb{R}$, there is a *unique* $H \in \mathbb{R}$ such that equation (4) holds for all ε. This says that the graph of f is a straight line passing through (0, f(0)) with slope H. Thus any function on Δ is what mathematicians term *affine*, and so this postulate is naturally termed the *principle of infinitesimal affineness.* It means that Δ *cannot be bent or broken*: it is subject only to *translations and rotations*—and yet is not (as it would have to be in ordinary analysis) identical with a point. Δ may be thought of as an entity possessing position and attitude, but lacking true extension.

If we think of a function $y = f(x)$ as defining a curve, then, for any a, the image under f of the "infinitesimal interval" $\Delta + a$ obtained by translating Δ to a is straight and coincides with the tangent to the curve at $x = a$ (see figure immediately below). In this sense each curve is "infinitesimally straight".



From the principle of infinitesimal affineness we deduce the important *principle of infinitesimal cancellation*, viz.

IF $\varepsilon a = \varepsilon b$ *FOR ALL* ε, THEN $a = b$.

For the premise asserts that the graph of the function g: $\Delta \to \mathbb{R}$ defined by $g(\varepsilon) = a\varepsilon$ has both slope a and slope b: the uniqueness condition in the principle of infinitesimal affineness then gives $a = b$. The principle of infinitesimal cancellation supplies the exact sense in which there are "enough" infinitesimals in smooth infinitesimal analysis.

From the principle of infinitesimal affineness it also follows that *all functions on* ℝ *are continuous*, that is, *send neighbouring points to neighbouring points.* Here two points x, y on ℝ are said to be neighbours if $x - y$ is in Δ, that is, if x and y differ by an infinitesimal. To see this, given f: $\mathbb{R} \to \mathbb{R}$ and neighbouring points x, y, note that $y = x + \varepsilon$ with ε in Δ , so that

$$f(y) - f(x) = f(x + \varepsilon) - f(x) = \varepsilon f'(x).$$

But clearly any multiple of an infinitesimal is also an infinitesimal, so $\varepsilon f'(x)$ is infinitesimal, and the result follows.

In fact, since equation (3) holds for any f, it also holds for its derivative f'; it follows that functions in smooth infinitesimal analysis are differentiable arbitrarily many times, thereby justifying the use of the term "smooth".

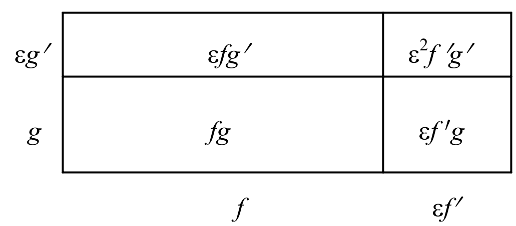
Let us derive a basic law of the differential calculus, the *product rule:*

$$(fg)' = f'g + fg'.$$

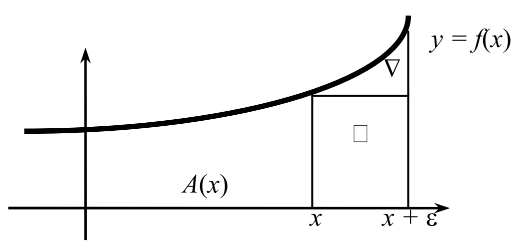
To do this we compute

$$(fg)(x + \varepsilon) = (fg)(x) + \varepsilon(fg)'(x) = f(x)g(x) + \varepsilon(fg)'(x),$$
$$(fg)(x + \varepsilon) = f(x + \varepsilon)g(x + \varepsilon) = [f(x) + \varepsilon f'(x)].[g(x) + \varepsilon g'(x)]$$
$$= f(x)g(x) + \varepsilon(f'g + fg') + \varepsilon^2 f'g'$$
$$= f(x)g(x) + \varepsilon(f'g + fg'),$$

since $\varepsilon^2 = 0$. Therefore $\varepsilon(fg)' = \varepsilon(f'g + fg')$, and the result follows by infinitesimal cancellation. This calculation is depicted in the diagram below.



Next, we derive the *Fundamental Theorem of the Calculus*.



Let J be a closed interval $\{x: a \leq x \leq b\}$ in $\mathbb{R}$ and $f: J \to \mathbb{R}$; let $A(x)$ be the area under the curve $y = f(x)$ as indicated above. Then, using equation (3),

$$\varepsilon A'(x) = A(x + \varepsilon) - A(x) = \square + \nabla = \varepsilon f(x) + \nabla.$$

Now by infinitesimal affineness ∇ is a triangle of area $\frac{1}{2}\varepsilon.\varepsilon f'(x) = 0$. Hence $\varepsilon A'(x) = \varepsilon f(x)$, so that, by infinitesimal cancellation,

$$A'(x) = f(x).$$

A *stationary point a* in $\mathbb{R}$ of a function $f\colon \mathbb{R} \to \mathbb{R}$ is defined to be one in whose vicinity "infinitesimal variations" fail to change the value of f, that is, such that $f(a + \varepsilon) = f(a)$ for all ε. This means that $f(a) + \varepsilon f'(a) = f(a)$, so that $\varepsilon f'(a) = 0$ for all ε, whence it follows from infinitesimal cancellation that $f'(a) = 0$. This is *Fermat's rule.*

An important postulate concerning stationary points that we adopt in smooth infinitesimal analysis is the

Constancy Principle. If every point in an interval J is a stationary point of $f\colon J \to \mathbb{R}$ (that is, if f' is identically 0), then f is constant.

Put succinctly, "universal local constancy implies global constancy". It follows from this that two functions with identical derivatives differ by at most a constant.

In ordinary analysis the continuum $\mathbb{R}$ is connected in the sense that it cannot be split into two nonempty subsets neither of which contains a limit point of the other. In smooth infinitesimal analysis it has the vastly stronger property of *indecomposability:* it cannot be split *in any way whatsoever* into two disjoint nonempty subsets. For suppose $\mathbb{R} = U \cup V$ with $U \cap V = \varnothing$. Define $f\colon \mathbb{R} \to \{0, 1\}$ by $f(x) = 1$ if $x \in U$, $f(x) = 0$ if $x \in V$. We claim that f is constant. For we have

$$(f(x) = 0 \text{ or } f(x) = 1) \quad \& \quad (f(x + \varepsilon) = 0 \text{ or } f(x + \varepsilon) = 1).$$

This gives 4 possibilities:

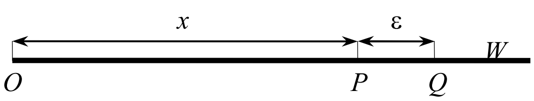
(i) $f(x) = 0$ & $f(x + \varepsilon) = 0$
(ii) $f(x) = 0$ & $f(x + \varepsilon) = 1$
(iii) $f(x) = 1$ & $f(x + \varepsilon) = 0$
(iv) $f(x) = 1$ & $f(x + \varepsilon) = 1$

Possibilities (ii) and (iii) may be ruled out because f is continuous. This leaves (i) and (iv), in either of which $f(x) = f(x + \varepsilon)$. So f is locally, and hence globally, constant, that is, constantly 1 or 0. In the first case $V = \varnothing$, and in the second $U = \varnothing$.

Partial derivatives can be defined in smooth infinitesimal analysis in a way similar to ordinary derivatives. For example, for arbitrary infinitesimals ε, η, we have the equations

$$f(x+\varepsilon, y) - f(x, y) = \varepsilon \frac{\partial f}{\partial x}, \qquad f(x, y+\eta) - f(x, y) = \eta \frac{\partial f}{\partial y}.$$

We use these in the derivation of the one-dimensional *heat equation.*



Thus suppose we are given a heated wire *W*. Let $T(x, t)$ be the temperature at a point *P* on it at time *t*. The heat content of the segment *PQ* is $k\varepsilon T_{\mathbf{average}}$, where *k* is a constant and $T_{\mathbf{average}}$ is the average temperature along *PQ*. Now

$$T_{\mathbf{average}} = \tfrac{1}{2}[T(x+\varepsilon, t) + T(x, t)] = T(x, t) + \tfrac{1}{2}\varepsilon \frac{\partial T}{\partial x}(x,t).$$

Therefore the heat content of *PQ* is

$$k\varepsilon\, [T(x, t) + \frac{\partial T}{\partial x}(x,t)\ ] = k\varepsilon\, T(x, t).$$

So the change in the heat content of *PQ* between time *t* and time $t + \eta$ is given by

$$k\varepsilon\, [T(x+\eta, t) - T(x, t)] = k\varepsilon\eta \frac{\partial T}{\partial t}(x,t). \qquad (5)$$

Now the rate of flow of heat across *P* is, according to a basic law of heat conduction, proportional to the temperature gradient there, that is, equal to

$$m\frac{\partial T}{\partial x}(x,t),$$

where *m* is a constant. Similarly, the rate of flow of heat across *Q* is

$$\frac{\partial T}{\partial x}(x+\varepsilon,t),$$

So the heat transfer across *P* between the times *t* and $t + \eta$ is

$$m\eta \frac{\partial T}{\partial x}(x,t)\,,$$

and across Q is

$$m\eta \frac{\partial T}{\partial x}(x+\varepsilon,t)\,.$$

So the net change of heat content in PQ between t and $t + \eta$ is

$$m\eta \left[\frac{\partial T}{\partial x}(x+\varepsilon,t)-\frac{\partial T}{\partial x}(x,t)\right] = m\eta\varepsilon \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = m\eta\varepsilon \frac{\partial^2 T}{\partial x^2}\,.$$

Equating this with (5), cancelling ε and η and writing $K = m/k$ yields the *one-dimensional heat equation*

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2}\,.$$

In conclusion, we observe that the postulates of smooth infinitesimal analysis are *incompatible with the law of excluded middle of classical logic* (*q.v.* Chapter 12). This incompatibility can be demonstrated in two ways, one informal and the other rigorous. First the informal argument. Consider the function f defined for real numbers x by $f(x) = 1$ if $x = 0$ and $f(x) = 0$ whenever $x \neq 0$. If the law of excluded middle held, each real number is then either equal or unequal to 0, so that the function f would be defined on the whole of $\mathbb{R}$. But, considered as a function with domain $\mathbb{R}$, f is clearly discontinuous. Since, as we know, in smooth infinitesimal analysis every function on $\mathbb{R}$ is continuous, f cannot have domain $\mathbb{R}$ there. So the law of excluded middle fails in smooth infinitesimal analysis. To put it succinctly, *universal continuity implies the failure of the law of excluded middle.*

Here now is the rigorous argument. We show that the failure of the law of excluded middle can be derived from the principle of infinitesimal cancellation. To begin with, if $x \neq 0$, then $x^2 \neq 0$, so that, if $x^2 = 0$, then necessarily not $x \neq 0$. This means that

$$\textit{for all infinitesimal } \varepsilon, \textit{ not } \varepsilon \neq 0. \qquad (*)$$

Now suppose that the law of excluded middle were to hold. Then we would have, for any ε, either $\varepsilon = 0$ or $\varepsilon \neq 0$. But (*) allows us to eliminate the second alternative, and we infer that, for all ε, $\varepsilon = 0$. This may be written

$$\textit{for all } \varepsilon,\ \ \varepsilon.1 = \varepsilon.0,$$

from which we derive by infinitesimal cancellation the falsehood 1 = 0. So again the law of excluded middle must fail.

The “internal” logic of smooth infinitesimal analysis is accordingly not full classical logic. It is, instead, *intuitionistic* logic, that is, the logic—described in Chapter 12—derived from the constructive interpretation of mathematical assertions. In our brief sketch we did not notice this “change of logic” because, like much of elementary mathematics, the topics we discussed are naturally treated by constructive means such as direct computation.