

An Option Value Problem from *Seinfeld**

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Abstract

This is a paper about nothing.

1 Introduction

In an episode of the sitcom *Seinfeld*, Elaine Benes uses a contraceptive sponge that gets taken off the market. She scours pharmacies in the neighborhood to stock a large supply, but it is finite. So she must “re-evaluate her whole screening process.” Every time she dates a new man, which happens very frequently, she has to consider a new issue: Is he “spongeworthy”? The purpose of this article is to quantify this concept of spongeworthiness.

When Elaine uses up a sponge, she is giving up the option to have it available when an even better man comes along. Therefore using the sponge amounts to exercising a real option to wait, and spongeworthiness is an option value. It can be calculated using standard option-pricing techniques. However, unlike the standard theory of financial or many real options, there are no complete markets and no replicating portfolios. Stochastic dynamic programming methods must be used.

2 The Model

Suppose Elaine believes herself to be infinitely lived; this is a good approximation in relation to the number of sponges she has and her time-discount factor or impatience. She meets a new man every day. Define the quality Q of each man as Elaine’s utility from having sex with him. This is independently and identically distributed, and drawn each day from a distribution which I assume to be uniform over $[0,1]$. Each day she observes the Q of that

* I developed this model many years ago, but kept it hidden because it seemed too politically incorrect. I hope that my advanced age now exempts me from the constraints of political correctness.

I thank Ricardo Guzmán for his abstract suggestion.

day's date. Actually this is only her estimate formed from observing and closely questioning the man (which is what she does in the episode), not the ex post facto outcome. But I assume that she has sufficient experience and expertise to make a very accurate estimate. Having observed Q , she makes her yes/no decision. Elaine's per-day discount factor is β . All these assumptions are to simplify the calculations; the method is perfectly general and many bells and whistles can be added to the analysis. I mention a few of them at the end.

If sponges were freely available for purchase at a constant price (which is small in relation to the potential value so I will ignore it), then Elaine's decision would be yes for any quality greater than zero. But when she has a finite stock and cannot buy any more, her optimal decision will be based on a "spongeworthiness threshold" of quality, Q_m , such that her decision will be yes if $Q > Q_m$. The threshold depends on the number m of sponges she has: the fewer sponges left, the higher the threshold needed to justify using up one of them.

Let V_m denote Elaine's expected present value of utility when she has a stock of m sponges. She meets a man and observes his quality Q . If she decides to use one of her sponges, she gets the immediate payoff Q and has continuation value V_{m-1} on the second day, which has present value βV_{m-1} . If she decides not to, there is no immediate payoff, only the present value of continuation with m sponges, namely βV_m . Therefore her decision rule is

$$\begin{aligned} \text{Spongeworthy if} & \quad Q + \beta V_{m-1} > \beta V_m \\ \text{that is, if} & \quad Q > Q_m \equiv \beta (V_m - V_{m-1}), \end{aligned} \tag{1}$$

and the value recursion formula of dynamic programming is

$$V_m = \mathbf{E} [\max \{ Q + \beta V_{m-1}, \beta V_m \}]. \tag{2}$$

Therefore

$$\begin{aligned} V_m &= \int_0^{Q_m} \beta V_m dq + \int_{Q_m}^1 (q + \beta V_{m-1}) dq \\ &= \beta V_m Q_m + \frac{1}{2} [1 - (Q_m)^2] + \beta V_{m-1} (1 - Q_m) \\ &= \beta (V_m - V_{m-1}) Q_m + \frac{1}{2} - \frac{1}{2} (Q_m)^2 + \beta V_{m-1} \\ &= (Q_m)^2 + \frac{1}{2} - \frac{1}{2} (Q_m)^2 + \beta V_{m-1} \\ &= \frac{1}{2} + \frac{1}{2} (Q_m)^2 + \beta V_{m-1} \\ &= \frac{1}{2} + \frac{1}{2} \beta^2 (V_m - V_{m-1})^2 + \beta V_{m-1}. \end{aligned}$$

Write this as

$$V_m - V_{m-1} = \frac{1}{2} + \frac{1}{2} \beta^2 (V_m - V_{m-1})^2 - (1 - \beta) V_{m-1},$$

or

$$\beta^2 (V_m - V_{m-1})^2 - 2 (V_m - V_{m-1}) + [1 - 2(1 - \beta) V_{m-1}] = 0.$$

Therefore

$$\begin{aligned} V_m - V_{m-1} &= \frac{2 \pm \sqrt{4 - 4\beta^2 [1 - 2(1 - \beta) V_{m-1}]}}{2\beta^2} \\ &= \frac{1 \pm \sqrt{1 - \beta^2 + 2\beta^2(1 - \beta) V_{m-1}}}{\beta^2}. \end{aligned}$$

The initial condition is $V_0 = 0$. Keeping the positive root would make $V_1 > 1/\beta^2 > 1$ which is impossible. Therefore keep the negative root and write the difference equation as

$$V_m = V_{m-1} + \frac{1 - \sqrt{1 - \beta^2 + 2\beta^2(1 - \beta) V_{m-1}}}{\beta^2}. \quad (3)$$

3 Solution

To solve this, examine the function

$$f(x) = x + \frac{1 - \sqrt{1 - \beta^2 + 2\beta^2(1 - \beta)x}}{\beta^2}. \quad (4)$$

We have

$$f(0) = \frac{1 - \sqrt{1 - \beta^2}}{\beta^2} > 0,$$

and for large x , the second term (built-up fraction) on the right hand side of (4) becomes negative so eventually $f(x) < x$. Also

$$\begin{aligned} f'(x) &= 1 - \frac{1}{\beta^2} \frac{1}{2} \left(1 - \beta^2 + 2\beta^2(1 - \beta)x\right)^{-1/2} 2\beta^2(1 - \beta) \\ &= 1 - (1 - \beta) \left(1 - \beta^2 + 2\beta^2(1 - \beta)x\right)^{-1/2}. \end{aligned}$$

This is increasing as x increases. At the extremes,

$$f'(0) = 1 - (1 - \beta) \left(1 - \beta^2\right)^{-1/2} = 1 - \left(\frac{1 - \beta}{1 + \beta}\right)^{1/2},$$

which is positive but less than one, and

$$f'(\infty) = 1.$$

Figure 1 shows this curve. Starting at $x = V_0 = 0$, we successively read off $V_1 = f(V_0)$, $V_2 = f(V_1)$ etc. As $m \rightarrow \infty$, $V_m \rightarrow V^*$, the fixed point $f(x) = x$. Solving (4) we find

$$V^* = \frac{1}{2(1 - \beta)}. \quad (5)$$

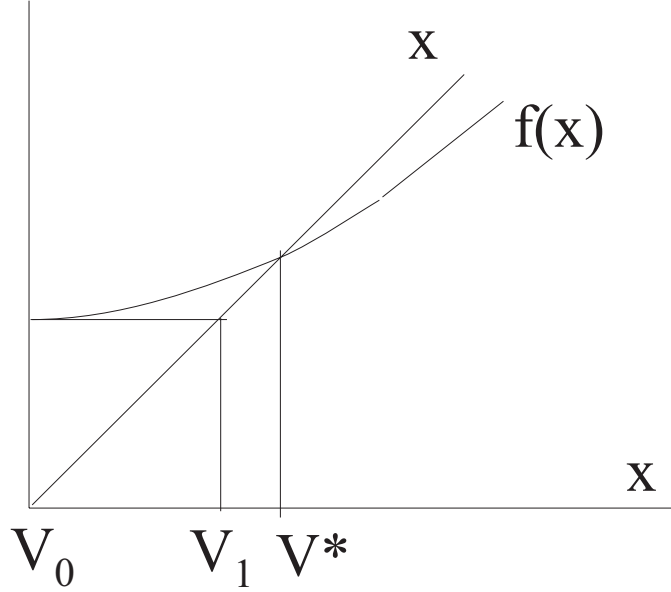


Figure 1: Graphical solution

Table 1: Numerical calculations

β	β^{365}	V_1	Q_1	V_{10}	Q_{10}	V_{100}	Q_{100}
0.999999	0.999635	0.998	0.998	9.968	0.996	99.054	0.986
0.999	0.694	0.957	0.956	9.044	0.866	73.333	0.617
0.990	0.026	0.876	0.868	7.272	0.619	35.416	0.168
0.900	1.988×10^{-17}	0.696	0.627	3.571	0.155	4.999	1.73×10^{-5}
0.500	0	0.536	0.268	0.999	4.27×10^{-4}	1.000	0.000

This is obviously correct: with infinitely many sponges Elaine can use one every day to have an expected value $\frac{1}{2}$ each day, and V^* is just the capitalized value of this.

Table 1 shows some numerical calculations for the values for the cases where Elaine has just one sponge left,

$$V_1 = \frac{1 - \sqrt{1 - \beta^2}}{\beta^2}, \quad (6)$$

where she has 10 sponges left (V_{10}), and 100 left (V_{100}) (there are no simple explicit formulas for the latter two), and the implied spongeworthiness thresholds for each case, for various discount factors. Note that β is the daily discount factor; therefore for better intuition I also show the annual discount factor β^{365} in each case.

Some limiting cases should be noted: [1] Elaine very patient: As $\beta \rightarrow 1$, from (3) we have $V_m - V_{m-1} \rightarrow 1$. Then $V_m \rightarrow m$. Therefore (1) gives $Q_m \rightarrow 1$: a completely patient

Elaine will accept no one except the best possible man every day. The first line of the table is a close approximation to this. [2] Elaine very impatient: As $\beta \rightarrow 0$, expanding the square root in (6) in its binomial series, we have $V_1 \rightarrow 1/2$ and then $Q_1 \rightarrow 0$. [3] Large stock: For the low values of β , namely 0.9 and 0.5, where events one year out are negligible, V_{100} is a very close approximation to the V^* in (5).

4 Some Extensions

[1] Elaine has a finite life of n days, where $n > m$ the number of sponges. Defining the value function $V(m, n)$ in the obvious way, the Bellman equation is

$$V(m, n) = \mathbf{E} [\max\{ Q + \beta V(m - 1, n - 1), \beta V(m, n - 1) \}] .$$

This yields the threshold

$$Q(m, n) = \beta [V(m, n - 1) - V(m - 1, n - 1)] .$$

The boundary conditions are $V(0, n) = V(m, 0) = 0$ for all m, n . This can be solved numerically quite easily at least for small m, n .

[2] If Elaine's estimation procedure is error-prone, she can calculate conditional probabilities and expectations based on her estimate Q . Other complexities such as a non-uniformity or serial correlation (positive or negative) in the distribution of Q can be handled using numerical methods for calculation.

[3] Elaine may have time-inconsistent preferences; for example, she may be a hyperbolic discounter, either naive or sophisticated (aware that next period she will have a different preference over the remaining time profile). The modifications to the analysis are relatively easy to make.

[4] In the episode, when George comes asking Elaine for one of her sponges, she should be willing to sell it for the price $V(m) - V(m - 1)$ (or $V(m, n) - V(m - 1, n)$ in the finite life case).

[5] At the end of the episode, when the man she has accepted for the night asks for a second helping, she says "Sorry, I can't afford two of them." This pins down the man's spongeworthiness quite precisely: greater than Q_m but less than Q_{m-1} . For example, if $m = 10$ and $\beta = 0.999$, the interval is $[0.866, 0.872]$.