The equivalence principle as a stepping stone from special to general relativity: A Socratic dialog

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We show how students can be led to an understanding of the connection between special relativity and general relativity by considering the time dilation effect of clocks placed on the surface of the Earth. This paper is written as a Socratic dialog between a lecturer and a student. © *2006 American Association of Physics Teachers.*
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I. SETTING THE SCENE

Sam is in the office and has just finished reading Plato's *Meno*[1] in which Socrates uses a self-discovery technique to teach a boy Pythagoras' theorem. Sam is inspired by this dialog and is pondering its applicability to lecturing undergraduate physics when a tap on the door breaks that chain of thought. Kim enters the room looking bleary eyed and pale. "Been out celebrating the last lecture of the year" Sam surmizes, little knowing that other things have kept Kim awake.

II. THE DIALOG

Kim: Your lectures on special relativity fascinated me, and when I got home I wondered if I could construct a simple experiment to prove or disprove time dilation, the aspect of special relativity that interests me the most. While lying in bed before dozing off, I realized that a clock placed at the equator should run slower than a clock placed at the pole. So I did a little calculation and found that special relativity predicts that a clock on the equator runs slower by about 100 ns per day with respect to a clock at the pole. Although this effect is not large, it is certainly measurable with modern atomic clocks. So I went onto the internet to see if I could find any reference to such an experiment and to my surprise I couldn't.

I was starting to get so frustrated that I couldn't sleep. I glanced at the clock (3 am). I thought to myself "How accurate is my clock?" I should check it against internet time. Then it occurred to me that the world timing standard organizations must mention a latitude effect on local clock accuracies. So I went onto the internet again and checked *The Bureau International des Poids et Mesures*[2] because the Bureau calculates the international atomic time from atomic clocks located in more than 30 countries around the world. I was sure that I must find something about the latitude effect on their web site. After spending hours trawling their site and then other sites on the web, I came up with nothing. There was a discussion of the relativistic effect of placing clocks at high altitudes, but nothing about latitude. In my despair I gave up and collapsed into a fitful sleep.

I came to see you today in the hope that you could cure my insomnia.

Sam: You are in good company in thinking that clocks at the equator and the pole should tick at different rates. Einstein himself predicted as much in his famous 1905 paper on the special theory of relativity.[3] Luckily for physics the effect was not measurable with the instruments of the day because Einstein's prediction would have failed to match experiment.

Let us return to your findings:

(1) According to the special theory of relativity a clock located at the equator should run slower than one at the pole and
(2) Empirically, all clocks located at sea-level on the Earth's surface tick at the same rate, regardless of latitude.

To help you understand how both apparently contradictory statements can be true I will ask you a question. If the Earth were a rotating perfect fluid and we could ignore the gravitational effects of the Sun and the Moon, what shape would the Earth be?

Kim: Well, I don't see how this question is relevant, but I would answer your question by drawing a free-body diagram. Can I use your blackboard?

Now let me see … consider a test mass placed on the surface of the Earth (see Fig. 1). We know that the forces acting on the test mass are the outward force due to the difference in pressure and the inward force due to gravity. If the test mass is in hydrostatic equilibrium, then the pressure gradient force must be perpendicular to the surface and the sum of the gravitational and pressure gradient forces is the centripetal force, which is perpendicular to the axis of rotation.[4] Hmmm … you would have a complicated integral equation to solve because the direction of the gravitational force vector would depend on the distribution of mass. Furthermore, the pressure gradient would be perpendicular to the surface we are trying to calculate. It seems to be a complicated problem and, to be honest, I am not sure that I could solve it.

Sam: It is a difficult problem whose solutions involve hyperbolic and elliptic functions. Chandrasekhar has devoted a whole book to the subject.[5] Before we travel that arduous mathematical road, let us see if we can use some physics to help us. Taking our model of Earth as a rotating perfect fluid, is the Earth an equipotential surface?

Kim: *Thinks* … Yes.

Sam: Why?

Kim: Because if it wasn't, the sea water would feel a force $\vec{F} = -m\nabla\Phi$ and would move until $\nabla\Phi = 0$ everywhere on the surface.

Sam: So if I told you what the Earth's gravitational field is, could you tell me the shape of the Earth?

Kim: Yes, I think I could.

Sam: How?

Kim: If you told me that the Earth's gravitational potential is $\Phi_g(r, \theta)$, where r is the distance from the center and θ is the colatitude,[6] then I could calculate the *effective* potential felt by an observer co-rotating with the Earth by including the centrifugal force
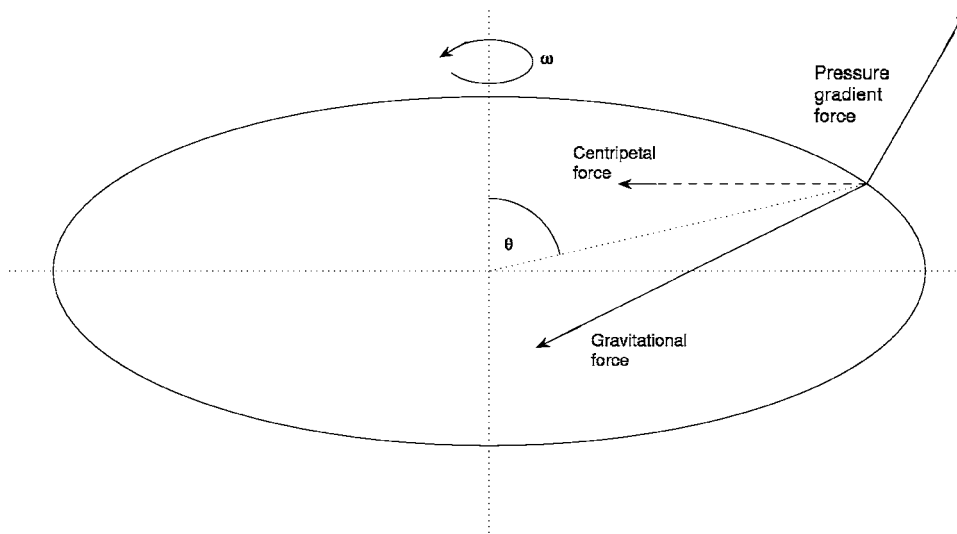
Fig. 1. Free body diagram for a mass placed on the surface of the Earth as seen from space.

$$\Phi_{\text{eff}} = \Phi_g(r,\theta) - \tfrac{1}{2}\omega^2 r^2 \sin^2\theta, \tag{1}$$

where $\theta \in [0,\pi]$, $\theta = 0$ at the north pole, $\pi/2$ at the equator, and π at the south pole; ω is the Earth's rotation rate; and r is the distance from the center to the Earth's surface. The second term on the right-hand side of Eq. (1) is the "centrifugal potential." Now we have already argued that a co-rotating observer on the surface of the Earth feels no change in the effective potential regardless of his/her latitude, that is, Φ_{eff} is constant. Furthermore, you have told me that we know the Earth's gravitational potential $\Phi_g(r,\theta)$, so all I need to do is rearrange Eq. (1) and, *voilà*, we have an expression for the shape of the Earth's surface. Mind you, because $\Phi_g(r,\theta)$ may be a complicated function, I am not sure that I can find an analytic expression for r anyway.

All this is very interesting, but I don't see how it answers my question about why clocks tick at the same rate on the Earth's surface.

Sam: Patience, we are coming to that. First let us investigate the discovery you have made, namely the shape of the Earth. Let me see, I know I have it in here somewhere …

Sam flicks through some notes in the filing cabinet.

Ah here it is. Despite the Earth's complicated shape with mountains and valleys, its gravitational field can be modeled to a fractional accuracy of 10^{-14} by[7]

$$\Phi_g(r,\theta) = \frac{-GM_e}{r} - \frac{J_2 GM_e a^2(1 - 3\cos^2\theta)}{2r^3}, \tag{2}$$

where $GM_e = 3.986\,004\,42 \times 10^{14}$ m^3 s^{-2} is the product of the gravitational constant and the mass of the Earth,[8] $J_2 = 1.082\,636 \times 10^{-3}$ is a measure of the Earth's equatorial bulge and is related to the Legendre polynomials,[9] and $a = 637\,813\,7$ m is the Earth's equatorial radius.[10]

To evaluate your equation for the Earth's surface (which incidentally is called the Geoid), you will need an accurate value of the Earth's rotation rate.

Sam shuffles through some files … Yes here it is[11]

$$\omega = 7.292\,116 \times 10^{-5} \text{ rad/s}. \tag{3}$$

Now your Geoid equation is going to be a bit tricky to solve analytically so instead of doing that let us see if we are on the right track. The easiest thing for us to do is to check that your equation for the Earth's effective potential Φ_{eff} is the same at the equator and the pole.

Φ_{eff} at the pole: The Earth's mean polar radius is $\bar{r} = 6356.76 \pm 0.07$ km (Ref. 11) and

$$\Phi_{\text{eff}}(r = \bar{r}, \theta = 0) = \frac{-GM_e}{\bar{r}} + \frac{J_2 GM_e a^2}{\bar{r}^3}$$

$$= -6.2637 \times 10^7 \text{ m}^2\text{ s}^{-2}. \tag{4}$$

Φ_{eff} at the equator: The Earth's mean equatorial radius is $\bar{a} = 6378.1 \pm 0.2$ km (Ref. 11) and

$$\Phi_{\text{eff}}(r = \bar{a}, \theta = \pi/2) = \frac{-GM_e}{\bar{a}} - \frac{J_2 GM_e a^2}{2\bar{a}^3} - \frac{1}{2}\omega^2 \bar{a}^2$$

$$= -6.2637 \times 10^7 \text{ m}^2\text{ s}^{-2}. \tag{5}$$

Look the two values for Φ_{eff} are the same! What have you shown?

Kim: We have shown that the Earth is indeed an equipotential surface with respect to an observer sitting on the surface. But Sam, this calculation has nothing to do with the question I originally asked you!

Sam: Doesn't it? What did you ask me again?

Kim: I asked you why all clocks tick at the same rate on the surface of the Earth when special relativity predicts that they should run slower at the equator than at the pole.

Sam: Kim do you remember how we derived Einstein's famous formula $E = mc^2$?

Kim: Yes and to be honest I was a little disappointed with it. Once we learned that a constant speed of light lead to the Lorentz transformations, the rest was just algebra.

Sam: Remind me of the algebra.

Kim: We got to the point where we realized that the proper time interval, $d\tau$ must be defined as

$$c^2 d\tau^2 = c^2 dt^2 - d\vec{x}^2, \tag{6}$$

with dt and $d\vec{x}$ the coordinate time and space interval, respectively. Then we multiplied Eq. (6) by $m^2 c^2/d\tau^2$ to obtain

$$m^2 c^4 = m^2 c^4 \left(\frac{dt}{d\tau} \right)^2 - m^2 \vec{u}^2 c^2. \tag{7}$$

If we equate $\vec{u}$ with $d\vec{x}/d\tau$, we obtain $m^2 c^4 \gamma^2 - \vec{p}^2 c^2$ because $\gamma = dt/d\tau$ and $\vec{p} = m\vec{u}$, and hence,

$$m^2 c^4 = E^2 - \vec{p}^2 c^2, \tag{8}$$

because relativistic kinetic energy is $mc^2 \gamma$. So if $\vec{p} = 0$, then $E = mc^2$. Like I said, just algebra.

Sam: Hmm, yes indeed. Suppose you are floating in a room with no windows or doors. All of a sudden, you feel a force that throws you against the wall. If there were two possible forces, gravitational and centrifugal, are you able to determine which force you are feeling?

Kim: I don't see how.

Sam: And what would you (sitting in this closed room) say your time dilation was with respect to an observer who was not feeling the centrifugal or gravitation force?

Kim: I think I see what you are getting at. I can't say whether the force is gravitational or centrifugal, so I must treat their effects as the same. If I knew the force was centrifugal, I would say that my time dilation with respect to a stationary observer depends only on my velocity v, that is, $\gamma = 1/\sqrt{1 - v^2/c^2}$. Because I don't know where the energy to thrust me against the wall has come from, to be consistent I must say that the time dilation depends only on the effective potential, which is the sum of the gravitational and centrifugal potentials.

Sam: Excellent! The idea that you can't know if the force is a uniform gravitational force or a combination of uniform forces, is called the *equivalence principle*.[12] What does it tell you about clocks on the surface of the Earth?

Kim: Yes, yes, of course. According to somebody standing anywhere on the surface of the Earth, all their energy is effective potential energy Φ_{eff}. The rate at which their clock ticks depends only on this effective potential. We already showed that the effective potential over the surface of the Earth is constant. So all clocks on the surface of the Earth tick at the same rate. Eureka, I can sleep again!

Sam: Yes, you can sleep well indeed because you have just discovered one of the fundamental arguments that led to the development of the general theory of relativity. Before you go, let me clarify one point. To determine the time dilation, you used the effective potential which came from Newtonian arguments about gravitational and centrifugal forces. According to general relativity the Newtonian effective potential is an approximation to the relativistic effective potential. This difference does not change our conclusion in any way, the effective potential is still constant, it just means that in general relativity we have a slightly different version of Φ_{eff} (see the Appendix). Having said that, you should note that for the Earth, the Newtonian and relativistic effective potentials are almost identical. To learn precisely what the difference is, you will have to take my general relativity course, unless you continue to derive general relativity by yourself!

After exchanging pleasantries, Kim leaves for the long cycle home. While cycling home Kim reflects on the fact that the thought experiment involving a person in a windowless room who didn't know if the force she/he felt was gravitational or centrifugal was very similar to the arguments about absolute and relative motion that they had learned in their special relativity course.

Sam contemplates this conversation with Kim and wonders if it should be entered into next year's general relativity lecture notes.

APPENDIX: GENERAL RELATIVISTIC CORRECTIONS TO THE EFFECTIVE POTENTIAL

According to the general theory of relativity, the proper time interval $(d\tau)$ for a clock in a weak gravitational field (such as the Earth's) is given by[12]

$$-c^2 d\tau^2 \approx -\left(1 + 2\frac{\Phi_g}{c^2}\right) c^2 dt^2 + \left(1 - 2\frac{\Phi_g}{c^2}\right) dr^2 + r^2 d\theta^2$$
$$+ r^2 \sin^2 \theta d\phi, \tag{A1}$$

where Φ_g is given by Eq. (2) and $\Phi_g/c^2 \ll 1$. For a clock sitting on the surface of the Earth

$$dr = d\theta = 0 \quad \text{and} \quad d\phi = \omega dt, \tag{A2}$$

so the proper time interval is

$$d\tau = dt \sqrt{1 + \underbrace{2\frac{\Phi_g}{c^2} - \frac{r^2 \omega^2}{c^2} \sin^2 \theta}_{2\Phi_{\text{eff}}/c^2}}, \tag{A3}$$

where Φ_{eff} is the Newtonian gravitational potential [see Eq. (1)]. The time dilation effect is obtained by rearranging Eq. (A3):

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 + 2\frac{\Phi_{\text{eff}}}{c^2}}}. \tag{A4}$$

We have shown that the weak equivalence principle effectively states that time dilation can be calculated in terms of the effective potential only, that is

$$\frac{dt}{d\tau} = 1 - \frac{\Phi_{\text{eff}}^{GR}}{c^2}, \tag{A5}$$

where Φ_{eff}^{GR} is the relativistic effective potential. The relativistic effective potential can be determined in terms of the Newtonian potential by expanding Eq. (A4) and equating it with Eq. (A5):

$$\frac{\Phi_{\text{eff}}^{GR}}{c^2} = \frac{\Phi_{\text{eff}}}{c^2} - \frac{3}{2}\frac{\Phi_{\text{eff}}^2}{c^4} + \mathcal{O}\left(\frac{\Phi_{\text{eff}}^3}{c^6}\right)$$
$$\approx \frac{\Phi_{\text{eff}}}{c^2} - \frac{3}{2}\frac{\Phi_{\text{eff}}^2}{c^4} \quad \text{if} \quad \frac{\Phi_{\text{eff}}}{c^2} \ll 1. \tag{A6}$$

If we compare Eq. (A6) with Eq. (1), using the values for Φ_{eff} as calculated in Eqs. (4) and (5), we see that the relativistic effective potential differs from the Newtonian effective potential to a fractional accuracy of 10^{-11}.

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[1] Meno by Plato translated with an introduction by Benjamin Jowett, <etext.library.adelaide.edu.au/p/p71mo/>.

[2] The Bureau International des Poids et Mesures, <www.bipm.fr/>.

[3] Alex Harvey and Engelbert Schucking, "A small puzzle from 1905," Phys. Today **58**(3), 34–36 (2005).

[4] Dan Bruton, "The shape of rotating stars," <observe.phy.sfasu.edu/downloads/StarLight/Word/StarSpin.pdf>; William K. Rose, *Astrophysics* (Holt, Rinehart and Winston, New York, 1973).

[5] Subrahmanyan Chandrasekhar, *Ellipsoidal Figures of Equilibrium*, 2nd ed. (Dover, New York, 1987).

[6] Colatitude is $\pi/2$ minus latitude. It is the coordinate of choice for most references on planetary gravitational potentials.

[7] Claude Audoin and Bernard Guinot, *The Measurement of Time* (Cambridge University Press, Cambridge, 2001).

[8] This is the total mass of the Earth including the atmosphere. Because we are calculating the gravitational potential at sea level, we should subtract the mass of the atmosphere, but we do not need to do this subtraction as the mass of the atmosphere is roughly one millionth of the total mass of the Earth.

[9] Guy Bomford, *Geodesy*, 5th ed. (Oxford University Press, Oxford, 1986).

[10] This value of a is defined to fit the ellipsoidal Earth model WGS84. See <earth-info.nga.mil/GandG/wgsegm/> for more details.

[11] David Lide, *CRC Handbook of Chemistry and Physics* (CRC Press, New York, 1997).

[12] Bernard F. Schutz, *A First Course in General Relativity* (Cambridge University Press, Cambridge, 1985).