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Phasing of harmonic components to optimize measured signal-to-noise ratios of transfer functions

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Abstract. Much can be learnt about a system by studying how its various transfer functions vary with frequency. The best measured signal-to-noise ratio (SNR), and hence the most precise results, will occur when a sequence of stimuli, each containing only a single frequency, is applied to the system. However, it is often advantageous to shorten the total measurement time, with a consequent decrease in the measured SNR, by applying a stimulus that simultaneously contains several different frequency components. This paper considers the various compromises between the measured SNR and the frequency components that are simultaneously present. It concentrates upon improvements in the measured SNR that result from optimum phasing of the harmonic components. Calculations show that the SNR of a system with a transfer function that is substantially independent of frequency can thus be improved by a factor of approximately $k^{0.4}$, where k denotes the number of frequencies that are simultaneously present. The measured SNR will thus vary approximately as $k^{-0.6}$ if optimum waveforms are used.

1. Introduction

Transfer functions (the ratio of a response to a stimulus) provide a useful method of characterizing linear systems (namely those for which a response is proportional to the stimulus). Many experiments involve studying how the transfer functions of a system vary with frequency. Because any measuring device will have a finite resolution, the best signal-to-noise ratio (SNR), and hence the most precise measurement, will occur when a stimulus at a single frequency is applied to the system. The value of the transfer function is then measured separately for each frequency. This approach also maximizes the time taken to measure the frequency-dependence.

The total measurement time can be shortened, however, with a consequent decrease in the measured SNR, by applying a stimulus that simultaneously contains several different frequencies. Common stimuli with multiple frequency components include square waves, saw-tooth waves, triangular waves and random 'noise' signals.

Most experiments now involve a digital approach in which a computer both generates the stimulus via a digital-to-analogue converter (DAC) and measures the response via an analogue-to-digital converter (ADC) (Bell *et al* 1975). Like any other transducers or measuring devices, the ADC and DAC will each have a fixed resolution within a restricted range. Consequently the addition of each extra frequency component to the stimulus must reduce the maximum relative amplitude of other frequency components present in the response. The SNR of each component will thus be decreased as the number of components increases.

This paper considers how experiments that measure transfer functions can be optimized by considering various compromises between the number of harmonic frequency components that are simultaneously present in the stimulus and the SNR of each component in the response. In particular it quantifies the improvement in SNR that can result from optimal phasing of the harmonic components in the stimulus and the selection of their relative frequencies. Although originally developed for four-terminal measurements of the electrical admittance of electrolyte–membrane systems at low frequencies, the technique is applicable to a wide variety of systems.

2. Theory

A stimulus V_S composed of k harmonic components of a fundamental frequency ω_0 will have the form

$$V_S(t) = \sum_{j=1}^{k} A_{Sj} \sin(n_j\omega_0 t + \phi_{Sj}) + \chi_S(t) \qquad (1)$$

where A_{Sj} and ϕ_{Sj} denote the amplitudes and phases respectively of the jth component of angular frequency $n_j\omega_0$ and n_j are positive integers. χ_S denotes the noise in the stimulus that arises from the conversion noise of the DAC and any interference. The measured response V_R in a linear system will thus be of the form

$$V_R(t) = \sum_{j=1}^{k} A_{Rj} \sin(n_j\omega_0 t + \phi_{Rj}) + \chi_R(t) \qquad (2)$$

where A_{Rj} and ϕ_{Rj} denote the amplitudes and phases respectively of the jth component of the response and χ_R denotes the noise in the response. This noise will be larger than χ_S because of noise from several additional sources including the experimental system itself, interference, amplification processes and the conversion noise of the ADC.

The treatment in this paper is restricted to situations in which the amplitudes of the components in the stimulus and response have reached a steady state. The relationship between stimulus and response depends upon the frequency-dependence of the transfer function. For a linear system the complex transfer function for the jth component can be written as

$$T_j = \frac{A_{Rj}}{A_{Sj}}[\cos(\phi_{Rj} - \phi_{Sj}) + \mathrm{i}\sin(\phi_{Rj} - \phi_{Sj})] \qquad (3)$$

where i denotes $\sqrt{-1}$. Comparison between experiments and theoretical models will generally require similar SNRs at each frequency. The relative amplitudes A_{Sj} in the stimulus required in order to produce the best overall SNR will then depend not only upon the frequency-dependence of the particular transfer function under study, but also upon the characteristics of the noise in the stimulus and response.

2.1. Transfer functions that are substantially independent of frequency

If the transfer function does not depend strongly on frequency, then any difference in phase between the stimulus and response would be expected to be small for most situations. Consequently the stimulus and response will have similar waveforms, and improving the SNR of the stimulus will also improve the SNR of the response. It is then convenient to use a stimulus with equal amplitudes of each components; the variation of the SNR of the response with frequency will then be similar to that of the transfer function if the noise is independent of frequency. Equation (1) then reduces to

$$V_S(t) = A_S \sum_{j=1}^{k} \sin(n_j\omega_0 t + \phi_{Sj}) \qquad (4)$$

where the amplitudes A_{Sj} have been set equal to a common value denoted by A_S. For simplicity the contribution of χ_S has been neglected. The magnitude of V_S should be as large as possible without exceeding the range of the ADC, denoted by $\pm A_M$, otherwise conversion errors will result. The amplitude of each component should thus be limited to $A_S \leqslant A_M/k$ if the phases are not considered. This worst-case value of A_S will be denoted by A_W.

However, A_S can be increased if the relative phases of the individual components can be adjusted so that the maxima of some components partially overlap with the minima of others. Optimum phasing can thus increase A_S to a value A_B, and consequently increase the measured SNR, by a factor denoted by α. This improvement factor α in the measured SNR produced by optimum phasing is defined by the equation

$$\alpha = \frac{\text{maximum amplitude with optimum phases}}{\text{maximum amplitude with worst phases}} = \frac{A_B}{A_W} = \frac{kA_B}{A_M}. \qquad (5)$$

The SNR of the response will be inferior to that of the stimulus and consequently it should be optimized at the expense of the SNR of the stimulus. If substantial phase differences occur in the transfer function, then it will be necessary to allow for these in generating the stimulus to ensure that the response has the optimal waveform.

2.2. Transfer functions that depend strongly upon frequency

Optimizing the SNR of the response is difficult in these situations because neither the stimulus nor the response will then have the optimal waveform. Calculating the stimulus requires consideration of the magnitudes of both χ_S and χ_R, as well as the complex frequency-dependence of the transfer function. Most of the power is often contained in only a few components, so the possibility of large improvements in SNR is reduced.

3. Calculations

In an experiment the stimulus would normally be generated by cyclically stepping through a table containing N values. Some possibilities are discussed by Smith and Sandler (1995). There are then $N^{(k-1)}$ possible phases for combining k harmonic components. The calculation procedure involved setting the amplitude of each component to a value denoted by A_0. V_S was then calculated for each of the N points in the table using equation (4) and the maximum absolute value of V_S in the table, denoted by V_A, was determined for each unique combination of phases for the k components. The smallest value found for V_A was then used to calculate α using $\alpha = A_0/V_A$. The speed of the calculation was increased by using 16-bit integer arithmetic and look-up tables. A particular combination of k components is indicated using the notation $[n_1, n_2, \ldots, n_k]$. For $k > 6$ there were usually too many possible combinations of phases to calculate V_S for every possibility. The approach then involved selecting the phase at random and calculating V_A as described above. This was repeated p times and an estimate of α, denoted by α', was then calculated from the smallest value of V_A. Because this approach only considers a sub-set of the possible phase combinations, $\alpha' \leqslant \alpha$.

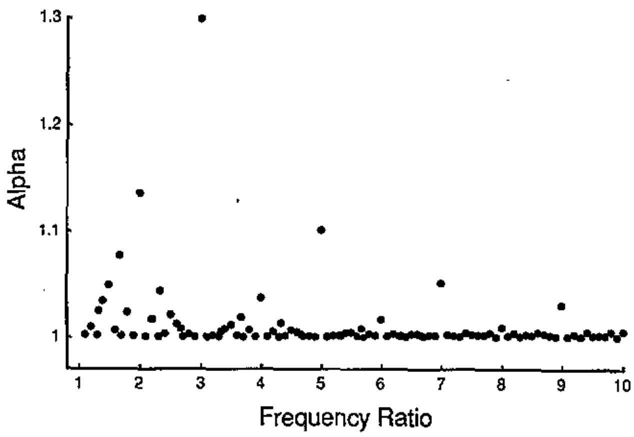

Figure 1. The variation of α, the increase in measured SNR produced by optimum phasing, when two harmonic components of equal amplitude are combined. α is shown as a function of n_2/n_1, the ratio of their frequencies. *Calculations assumed equal amplitudes with $N = 2048$.*

4. Results

Results are only presented for waveforms with components of equal amplitude; these would be suitable stimuli for systems with transfer functions that do not vary strongly with frequency. The calculations involved are computationally intensive, for example, there are 2^{32} possible phases to consider for each combination in table 4. Consequently phases for the best combinations are tabulated for each value of k; this allows experimenters to produce optimum phasing without further intensive calculation.

4.1. Improvements in SNR for stimuli with six or less components

Figure 1 shows how α varies when two harmonic components of equal amplitude are combined. Maxima occur when their frequency ratio n_2/n_1 is the ratio of two simple integers, the best combinations in order of decreasing α being [1,3], [1,2], [1,5], [3,5], [1,7], [2,3], [3,7] and [1,4]. Table 1 indicates that an increase in SNR of up to 30% is possible for the combination [1,3]. Further increases in the frequency of the higher harmonic reduce the chance of a suitable overlap between the minima and maxima of the two components, and consequently reduce the possible improvement in α when their relative phase is varied. This will become significant for the entire waveform when the components are reasonably close in frequency. The fastest measurements require that only one cycle of the lowest frequency be measured. Consequently the treatment in this paper will be restricted to the case $n_1 = 1$.

Table 2 shows that suitable phasing can further increase the SNR when a third component is present, the improvement in SNR now exceeding 72% for the combination [1,3,5]. Improvements in the SNR produced by phasing become increasingly significant as k, the number of components, increases. Table 3 shows that the combinations [1,2,3,4], [1,3,5,7], [1,3,5,9] and [1,3,7,9] can produce values of $\alpha \approx 1.95$. Further improvements in α for $k = 5$ and 6 are evident in tables 4 and 5 respectively. These stimuli with optimum phasing have unusual waveforms, but are simple to synthesize using digital techniques. Figure 2 shows the waveforms for the combination [1,2,3,4,5,6] with phases that produced the worst SNR (that is the lowest α) and the best SNR (the highest α) for these six components.

Table 1. *Optimum phases for adding two harmonic components of equal amplitude, calculated assuming that $\phi_1 = 0$ and $N = 1024$.*

n_1	n_2	ϕ_2	α
1	2	0	1.136
1	3	0	1.299
1	5	3.1416	1.102

Table 2. Optimum phases for adding three harmonic components of equal amplitude, calculated assuming that $\phi_1 = 0$ and $N = 1024$.

n_1	n_2	n_3	ϕ_2	ϕ_3	α
1	2	3	1.2885	5.2892	1.515
1	2	4	1.5217	1.5953	1.483
1	2	5	0.11045	0.28839	1.445
1	3	5	3.792	4.0804	1.725
1	3	7	0.45406	0.71177	1.548

Table 3. Optimum phases for adding four harmonic components of equal amplitude, calculated assuming that $\phi_1 = 0$ and $N = 512$.

n_1	n_2	n_3	n_4	ϕ_2	ϕ_3	ϕ_4	α
1	2	3	4	0.2454	2.1353	6.0378	1.959
1	2	3	5	0.0491	4.4547	5.6573	1.821
1	3	5	7	1.3622	3.4484	2.3930	1.952
1	3	5	9	5.7064	1.1536	6.2096	1.954
1	3	7	9	2.5035	5.6819	5.4487	1.946

4.2. The best combinations of frequencies

In many experiments it is only necessary to measure the frequency-dependence of the appropriate transfer function, the exact frequencies being unimportant provided that they provide sufficient resolution. It is thus important to know whether certain combinations of components will always produce high values of α. As an example figure 3 shows the values of α for all possible combinations of six harmonic components with n_j ranging from 1 to 9. No obvious pattern is evident in these data apart from a tendency for α to decrease as the frequency range of the combined components increases. It is apparent, however, that optimum phasing can produce significant improvements ($\alpha \geqslant 2.05$) for any combination of frequencies within this range.

Table 4. Optimum phases for adding five harmonic components of equal amplitude, calculated assuming that $\phi_1 = 0$ and $N = 256$.

n_1	n_2	n_3	n_4	n_5	ϕ_2	ϕ_3	ϕ_4	ϕ_5	α
1	2	3	4	5	0.49087	1.03084	4.90874	1.22718	2.124
1	2	3	4	6	1.27627	0.71177	3.5343	2.35619	2.100
1	2	3	5	8	0.7854	4.9087	4.9333	5.4978	1.986

Table 5. Optimum phases for adding six harmonic components of equal amplitude, calculated assuming that $\phi_1 = 0$ and $N = 64$.

n_1	n_2	n_3	n_4	n_5	n_6	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	α
1	2	3	4	5	6	1.1964	5.1051	5.1051	1.5708	5.3014	2.316
1	2	3	4	5	7	1.1781	1.1781	5.4978	2.6507	1.8653	2.320
1	2	3	4	6	7	1.3745	4.5160	1.1781	2.7489	2.2580	2.339
1	2	3	5	6	8	1.3745	4.8106	4.4179	3.9270	5.4978	2.347
1	2	3	5	7	8	0.9817	0.9817	5.8905	3.3379	0.7854	2.360
1	2	3	6	7	9	1.3745	1.6690	1.7672	4.9087	3.8288	2.324

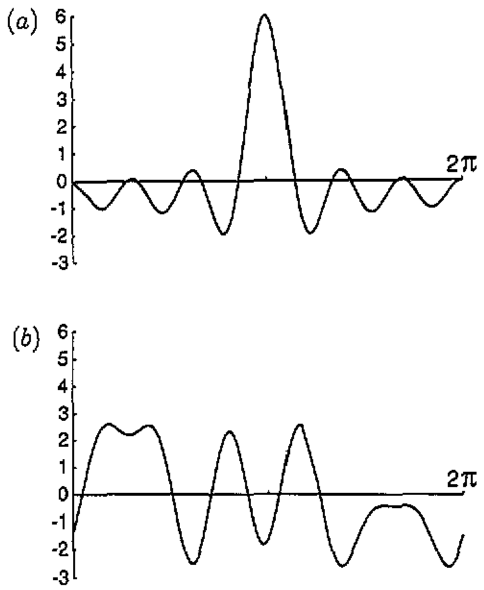

Figure 2. One complete cycle of two stimuli each containing six harmonic components with unity amplitude in the combination [1, 2, 3, 4, 5, 6]. In (*a*) is shown the combination of phases that produce the worst possible SNR with maximum amplitude equal to 6. In (*b*) is shown the combination of phases that produces the best possible SNR with maximum amplitude equal to 2.5906. Consequently $\alpha = 6/2.5906 = 2.316$.

Some particular combinations of frequencies can speed analysis, for example the combinations [1, 2, 4,…, 2^{k-1}] allow use of a simplified version of the Goertzel algorithm (Smith and Sandler 1995). However, when $k > 3$ the frequency of additional components is sufficiently large to limit the potential improvement in SNR. For example $\alpha = 1.136$, 1.483, 1.62 and 1.688 for $k = 2$, 3, 4 and 5 respectively.

4.3. Stimuli with six or more components

The large number of possibilities for $k > 6$ with any realistic value of N make it impossible to examine all possible phasings. One approach is simply to pick phases at random. The best result from p trials, denoted by α', can then be used as an estimate of α. The suitability of this approach was exhaustively tested for $k = 6$, for which the result was calculated for all possible phases. The results are shown in figure 4 in which α' is plotted against p, the number of trials used to produce that value. It is evident that a set of phases that produce a reasonably close estimate of the best value for α can be found by sampling only a small sub-set of the possible phases. This random sampling approach was used to investigate α for values of $k > 6$.

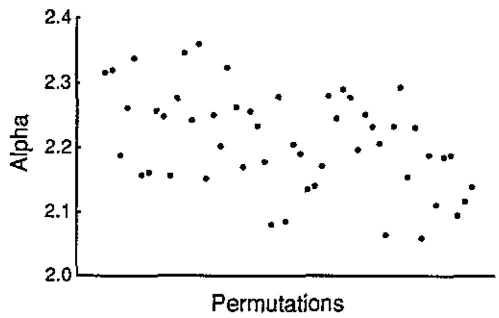

Figure 3. The variation of α, the increase in measured SNR produced by optimum phasing, for every combination of six harmonic components of equal amplitude with $1 \leqslant n_i \leqslant 9$. Permuations are arranged in increasing order, that is [1, 2, 3, 4, 5, 6] at the far-left hand side and [1, 5, 6, 7, 8, 9] at the far right-hand side. Calculations used $N = 64$.

4.4. The variation of α with k

Figure 5 shows the dependence of α' upon k, the number of harmonic components, estimated using random sampling. Combinations tested had the form $[1, 2, 3, \ldots, k]$. The relationship between α and k was described reasonably well by the function $\alpha = k^{0.4}$. The measured SNR will thus vary approximately as $k^{-0.6}$ if optimum waveforms are used.

Table 6. A comparison of the fractional reduction in SNR and total time when measurements of k components are taken simultaneously rather than sequentially. Calculations were for the combinations $[1, 2, \ldots, k]$.

	k				
	2	3	4	5	6
Reduction in time	0.667	0.545	0.48	0.438	0.408
Reduction in SNR	0.568	0.505	0.49	0.425	0.386

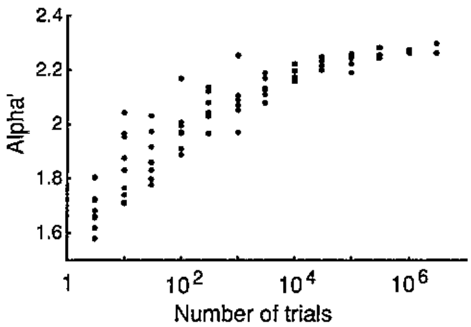

Figure 4. Variations in α', the estimated increase in SNR, for repeated estimates with different values of p, the number of trials with randomly selected phases. Calculations were for the combination $[1, 2, 3, 4, 5, 6]$ with $N = 1024$ ($\alpha = 2.316$ when all possible phase combinations for $N = 64$ are considered).

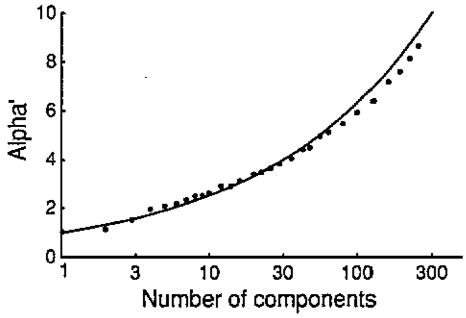

Figure 5. The dependence of α', the estimated increase in SNR, upon the number of harmonic components k. Each point was calculated for the combination $[1, 2, 3, \ldots, k]$ using 10^4 trials with $N = 1024$. The full line shows the relationship $\alpha = k^{0.4}$.

4.5. Compromises between SNR and total measurement time

An important consideration is how much time is saved when stimuli with simultaneous components are used. Consider combinations of the form $[1, n_2, \ldots, n_k]$. The measurement time for k simultaneous components will be shorter than that for k sequential components by a factor given by $1/(1 + n_2^{-1} + \ldots + n_k^{-1})$.

Table 6 shows the fractional decrease in measurement time for stimuli containing k simultaneous components with combinations of the form $[1, 2, \ldots, k]$ and also shows the concomitant decrease in SNR ($= \alpha/k$). For $k > 2$ the decrease in measurement time is similar to the decrease in SNR. (The measurement time for simultaneous components will be even shorter than that for sequential components when the time taken for the response to reach a steady state is considered.)

5. Discussion

The results show that optimum phasing can significantly improve the SNR for simultaneous measurements at multiple harmonic frequencies when the transfer functions are relatively independent of frequency. In most cases these improvements can be made simply by programming changes. The waveforms so generated are superior to square or saw-tooth waves, for which the amplitude of components decreases inversely with frequency.

For waveforms with less than seven components ($k < 7$) it is simple just to use the phases given in tables 1–5. Although these have been calculated for the largest values of N that were computationally realistic, calculations indicate that neither the optimum phases nor α depend significantly upon N when N is sufficiently large. Random trials of phases will probably be required for waveforms with additional components ($k > 6$). Random trials also allow the relative amplitudes of components to be varied during an experiment if the transfer function varies with frequency (Wolfe *et al* 1995).

5.1. Advantages

There are several situations in which simultaneous measurement at several frequencies can be advantageous. These include the following.

(i) Situations in which the maximum possible precision of the apparatus is not required.

(ii) Experimental systems involving measurements at very low frequencies (of the order of millihertz). Some examples include measurements of the electrical conductances of glasses, electrolytes or artificial lipid bilayers. The time taken to examine individually several very low frequencies can become inordinately long because at least one complete cycle of responses is required at each frequency to determine the in- and out-of-phase components.

(iii) Studies of unstable systems with a short life or transient states. For example, a major problem with studies of lipid bilayers is that they have unpredictable lifetimes ranging between minutes and days. It is thus best to measure their frequency-dependence as rapidly as possible, even if the SNR ratio suffers.

(iv) Systems with properties that drift or vary with time.

5.2. Limitations

One limitation of using stimuli with multiple harmonic components is that any nonlinearity in the system under investigation will produce additional harmonics of each component in the stimulus. Many of these additional harmonics will be at the same frequency as the original components; their contribution will thus be indistinguishable in the response and the precision of measurements will be reduced. This can be tested by injecting only the fundamental and measuring the amplitude of the harmonics. Nonlinearities can also produce components at frequencies equal to the sum and difference of components in the stimulus.

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