SOLUTIONS TO PROBLEMS

ELEMENTARY

LINEAR ALGEBRA

K. R. MATTHEWS

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF QUEENSLAND

First Printing, 1991

CONTENTS

SECTION 1.6

2. (i) $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_1 \to \frac{1}{2}R_1 \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;

(ii) $\begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} R_1 \to R_1 - 2R_2 \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$;

(iii) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix}$

$\begin{matrix} R_1 \to R_1 + R_3 \\ R_3 \to -R_3 \\ R_2 \leftrightarrow R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_3 \\ R_3 \to -R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

(iv) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 + 2R_1 \\ R_1 \to \frac{1}{2}R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

3. (a) $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & -1 & 8 \\ 1 & -1 & -1 & -8 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 4 \\ 0 & -2 & -2 & -10 \end{bmatrix}$

$\begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + 2R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 4 & -2 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & -8 & -2 \end{bmatrix} R_3 \to \frac{-1}{8}R_3 \begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 1 & \frac{1}{4} \end{bmatrix}$

$\begin{matrix} R_1 \to R_1 - 4R_3 \\ R_2 \to R_2 + 3R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & \frac{19}{4} \\ 0 & 0 & 1 & \frac{1}{4} \end{bmatrix}$.

The augmented matrix has been converted to reduced row–echelon form and we read off the unique solution $x = -3,\ y = \frac{19}{4},\ z = \frac{1}{4}$.

(b) $\begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 3 & -1 & 7 & 4 & 1 \\ -5 & 3 & -15 & -6 & 9 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + 5R_1 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 0 & -4 & 10 & -2 & -29 \\ 0 & 8 & -20 & 4 & 59 \end{bmatrix}$

$R_3 \to R_3 + 2R_2 \begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 0 & -4 & 10 & -2 & -29 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

From the last matrix we see that the original system is inconsistent.

(c) $\begin{bmatrix} 3 & -1 & 7 & 0 \\ 2 & -1 & 4 & \frac{1}{2} \\ 1 & -1 & 1 & 1 \\ 6 & -4 & 10 & 3 \end{bmatrix} R_1 \leftrightarrow R_3 \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -1 & 4 & \frac{1}{2} \\ 3 & -1 & 7 & 0 \\ 6 & -4 & 10 & 3 \end{bmatrix}$

$$\begin{array}{c} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{array} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & \frac{-3}{2} \\ 0 & 2 & 4 & -3 \\ 0 & 2 & 4 & -3 \end{bmatrix} \begin{array}{c} R_1 \to R_1 + R_2 \\ R_4 \to R_4 - R_3 \\ R_3 \to R_3 - 2R_2 \end{array} \begin{bmatrix} 1 & 0 & 3 & \frac{-1}{2} \\ 0 & 1 & 2 & \frac{-3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The augmented matrix has been converted to reduced row–echelon form and we read off the complete solution $x = -\frac{1}{2} - 3z,\ y = -\frac{3}{2} - 2z$, with z arbitrary.

4. $\begin{bmatrix} 2 & -1 & 3 & a \\ 3 & 1 & -5 & b \\ -5 & -5 & 21 & c \end{bmatrix} R_2 \to R_2 - R_1 \begin{bmatrix} 2 & -1 & 3 & a \\ 1 & 2 & -8 & b-a \\ -5 & -5 & 21 & c \end{bmatrix}$

$$R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 2 & -8 & b-a \\ 2 & -1 & 3 & a \\ -5 & -5 & 21 & c \end{bmatrix} \begin{array}{c} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 5R_1 \end{array} \begin{bmatrix} 1 & 2 & -8 & b-a \\ 0 & -5 & 19 & -2b+3a \\ 0 & 5 & -19 & 5b-5a+c \end{bmatrix}$$

$$\begin{array}{c} R_3 \to R_3 + R_2 \\ R_2 \to \frac{-1}{5}R_2 \end{array} \begin{bmatrix} 1 & 2 & -8 & b-a \\ 0 & 1 & \frac{-19}{5} & \frac{2b-3a}{5} \\ 0 & 0 & 0 & 3b-2a+c \end{bmatrix}$$

$$R_1 \to R_1 - 2R_2 \begin{bmatrix} 1 & 0 & \frac{-2}{5} & \frac{(b+a)}{5} \\ 0 & 1 & \frac{-19}{5} & \frac{2b-3a}{5} \\ 0 & 0 & 0 & 3b-2a+c \end{bmatrix}.$$

From the last matrix we see that the original system is inconsistent if $3b - 2a + c \neq 0$. If $3b - 2a + c = 0$, the system is consistent and the solution is

$$x = \frac{(b+a)}{5} + \frac{2}{5}z,\ y = \frac{(2b-3a)}{5} + \frac{19}{5}z,$$

where z is arbitrary.

5. $\begin{bmatrix} 1 & 1 & 1 \\ t & 1 & t \\ 1+t & 2 & 3 \end{bmatrix} \begin{array}{c} R_2 \to R_2 - tR_1 \\ R_3 \to R_3 - (1+t)R_1 \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1-t & 0 \\ 0 & 1-t & 2-t \end{bmatrix}$

$$R_3 \to R_3 - R_2 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1-t & 0 \\ 0 & 0 & 2-t \end{bmatrix} = B.$$

Case 1. $t \neq 2$. No solution.

Case 2. $t = 2$. $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

We read off the unique solution $x = 1,\ y = 0$.

6. Method 1.

$$\begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix} \begin{array}{c} R_1 \to R_1 - R_4 \\ R_2 \to R_2 - R_4 \\ R_3 \to R_3 - R_4 \end{array} \begin{bmatrix} -4 & 0 & 0 & 4 \\ 0 & -4 & 0 & 4 \\ 0 & 0 & -4 & 4 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & -3 \end{bmatrix} R_4 \to R_4 - R_3 - R_2 - R_1 \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence the given homogeneous system has complete solution

$$x_1 = x_4,\ x_2 = x_4,\ x_3 = x_4,$$

with x_4 arbitrary.

Method 2. Write the system as

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 4x_1 \\ x_1 + x_2 + x_3 + x_4 &= 4x_2 \\ x_1 + x_2 + x_3 + x_4 &= 4x_3 \\ x_1 + x_2 + x_3 + x_4 &= 4x_4. \end{aligned}$$

Then it is immediate that any solution must satisfy $x_1 = x_2 = x_3 = x_4$. Conversely, if x_1, x_2, x_3, x_4 satisfy $x_1 = x_2 = x_3 = x_4$, we get a solution.

7.

$$\begin{bmatrix} \lambda - 3 & 1 \\ 1 & \lambda - 3 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & \lambda - 3 \\ \lambda - 3 & 1 \end{bmatrix}$$

$$R_2 \to R_2 - (\lambda - 3)R_1 \begin{bmatrix} 1 & \lambda - 3 \\ 0 & -\lambda^2 + 6\lambda - 8 \end{bmatrix} = B.$$

Case 1: $-\lambda^2 + 6\lambda - 8 \neq 0$. That is $-(\lambda - 2)(\lambda - 4) \neq 0$ or $\lambda \neq 2, 4$. Here B is row equivalent to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

$R_2 \to \frac{1}{-\lambda^2+6\lambda-8} R_2 \begin{bmatrix} 1 & \lambda - 3 \\ 0 & 1 \end{bmatrix} R_1 \to R_1 - (\lambda - 3)R_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Hence we get the trivial solution $x = 0,\ y = 0$.

Case 2: $\lambda = 2$. Then $B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ and the solution is $x = y$, with y arbitrary.

Case 3: $\lambda = 4$. Then $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and the solution is $x = -y$, with y arbitrary.

8.

$$\begin{aligned}
\begin{bmatrix} 3 & 1 & 1 & 1 \\ 5 & -1 & 1 & -1 \end{bmatrix} R_1 &\rightarrow \frac{1}{3}R_1 \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 5 & -1 & 1 & -1 \end{bmatrix} \\
R_2 &\rightarrow R_2 - 5R_1 \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{8}{3} \end{bmatrix} \\
R_2 &\rightarrow \frac{-3}{8}R_2 \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{4} & 1 \end{bmatrix} \\
R_1 &\rightarrow R_1 - \frac{1}{3}R_2 \begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{4} & 1 \end{bmatrix}.
\end{aligned}$$

Hence the solution of the associated homogeneous system is

$$x_1 = -\frac{1}{4}x_3, \; x_2 = -\frac{1}{4}x_3 - x_4,$$

with x_3 and x_4 arbitrary.

9.

$$A = \begin{bmatrix} 1-n & 1 & \cdots & 1 \\ 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1-n \end{bmatrix} \quad \begin{array}{c} R_1 \rightarrow R_1 - R_n \\ R_2 \rightarrow R_2 - R_n \\ \vdots \\ R_{n-1} \rightarrow R_{n-1} - R_n \end{array} \quad \begin{bmatrix} -n & 0 & \cdots & n \\ 0 & -n & \cdots & n \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1-n \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & \cdots & -1 \\ 0 & 1 & \cdots & -1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1-n \end{bmatrix} R_n \rightarrow R_n - R_{n-1} \cdots - R_1 \begin{bmatrix} 1 & 0 & \cdots & -1 \\ 0 & 1 & \cdots & -1 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The last matrix is in reduced row–echelon form.

Consequently the homogeneous system with coefficient matrix A has the solution

$$x_1 = x_n, \; x_2 = x_n, \ldots, x_{n-1} = x_n,$$

with x_n arbitrary.

Alternatively, writing the system in the form

$$\begin{aligned} x_1 + \cdots + x_n &= nx_1 \\ x_1 + \cdots + x_n &= nx_2 \\ &\vdots \\ x_1 + \cdots + x_n &= nx_n \end{aligned}$$

shows that any solution must satisfy $nx_1 = nx_2 = \cdots = nx_n$, so $x_1 = x_2 = \cdots = x_n$. Conversely if $x_1 = x_n, \ldots, x_{n-1} = x_n$, we see that $x_1, \ldots, x_n$ is a solution.

10. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and assume that $ad - bc \neq 0$.

Case 1: $a \neq 0$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} R_1 \to \tfrac{1}{a}R_1 \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix} R_2 \to R_2 - cR_1 \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix}$$

$$R_2 \to \tfrac{a}{ad-bc}R_2 \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix} R_1 \to R_1 - \tfrac{b}{a}R_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Case 2: $a = 0$. Then $bc \neq 0$ and hence $c \neq 0$.

$$A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} \to \begin{bmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So in both cases, A has reduced row–echelon form equal to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

11. We simplify the augmented matrix of the system using row operations:

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix}$$

$$\begin{matrix} R_3 \to R_3 - R_2 \\ R_2 \to \frac{-1}{7}R_2 \\ R_1 \to R_1 - 2R_2 \end{matrix} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix} R_1 \to R_1 - 2R_2 \begin{bmatrix} 1 & 0 & 1 & \frac{8}{7} \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}.$$

Denote the last matrix by B.

Case 1: $a^2 - 16 \neq 0$. i.e. $a \neq \pm 4$. Then

$$\begin{array}{l} R_3 \to \frac{1}{a^2-16} R_3 \\ R_1 \to R_1 - R_3 \\ R_2 \to R_2 + 2R_3 \end{array} \left[\begin{array}{cccc} 1 & 0 & 0 & \frac{8a+25}{7(a+4)} \\ 0 & 1 & 0 & \frac{10a+54}{7(a+4)} \\ 0 & 0 & 1 & \frac{1}{a+4} \end{array}\right]$$

and we get the unique solution

$$x = \frac{8a+25}{7(a+4)}, \; y = \frac{10a+54}{7(a+4)}, \; z = \frac{1}{a+4}.$$

Case 2: $a = -4$. Then $B = \left[\begin{array}{cccc} 1 & 0 & 1 & \frac{8}{7} \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & 0 & -8 \end{array}\right]$, so our system is inconsistent.

Case 3: $a = 4$. Then $B = \left[\begin{array}{cccc} 1 & 0 & 1 & \frac{8}{7} \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$. We read off that the system is consistent, with complete solution $x = \frac{8}{7} - z$, $y = \frac{10}{7} + 2z$, where z is arbitrary.

12. We reduce the augmented array of the system to reduced row–echelon form:

$$\left[\begin{array}{ccccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \quad R_3 \to R_3 + R_1 \quad \left[\begin{array}{ccccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$R_3 \to R_3 + R_2 \quad \left[\begin{array}{ccccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{c} R_1 \to R_1 + R_4 \\ R_3 \leftrightarrow R_4 \end{array} \quad \left[\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The last matrix is in reduced row–echelon form and we read off the solution of the corresponding homogeneous system:

$$\begin{aligned} x_1 &= -x_4 - x_5 = x_4 + x_5 \\ x_2 &= -x_4 - x_5 = x_4 + x_5 \\ x_3 &= -x_4 = x_4, \end{aligned}$$

where x_4 and x_5 are arbitrary elements of $\mathbb{Z}_2$. Hence there are four solutions:

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}.$$

13. (a) We reduce the augmented matrix to reduced row–echelon form:

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 1 & 4 & 1 \\ 3 & 1 & 2 & 0 \end{bmatrix} \quad R_1 \to 3R_1 \quad \begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 1 & 4 & 1 \\ 3 & 1 & 2 & 0 \end{bmatrix}$$

$$\begin{array}{c} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + 2R_1 \end{array} \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 4 & 3 & 3 \\ 0 & 2 & 0 & 4 \end{bmatrix} \quad R_2 \to 4R_2 \quad \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

$$\begin{array}{c} R_1 \to R_1 + 2R_2 \\ R_3 \to R_3 + 3R_2 \end{array} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{array}{c} R_1 \to R_1 + 2R_3 \\ R_2 \to R_2 + 3R_3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Consequently the system has the unique solution $x = 1$, $y = 2$, $z = 0$.

(b) Again we reduce the augmented matrix to reduced row–echelon form:

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 1 & 4 & 1 \\ 1 & 1 & 0 & 3 \end{bmatrix} \quad R_1 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 1 & 0 & 3 \\ 4 & 1 & 4 & 1 \\ 2 & 1 & 3 & 4 \end{bmatrix}$$

$$\begin{array}{c} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + 3R_1 \end{array} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 2 & 4 & 4 \\ 0 & 4 & 3 & 3 \end{bmatrix} \quad R_2 \to 3R_2 \quad \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 4 & 3 & 3 \end{bmatrix}$$

$$\begin{array}{c} R_1 \to R_1 + 4R_2 \\ R_3 \to R_3 + R_2 \end{array} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We read off the complete solution

$$\begin{aligned} x &= 1 - 3z = 1 + 2z \\ y &= 2 - 2z = 2 + 3z, \end{aligned}$$

where z is an arbitrary element of $\mathbb{Z}_5$.

14. Suppose that $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ are solutions of the system of linear equations

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad 1 \leq i \leq m.$$

Then

$$\sum_{j=1}^{n} a_{ij} \alpha_j = b_i \quad \text{and} \quad \sum_{j=1}^{n} a_{ij} \beta_j = b_i$$

for $1 \leq i \leq m$.

Let $\gamma_i = (1-t)\alpha_i + t\beta_i$ for $1 \leq i \leq m$. Then $(\gamma_1, \ldots, \gamma_n)$ is a solution of the given system. For

$$\begin{aligned}
\sum_{j=1}^{n} a_{ij} \gamma_j &= \sum_{j=1}^{n} a_{ij} \{(1-t)\alpha_j + t\beta_j\} \\
&= \sum_{j=1}^{n} a_{ij}(1-t)\alpha_j + \sum_{j=1}^{n} a_{ij} t\beta_j \\
&= (1-t)b_i + tb_i \\
&= b_i.
\end{aligned}$$

15. Suppose that $(\alpha_1, \ldots, \alpha_n)$ is a solution of the system of linear equations

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad 1 \leq i \leq m. \tag{1}$$

Then the system can be rewritten as

$$\sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} a_{ij} \alpha_j, \quad 1 \leq i \leq m,$$

or equivalently

$$\sum_{j=1}^{n} a_{ij}(x_j - \alpha_j) = 0, \quad 1 \leq i \leq m.$$

So we have

$$\sum_{j=1}^{n} a_{ij} y_j = 0, \quad 1 \leq i \leq m.$$

where $x_j - \alpha_j = y_j$. Hence $x_j = \alpha_j + y_j$, $1 \leq j \leq n$, where $(y_1, \ldots, y_n)$ is a solution of the associated homogeneous system. Conversely if $(y_1, \ldots, y_n)$

is a solution of the associated homogeneous system and $x_j = \alpha_j + y_j$, $1 \le j \le n$, then reversing the argument shows that $(x_1, \ldots, x_n)$ is a solution of the system 1 .

16. We simplify the augmented matrix using row operations, working towards row–echelon form:

$$\left[\begin{array}{ccccc} 1 & 1 & -1 & 1 & 1 \\ a & 1 & 1 & 1 & b \\ 3 & 2 & 0 & a & 1+a \end{array}\right] \begin{array}{c} R_2 \to R_2 - aR_1 \\ R_3 \to R_3 - 3R_1 \end{array} \left[\begin{array}{ccccc} 1 & 1 & -1 & 1 & 1 \\ 0 & 1-a & 1+a & 1-a & b-a \\ 0 & -1 & 3 & a-3 & a-2 \end{array}\right]$$

$$\begin{array}{c} R_2 \leftrightarrow R_3 \\ R_2 \to -R_2 \end{array} \left[\begin{array}{ccccc} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -3 & 3-a & 2-a \\ 0 & 1-a & 1+a & 1-a & b-a \end{array}\right]$$

$$R_3 \to R_3 + (a-1)R_2 \left[\begin{array}{ccccc} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -3 & 3-a & 2-a \\ 0 & 0 & 4-2a & (1-a)(a-2) & -a^2+2a+b-2 \end{array}\right] = B.$$

Case 1: $a \neq 2$. Then $4 - 2a \neq 0$ and

$$B \to \left[\begin{array}{ccccc} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -3 & 3-a & 2-a \\ 0 & 0 & 1 & \frac{a-1}{2} & \frac{-a^2+2a+b-2}{4-2a} \end{array}\right].$$

Hence we can solve for x, y and z in terms of the arbitrary variable w.

Case 2: $a = 2$. Then

$$B = \left[\begin{array}{ccccc} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & b-2 \end{array}\right].$$

Hence there is no solution if $b \neq 2$. However if $b = 2$, then

$$B = \left[\begin{array}{ccccc} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccccc} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and we get the solution $x = 1 - 2z$, $y = 3z - w$, where w is arbitrary.

17. (a) We first prove that $1 + 1 + 1 + 1 = 0$. Observe that the elements

$$1 + 0, \quad 1 + 1, \quad 1 + a, \quad 1 + b$$

are distinct elements of F by virtue of the *cancellation law for addition*. For this law states that $1+x=1+y \Rightarrow x=y$ and hence $x \neq y \Rightarrow 1+x \neq 1+y$.

Hence the above four elements are just the elements $0,\ 1,\ a,\ b$ in some order. Consequently

$$\begin{aligned}(1+0)+(1+1)+(1+a)+(1+b) &= 0+1+a+b\\ (1+1+1+1)+(0+1+a+b) &= 0+(0+1+a+b),\end{aligned}$$

so $1+1+1+1=0$ after cancellation.

Now $1+1+1+1=(1+1)(1+1)$, so we have $x^2=0$, where $x=1+1$. Hence $x=0$. Then $a+a=a(1+1)=a\cdot 0=0$.

Next $a+b=1$. For $a+b$ must be one of 0, 1, a, b. Clearly we can't have $a+b=a$ or b; also if $a+b=0$, then $a+b=a+a$ and hence $b=a$; hence $a+b=1$. Then

$$a+1=a+(a+b)=(a+a)+b=0+b=b.$$

Similarly $b+1=a$. Consequently the addition table for F is

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

.

We now find the multiplication table. First, ab must be one of 1, a, b; however we can't have $ab=a$ or b, so this leaves $ab=1$.

Next $a^2=b$. For a^2 must be one of 1, a, b; however $a^2=a \Rightarrow a=0$ or $a=1$; also

$$a^2=1 \Rightarrow a^2-1=0 \Rightarrow (a-1)(a+1)=0 \Rightarrow (a-1)^2=0 \Rightarrow a=1;$$

hence $a^2=b$. Similarly $b^2=a$. Consequently the multiplication table for F is

×	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

.

(b) We use the addition and multiplication tables for F:

$$A=\begin{bmatrix}1&a&b&a\\a&b&b&1\\1&1&1&a\end{bmatrix}\quad\begin{array}{c}R_2\to R_2+aR_1\\R_3\to R_3+R_1\end{array}\quad\begin{bmatrix}1&a&b&a\\0&0&a&a\\0&b&a&0\end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & a & b & a \\ 0 & b & a & 0 \\ 0 & 0 & a & a \end{bmatrix} \quad \begin{array}{l} R_2 \to aR_2 \\ R_3 \to bR_3 \end{array} \quad \begin{bmatrix} 1 & a & b & a \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_1 + aR_2 \quad \begin{bmatrix} 1 & 0 & a & a \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 + aR_3 \\ R_2 \to R_2 + bR_3 \end{array} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The last matrix is in reduced row–echelon form.

Section 2.4

2. Suppose $B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ and that $AB = I_2$. Then

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -a+e & -b+f \\ c+e & d+f \end{bmatrix}.$$

Hence

$$\begin{array}{c} -a+e=1 \\ c+e=0 \end{array}, \quad \begin{array}{c} -b+f=0 \\ d+f=1 \end{array};$$

$$\begin{array}{c} e=a+1 \\ c=-e=-(a+1) \end{array}, \quad \begin{array}{c} f=b \\ d=1-f=1-b \end{array};$$

$$B = \begin{bmatrix} a & b \\ -a-1 & 1-b \\ a+1 & b \end{bmatrix}.$$

Next,

$$(BA)^2B = (BA)(BA)B = B(AB)(AB) = BI_2I_2 = BI_2 = B.$$

4. Let p_n denote the statement

$$A^n = \tfrac{(3^n-1)}{2}A + \tfrac{(3-3^n)}{2}I_2.$$

Then p_1 asserts that $A = \frac{(3-1)}{2}A + \frac{(3-3)}{2}I_2$, which is true. So let $n \geq 1$ and assume p_n. Then from (1),

$$\begin{aligned} A^{n+1} &= A \cdot A^n = A\left\{\tfrac{(3^n-1)}{2}A + \tfrac{(3-3^n)}{2}I_2\right\} = \tfrac{(3^n-1)}{2}A^2 + \tfrac{(3-3^n)}{2}A \\ &= \tfrac{(3^n-1)}{2}(4A - 3I_2) + \tfrac{(3-3^n)}{2}A = \tfrac{(3^n-1)4+(3-3^n)}{2}A + \tfrac{(3^n-1)(-3)}{2}I_2 \\ &= \tfrac{(4\cdot3^n-3^n)-1}{2}A + \tfrac{(3-3^{n+1})}{2}I_2 \\ &= \tfrac{(3^{n+1}-1)}{2}A + \tfrac{(3-3^{n+1})}{2}I_2. \end{aligned}$$

Hence p_{n+1} is true and the induction proceeds.

5. The equation $x_{n+1} = ax_n + bx_{n-1}$ is seen to be equivalent to

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

or

$$X_n = AX_{n-1},$$

where $X_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$. Then

$$X_n = A^n X_0$$

if $n \geq 1$. Hence by Question 3,

$$\begin{aligned}
\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} &= \left\{ \frac{(3^n - 1)}{2} A + \frac{(3 - 3^n)}{2} I_2 \right\} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} \\
&= \left\{ \frac{(3^n - 1)}{2} \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} \frac{3-3^n}{2} & 0 \\ 0 & \frac{3-3^n}{2} \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} \\
&= \begin{bmatrix} (3^n - 1)2 + \frac{3-3^n}{2} & \frac{(3^n-1)(-3)}{2} \\ \frac{3^n-1}{2} & \frac{3-3^n}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}
\end{aligned}$$

Hence, equating the $(2, 1)$ elements gives

$$x_n = \frac{(3^n - 1)}{2} x_1 + \frac{(3 - 3^n)}{2} x_0 \qquad \text{if } n \geq 1$$

7. Note: $\lambda_1 + \lambda_2 = a + d$ and $\lambda_1\lambda_2 = ad - bc$.
Then

$$\begin{aligned}
(\lambda_1 + \lambda_2)k_n - \lambda_1\lambda_2 k_{n-1} &= (\lambda_1 + \lambda_2)(\lambda_1^{n-1} + \lambda_1^{n-2}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-2} + \lambda_2^{n-1}) \\
&\quad -\lambda_1\lambda_2(\lambda_1^{n-2} + \lambda_1^{n-3}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-3} + \lambda_2^{n-2})
\end{aligned}$$

$$\begin{aligned}
&= (\lambda_1^n + \lambda_1^{n-1}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-1}) \\
&\quad +(\lambda_1^{n-1}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-1} + \lambda_2^n) \\
&\quad -(\lambda_1^{n-1}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-1}) \\
&= \lambda_1^n + \lambda_1^{n-1}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-1} + \lambda_2^n = k_{n+1}
\end{aligned}$$

If $\lambda_1 = \lambda_2$, we see

$$\begin{aligned}
k_n &= \lambda_1^{n-1} + \lambda_1^{n-2}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-2} + \lambda_2^{n-1} \\
&= \lambda_1^{n-1} + \lambda_1^{n-2}\lambda_1 + \cdots + \lambda_1\lambda_1^{n-2} + \lambda_1^{n-1} \\
&= n\lambda_1^{n-1}
\end{aligned}$$

If $\lambda_1 \neq \lambda_2$, we see that

$$\begin{aligned}
(\lambda_1 - \lambda_2)k_n &= (\lambda_1 - \lambda_2)(\lambda_1^{n-1} + \lambda_1^{n-2}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-2} + \lambda_2^{n-1}) \\
&= \lambda_1^n + \lambda_1^{n-1}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-1} \\
&\quad -(\lambda_1^{n-1}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-1} + \lambda_2^n) \\
&= \lambda_1^n - \lambda_2^n.
\end{aligned}$$

Hence $k_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$.

We have to prove

$$A^n = k_nA - \lambda_1\lambda_2k_{n-1}I_2. \quad *$$

n=1:

$$\begin{aligned}
A^1 = A;\ \text{also } k_1A - \lambda_1\lambda_2k_0I_2 &= k_1A - \lambda_1\lambda_2 0I_2 \\
&= A.
\end{aligned}$$

Let $n \geq 1$ and assume equation $*$ holds. Then

$$\begin{aligned}
A^{n+1} = A^n \cdot A &= (k_nA - \lambda_1\lambda_2k_{n-1}I_2)A \\
&= k_nA^2 - \lambda_1\lambda_2k_{n-1}A.
\end{aligned}$$

Now $A^2 = (a+d)A - (ad-bc)I_2 = (\lambda_1 + \lambda_2)A - \lambda_1\lambda_2I_2$. Hence

$$\begin{aligned}
A^{n+1} &= k_n(\lambda_1 + \lambda_2)A - \lambda_1\lambda_2I_2 - \lambda_1\lambda_2k_{n-1}A \\
&= \{k_n(\lambda_1 + \lambda_2) - \lambda_1\lambda_2k_{n-1}\}A - \lambda_1\lambda_2k_nI_2 \\
&= k_{n+1}A - \lambda_1\lambda_2k_nI_2,
\end{aligned}$$

and the induction goes through.

8. Here λ_1, λ_2 are the roots of the polynomial $x^2 - 2x - 3 = (x-3)(x+1)$. So we can take $\lambda_1 = 3, \lambda_2 = -1$. Then

$$k_n = \frac{3^n - (-1)^n}{3 - (-1)} = \frac{3^n + (-1)^{n+1}}{4}.$$

Hence

$$\begin{aligned}
A^n &= \left\{\frac{3^n + (-1)^{n+1}}{4}\right\}A - (-3)\left\{\frac{3^{n-1} + (-1)^n}{4}\right\}I_2 \\
&= \frac{3^n + (-1)^{n+1}}{4}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + 3\left\{\frac{3^{n-1} + (-1)^n}{4}\right\}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\end{aligned}$$

which is equivalent to the stated result.

9. In terms of matrices, we have

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} \text{ for } n \geq 1.$$

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Now λ_1, λ_2 are the roots of the polynomial $x^2 - x - 1$ here.
Hence $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$ and

$$\begin{aligned} k_n &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\frac{1+\sqrt{5}}{2} - \left(\frac{1-\sqrt{5}}{2}\right)} \\ &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}}. \end{aligned}$$

Hence

$$\begin{aligned} A^n &= k_n A - \lambda_1 \lambda_2 k_{n-1} I_2 \\ &= k_n A + k_{n-1} I_2 \end{aligned}$$

So

$$\begin{aligned} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} &= (k_n A + k_{n-1} I_2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= k_n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + k_{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} k_n + k_{n-1} \\ k_n \end{bmatrix}. \end{aligned}$$

Hence

$$F_n = k_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}}.$$

10. From Question 5, we know that

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & r \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} a \\ b \end{bmatrix}.$$

Now by Question 7, with $A = \begin{bmatrix} 1 & r \\ 1 & 1 \end{bmatrix}$,

$$\begin{aligned} A^n &= k_n A - \lambda_1\lambda_2 k_{n-1} I_2 \\ &= k_n A - (1-r)k_{n-1} I_2, \end{aligned}$$

where $\lambda_1 = 1 + \sqrt{r}$ and $\lambda_2 = 1 - \sqrt{r}$ are the roots of the polynomial $x^2 - 2x + (1-r)$ and

$$k_n = \frac{\lambda_1^n - \lambda_2^n}{2\sqrt{r}}.$$

Hence

$$\begin{aligned} \begin{bmatrix} x_n \\ y_n \end{bmatrix} &= (k_n A - (1-r)k_{n-1} I_2) \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \left(\begin{bmatrix} k_n & k_n r \\ k_n & k_n \end{bmatrix} - \begin{bmatrix} (1-r)k_{n-1} & 0 \\ 0 & (1-r)k_{n-1} \end{bmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} k_n - (1-r)k_{n-1} & k_n r \\ k_n & k_n - (1-r)k_{n-1} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} a(k_n - (1-r)k_{n-1}) + bk_n r \\ ak_n + b(k_n - (1-r)k_{n-1}) \end{bmatrix}. \end{aligned}$$

Hence, in view of the fact that

$$\frac{k_n}{k_{n-1}} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1^{n-1} - \lambda_2^{n-1}} = \frac{\lambda_1^n (1 - \{\frac{\lambda_2}{\lambda_1}\}^n)}{\lambda_1^{n-1}(1 - \{\frac{\lambda_2}{\lambda_1}\}^{n-1})} \to \lambda_1, \quad \text{as } n \to \infty,$$

we have

$$\begin{aligned} \begin{bmatrix} x_n \\ y_n \end{bmatrix} &= \frac{a(k_n - (1-r)k_{n-1}) + bk_n r}{ak_n + b(k_n - (1-r)k_{n-1})} \\ &= \frac{a(\frac{k_n}{k_{n-1}} - (1-r)) + b\frac{k_n}{k_{n-1}} r}{a\frac{k_n}{k_{n-1}} + b(\frac{k_n}{k_{n-1}} - (1-r))} \\ &\to \frac{a(\lambda_1 - (1-r)) + b\lambda_1 r}{a\lambda_1 + b(\lambda_1 - (1-r))} \\ &= \frac{a(\sqrt{r} + r) + b(1+\sqrt{r})r}{a(1+\sqrt{r}) + b(\sqrt{r} + r)} \\ &= \frac{\sqrt{r}\{a(1+\sqrt{r}) + b(1+\sqrt{r})\sqrt{r}\}}{a(1+\sqrt{r}) + b(\sqrt{r}+r)} \\ &= \sqrt{r}. \end{aligned}$$

Section 2.7

1. $[A|I_2] = \left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{array}\right]$ $R_2 \to R_2 + 3R_1$ $\left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ 0 & 13 & 3 & 1 \end{array}\right]$

$R_2 \to \frac{1}{13}R_2$ $\left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ 0 & 1 & 3/13 & 1/13 \end{array}\right]$ $R_1 \to R_1 - 4R_2$ $\left[\begin{array}{rr|rr} 1 & 0 & 1/13 & -4/13 \\ 0 & 1 & 3/13 & 1/13 \end{array}\right]$.

Hence A is non–singular and $A^{-1} = \begin{bmatrix} 1/13 & -4/13 \\ 3/13 & 1/13 \end{bmatrix}$.

Moreover

$$E_{12}(-4)E_2(1/13)E_{21}(3)A = I_2,$$

so

$$A^{-1} = E_{12}(-4)E_2(1/13)E_{21}(3).$$

Hence

$$A = \{E_{21}(3)\}^{-1}\{E_2(1/13)\}^{-1}\{E_{12}(-4)\}^{-1} = E_{21}(-3)E_2(13)E_{12}(4).$$

2. Let $D = [d_{ij}]$ be an $m \times m$ diagonal matrix and let $A = [a_{jk}]$ be an $m \times n$ matrix. Then

$$(DA)_{ik} = \sum_{j=1}^{n} d_{ij}a_{jk} = d_{ii}a_{ik},$$

as $d_{ij} = 0$ if $i \neq j$. It follows that the ith row of DA is obtained by multiplying the ith row of A by d_{ii}.

Similarly, post–multiplication of a matrix by a diagonal matrix D results in a matrix whose columns are those of A, multiplied by the respective diagonal elements of D.

In particular,

$$\mathrm{diag}\,(a_1, \ldots, a_n)\mathrm{diag}\,(b_1, \ldots, b_n) = \mathrm{diag}\,(a_1b_1, \ldots, a_nb_n),$$

as the left–hand side can be regarded as pre–multiplication of the matrix $\mathrm{diag}\,(b_1, \ldots, b_n)$ by the diagonal matrix $\mathrm{diag}\,(a_1, \ldots, a_n)$.

Finally, suppose that each of $a_1, \ldots, a_n$ is non–zero. Then $a_1^{-1}, \ldots, a_n^{-1}$ all exist and we have

$$\begin{aligned} \mathrm{diag}\,(a_1, \ldots, a_n)\mathrm{diag}\,(a_1^{-1}, \ldots, a_n^{-1}) &= \mathrm{diag}\,(a_1a_1^{-1}, \ldots, a_na_n^{-1}) \\ &= \mathrm{diag}\,(1, \ldots, 1) = I_n. \end{aligned}$$

Hence $\mathrm{diag}\,(a_1, \ldots, a_n)$ is non–singular and its inverse is $\mathrm{diag}\,(a_1^{-1}, \ldots, a_n^{-1})$.

Next suppose that $a_i = 0$. Then diag $(a_1, \ldots, a_n)$ is row–equivalent to a matix containing a zero row and is hence singular.

3. $[A|I_3] = \left[\begin{array}{ccc|ccc} 0 & 0 & 2 & 1 & 0 & 0 \\ 1 & 2 & 6 & 0 & 1 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_2 \quad \left[\begin{array}{cccccc} 1 & 2 & 6 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{array}\right]$

$R_3 \to R_3 - 3R_1 \quad \left[\begin{array}{cccccc} 1 & 2 & 6 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -9 & 0 & -3 & 1 \end{array}\right] \quad R_2 \leftrightarrow R_3 \quad \left[\begin{array}{cccccc} 1 & 2 & 6 & 0 & 1 & 0 \\ 0 & 1 & -9 & 0 & -3 & 1 \\ 0 & 0 & 2 & 1 & 0 & 0 \end{array}\right]$

$R_3 \to \frac{1}{2}R_3 \quad \left[\begin{array}{cccccc} 1 & 2 & 6 & 0 & 1 & 0 \\ 0 & 1 & -9 & 0 & -3 & 1 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \end{array}\right] \quad R_1 \to R_1 - 2R_2 \quad \left[\begin{array}{cccccc} 1 & 0 & 24 & 0 & 7 & -2 \\ 0 & 1 & -9 & 0 & -3 & 1 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \end{array}\right]$

$\begin{array}{c} R_1 \to R_1 - 24R_3 \\ R_2 \to R_2 + 9R_3 \end{array} \quad \left[\begin{array}{cccccc} 1 & 0 & 0 & -12 & 7 & -2 \\ 0 & 1 & 0 & 9/2 & -3 & 1 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \end{array}\right].$

Hence A is non–singular and $A^{-1} = \left[\begin{array}{ccc} -12 & 7 & -2 \\ 9/2 & -3 & 1 \\ 1/2 & 0 & 0 \end{array}\right]$.

Also

$$E_{23}(9)E_{13}(-24)E_{12}(-2)E_3(1/2)E_{23}E_{31}(-3)E_{12}A = I_3.$$

Hence

$$A^{-1} = E_{23}(9)E_{13}(-24)E_{12}(-2)E_3(1/2)E_{23}E_{31}(-3)E_{12},$$

so

$$A = E_{12}E_{31}(3)E_{23}E_3(2)E_{12}(2)E_{13}(24)E_{23}(-9).$$

4.

$$A = \left[\begin{array}{ccc} 1 & 2 & k \\ 3 & -1 & 1 \\ 5 & 3 & -5 \end{array}\right] \to \left[\begin{array}{ccc} 1 & 2 & k \\ 0 & -7 & 1-3k \\ 0 & -7 & -5-5k \end{array}\right] \to \left[\begin{array}{ccc} 1 & 2 & k \\ 0 & -7 & 1-3k \\ 0 & 0 & -6-2k \end{array}\right] = B.$$

Hence if $-6 - 2k \neq 0$, i.e. if $k \neq -3$, we see that B can be reduced to I_3 and hence A is non–singular.

If $k = -3$, then $B = \left[\begin{array}{ccc} 1 & 2 & -3 \\ 0 & -7 & 10 \\ 0 & 0 & 0 \end{array}\right] = B$ and consequently A is singular, as it is row–equivalent to a matrix containing a zero row.

5. $E_{21}(2)\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. Hence, as in the previous question, $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$ is singular.

6. Starting from the equation $A^2 - 2A + 13I_2 = 0$, we deduce

$$A(A - 2I_2) = -13I_2 = (A - 2I_2)A.$$

Hence $AB = BA = I_2$, where $B = \frac{-1}{13}(A - 2I_2)$. Consequently A is non–singular and $A^{-1} = B$.

7. We assume the equation $A^3 = 3A^2 - 3A + I_3$.

$$\begin{aligned} \text{(ii)}\quad A^4 &= A^3A = (3A^2 - 3A + I_3)A = 3A^3 - 3A^2 + A \\ &= 3(3A^2 - 3A + I_3) - 3A^2 + A = 6A^2 - 8A + 3I_3. \end{aligned}$$

(iii) $A^3 - 3A^2 + 3A = I_3$. Hence

$$A(A^2 - 3A + 3I_3) = I_3 = (A^2 - 3A + 3I_3)A.$$

Hence A is non–singular and

$$\begin{aligned} A^{-1} &= A^2 - 3A + 3I_3 \\ &= \begin{bmatrix} -1 & -3 & 1 \\ 2 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \end{aligned}$$

8. (i) If $B^3 = 0$ then

$$\begin{aligned} (I_n - B)(I_n + B + B^2) &= I_n(I_n + B + B^2) - B(I_n + B + B^2) \\ &= (I_n + B + B^2) - (B + B^2 + B^3) \\ &= I_n - B^3 = I_n - 0 = I_n. \end{aligned}$$

Similarly $(I_n + B + B^2)(I_n - B) = I_n$.

Hence $A = I_n - B$ is non–singular and $A^{-1} = I_n + B + B^2$.

It follows that the system $AX = b$ has the unique solution

$$X = A^{-1}b = (I_n + B + B^2)b = b + Bb + B^2b.$$

(ii) Let $B = \begin{bmatrix} 0 & r & s \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix}$. Then $B^2 = \begin{bmatrix} 0 & 0 & rt \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B^3 = 0$. Hence from the preceding question

$$\begin{aligned} (I_3 - B)^{-1} &= I_3 + B + B^2 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & r & s \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & rt \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & r & s+rt \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

9. (i) Suppose that $A^2 = 0$. Then if A^{-1} exists, we deduce that $A^{-1}(AA) = A^{-1}0$, which gives $A = 0$ and this is a contradiction, as the zero matrix is singular. We conclude that A does not have an inverse.

(ii). Suppose that $A^2 = A$ and that A^{-1} exists. Then

$$A^{-1}(AA) = A^{-1}A,$$

which gives $A = I_n$. Equivalently, if $A^2 = A$ and $A \neq I_n$, then A does not have an inverse.

10. The system of linear equations

$$\begin{aligned} x + y - z &= a \\ z &= b \\ 2x + y + 2z &= c \end{aligned}$$

is equivalent to the matrix equation $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

By Question 7, A^{-1} exists and hence the system has the unique solution

$$X = \begin{bmatrix} -1 & -3 & 1 \\ 2 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -a - 3b + c \\ 2a + 4b - c \\ b \end{bmatrix}.$$

Hence $x = -a - 3b + c,\ y = 2a + 4b - c,\ z = b$.

12.

$$\begin{aligned} A &= E_3(2)E_{14}E_{42}(3) = E_3(2)E_{14}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix} \\ &= E_3(2)\begin{bmatrix} 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Also

$$\begin{aligned} A^{-1} &= (E_3(2)E_{14}E_{42}(3))^{-1} \\ &= (E_{42}(3))^{-1}E_{14}^{-1}(E_3(2))^{-1} \\ &= E_{42}(-3)E_{14}E_3(1/2) \\ &= E_{42}(-3)E_{14}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= E_{42}(-3)\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 1 & -3 & 0 & 0 \end{bmatrix}. \end{aligned}$$

13. (All matrices in this question are over $\mathbb{Z}_2$.)

(a)

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\rightarrow \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right].$$

Hence A is non–singular and

$$A^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

(b) $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \quad R_4 \rightarrow R_4 + R_1 \quad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so A is singular.

14.

(a) $$\left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{c} R_3 \rightarrow \frac{1}{2}R_3 \\ R_1 \rightarrow R_1 - R_3 \\ R_2 \rightarrow R_2 + R_3 \\ R_1 \leftrightarrow R_3 \end{array} \quad \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1 & 1/2 \\ 0 & 1 & 1 & 1 & 0 & -1/2 \end{array}\right]$$

$$R_3 \rightarrow R_3 - R_2 \quad \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right].$$

Hence A^{-1} exists and

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 1 & -1 & -1 \end{bmatrix}.$$

(b) $$\left[\begin{array}{rrr|rrr} 2 & 2 & 4 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{c} R_1 \rightarrow R_1 - 2R_2 \\ R_1 \leftrightarrow R_2 \\ R_2 \leftrightarrow R_3 \end{array} \quad \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 1 & -2 & 0 \end{array}\right]$$

$$R_3 \rightarrow R_3 - 2R_2 \quad \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & -2 & -2 \end{array}\right]$$

$$R_3 \rightarrow \tfrac{1}{2}R_3 \quad \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & -1 & -1 \end{array}\right]$$

$$R_1 \to R_1 - R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & -1 & -1 \end{array}\right].$$

Hence A^{-1} exists and

$$A^{-1} = \begin{bmatrix} -1/2 & 2 & 1 \\ 0 & 0 & 1 \\ 1/2 & -1 & -1 \end{bmatrix}.$$

(c) $$\begin{bmatrix} 4 & 6 & -3 \\ 0 & 0 & 7 \\ 0 & 0 & 5 \end{bmatrix} \begin{array}{c} R_2 \to \frac{1}{7}R_2 \\ R_3 \to \frac{1}{5}R_3 \end{array} \begin{bmatrix} 4 & 6 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \to R_3 - R_2 \quad \begin{bmatrix} 4 & 6 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence A is singular by virtue of the zero row.

(d) $$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & -5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 \to \frac{1}{2}R_1 \\ R_2 \to \frac{-1}{5}R_2 \\ R_3 \to \frac{1}{7}R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/7 \end{array}\right].$$

Hence A^{-1} exists and $A^{-1} = \mathrm{diag}\,(1/2,\ -1/5,\ 1/7)$.

(Of course this was also immediate from Question 2.)

(e) $$\left[\begin{array}{cccc|cccc} 1 & 2 & 4 & 6 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_1 \to R_1 - 2R_2 \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 6 & 1 & -2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 \to R_2 - 2R_3 \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 6 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_1 \to R_1 - 3R_4 \\ R_2 \to R_2 + 2R_4 \\ R_3 \to R_3 - R_4 \\ R_4 \to \frac{1}{2}R_4 \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/2 \end{array}\right].$$

Hence A^{-1} exists and

$$A^{-1} = \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}.$$

(f)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 5R_1 \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -3 & -6 \end{bmatrix} \; R_3 \to R_3 - R_2 \; \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence A is singular by virtue of the zero row.

15. Suppose that A is non–singular. Then

$$AA^{-1} = I_n = A^{-1}A.$$

Taking transposes throughout gives

$$\begin{aligned} (AA^{-1})^t &= I_n^t = (A^{-1}A)^t \\ (A^{-1})^t A^t &= I_n = A^t(A^{-1})^t, \end{aligned}$$

so A^t is non–singular and $(A^t)^{-1} = (A^{-1})^t$.

16. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $ad - bc = 0$. Then the equation

$$A^2 - (a+d)A + (ad - bc)I_2 = 0$$

reduces to $A^2 - (a+d)A = 0$ and hence $A^2 = (a+d)A$. From the last equation, if A^{-1} exists, we deduce that $A = (a+d)I_2$, or

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+d & 0 \\ 0 & a+d \end{bmatrix}.$$

Hence $a = a+d,\ b = 0,\ c = 0,\ d = a+d$ and $a = b = c = d = 0$, which contradicts the assumption that A is non–singular.

17.

$$\begin{aligned} A = \begin{bmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{bmatrix} \quad & \begin{array}{l} R_2 \to R_2 + aR_1 \\ R_3 \to R_3 + bR_1 \end{array} \begin{bmatrix} 1 & a & b \\ 0 & 1+a^2 & c+ab \\ 0 & ab-c & 1+b^2 \end{bmatrix} \\ & R_2 \to \tfrac{1}{1+a^2}R_2 \begin{bmatrix} 1 & a & b \\ 0 & 1 & \frac{c+ab}{1+a^2} \\ 0 & ab-c & 1+b^2 \end{bmatrix} \\ & R_3 \to R_3 - (ab-c)R_2 \begin{bmatrix} 1 & a & b \\ 0 & 1 & \frac{c+ab}{1+a^2} \\ 0 & 0 & 1+b^2+\frac{(c-ab)(c+ab)}{1+a^2} \end{bmatrix} = B. \end{aligned}$$

Now

$$\begin{aligned} 1+b^2+\frac{(c-ab)(c+ab)}{1+a^2} &= 1+b^2+\frac{c^2-(ab)^2}{1+a^2} \\ &= \frac{1+a^2+b^2+c^2}{1+a^2} \neq 0. \end{aligned}$$

Hence B can be reduced to I_3 using four more row operations and consequently A is non–singular.

18. The proposition is clearly true when $n = 1$. So let $n \geq 1$ and assume $(P^{-1}AP)^n = P^{-1}A^nP$. Then

$$\begin{aligned} (P^{-1}AP)^{n+1} &= (P^{-1}AP)^n(P^{-1}AP) \\ &= (P^{-1}A^nP)(P^{-1}AP) \\ &= P^{-1}A^n(PP^{-1})AP \\ &= P^{-1}A^nIAP \\ &= P^{-1}(A^nA)P \\ &= P^{-1}A^{n+1}P \end{aligned}$$

and the induction goes through.

19. Let $A = \begin{bmatrix} 2/3 & 1/4 \\ 1/3 & 3/4 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$. Then $P^{-1} = \frac{1}{7}\begin{bmatrix} 4 & -3 \\ 1 & 1 \end{bmatrix}$. We then verify that $P^{-1}AP = \begin{bmatrix} 5/12 & 0 \\ 0 & 1 \end{bmatrix}$. Then from the previous question,

$$P^{-1}A^nP = (P^{-1}AP)^n = \begin{bmatrix} 5/12 & 0 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} (5/12)^n & 0 \\ 0 & 1^n \end{bmatrix} = \begin{bmatrix} (5/12)^n & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence

$$\begin{aligned} A^n &= P\begin{bmatrix} (5/12)^n & 0 \\ 0 & 1 \end{bmatrix}P^{-1} = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} (5/12)^n & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 4 & -3 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} (5/12)^n & 3 \\ -(5/12)^n & 4 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 4(5/12)^n+3 & (-3)(5/12)^n+3 \\ -4(5/12)^n+4 & 3(5/12)^n+4 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} + \frac{1}{7}(5/12)^n\begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix}. \end{aligned}$$

Notice that $A^n \to \frac{1}{7}\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}$ as $n \to \infty$. This problem is a special case of a more general result about Markov matrices.

20. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix whose elements are non–negative real numbers satisfying

$$a \geq 0,\ b \geq 0,\ c \geq 0,\ d \geq 0,\ a + c = 1 = b + d.$$

Also let $P = \begin{bmatrix} b & 1 \\ c & -1 \end{bmatrix}$ and suppose that $A \neq I_2$.

(i) $\det P = -b - c = -(b + c)$. Now $b + c \geq 0$. Also if $b + c = 0$, then we would have $b = c = 0$ and hence $d = a = 1$, resulting in $A = I_2$. Hence $\det P < 0$ and P is non–singular.

Next,

$$\begin{aligned} P^{-1}AP &= \frac{-1}{b+c}\begin{bmatrix} -1 & -1 \\ -c & b \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} b & 1 \\ c & -1 \end{bmatrix} \\ &= \frac{-1}{b+c}\begin{bmatrix} -a-c & -b-d \\ -ac+bc & -cb+bd \end{bmatrix}\begin{bmatrix} b & 1 \\ c & -1 \end{bmatrix} \\ &= \frac{-1}{b+c}\begin{bmatrix} -1 & -1 \\ -ac+bc & -cb+bd \end{bmatrix}\begin{bmatrix} b & 1 \\ c & -1 \end{bmatrix} \\ &= \frac{-1}{b+c}\begin{bmatrix} -b-c & 0 \\ (-ac+bc)b+(-cb+bd)c & -ac+bc+cb-bd \end{bmatrix}. \end{aligned}$$

Now

$$\begin{aligned} -acb + b^2c - c^2b + bdc &= -cb(a+c) + bc(b+d) \\ &= -cb + bc = 0. \end{aligned}$$

Also

$$\begin{aligned} -(a+d-1)(b+c) &= -ab - ac - db - dc + b + c \\ &= -ac + b(1-a) + c(1-d) - bd \\ &= -ac + bc + cb - bd. \end{aligned}$$

Hence

$$P^{-1}AP = \frac{-1}{b+c}\begin{bmatrix} -(b+c) & 0 \\ 0 & -(a+d-1)(b+c) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & a+d-1 \end{bmatrix}.$$

(ii) We next prove that if we impose the extra restriction that $A \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $|a+d-1| < 1$. This will then have the following consequence:

$$\begin{aligned}
A &= P \begin{bmatrix} 1 & 0 \\ 0 & a+d-1 \end{bmatrix} P^{-1} \\
A^n &= P \begin{bmatrix} 1 & 0 \\ 0 & a+d-1 \end{bmatrix}^n P^{-1} \\
&= P \begin{bmatrix} 1 & 0 \\ 0 & (a+d-1)^n \end{bmatrix} P^{-1} \\
&\to P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \\
&= \begin{bmatrix} b & 1 \\ c & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{-1}{b+c} \begin{bmatrix} -1 & -1 \\ -c & b \end{bmatrix} \\
&= \frac{-1}{b+c} \begin{bmatrix} b & 0 \\ c & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -c & b \end{bmatrix} \\
&= \frac{-1}{b+c} \begin{bmatrix} -b & -b \\ -c & -c \end{bmatrix} \\
&= \frac{1}{b+c} \begin{bmatrix} b & b \\ c & c \end{bmatrix},
\end{aligned}$$

where we have used the fact that $(a+d-1)^n \to 0$ as $n \to \infty$.

We first prove the inequality $|a+d-1| \leq 1$:

$$\begin{aligned}
a+d-1 &\leq 1+d-1 = d \leq 1 \\
a+d-1 &\geq 0+0-1 = -1.
\end{aligned}$$

Next, if $a+d-1 = 1$, we have $a+d = 2$; so $a = 1 = d$ and hence $c = 0 = b$, contradicting our assumption that $A \neq I_2$. Also if $a+d-1 = -1$, then $a+d = 0$; so $a = 0 = d$ and hence $c = 1 = b$ and hence $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

22. The system is inconsistent: We work towards reducing the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 1 & 5 \\ 3 & 5 & 12 \end{array}\right] \begin{array}{c} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{array} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{array}\right]$$

$$R_3 \to R_3 - R_2 \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{array}\right].$$

The last row reveals inconsistency.

The system in matrix form is $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 5 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 5 \\ 12 \end{bmatrix}.$$

The normal equations are given by the matrix equation

$$A^t AX = A^t B.$$

Now

$$\begin{aligned} A^t A &= \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 11 & 18 \\ 18 & 30 \end{bmatrix} \\ A^t B &= \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 12 \end{bmatrix} = \begin{bmatrix} 45 \\ 73 \end{bmatrix}. \end{aligned}$$

Hence the normal equations are

$$\begin{aligned} 11x + 18y &= 45 \\ 18x + 30y &= 73. \end{aligned}$$

These may be solved, for example, by Cramer's rule:

$$\begin{aligned} x &= \frac{\begin{vmatrix} 45 & 18 \\ 73 & 30 \end{vmatrix}}{\begin{vmatrix} 11 & 18 \\ 18 & 30 \end{vmatrix}} = \frac{36}{6} = 6 \\ y &= \frac{\begin{vmatrix} 11 & 45 \\ 18 & 73 \end{vmatrix}}{\begin{vmatrix} 11 & 18 \\ 18 & 30 \end{vmatrix}} = \frac{-7}{6}. \end{aligned}$$

23. Substituting the coordinates of the five points into the parabola equation gives the following equations:

$$\begin{aligned} a &= 0 \\ a + b + c &= 0 \\ a + 2b + 4c &= -1 \\ a + 3b + 9c &= 4 \\ a + 4b + 16c &= 8. \end{aligned}$$

The associated normal equations are given by

$$\begin{bmatrix} 5 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 11 \\ 42 \\ 160 \end{bmatrix},$$

which have the solution $a = 1/5,\ b = -2,\ c = 1$.

24. Suppose that A is symmetric, i.e. $A^t = A$ and that AB is defined. Then

$$(B^t AB)^t = B^t A^t (B^t)^t = B^t AB,$$

so $B^t AB$ is also symmetric.

25. Let A be $m \times n$ and B be $n \times m$, where $m > n$. Then the homogeneous system $BX = 0$ has a non–trivial solution X_0, as the number of unknowns is greater than the number of equations. Then

$$(AB)X_0 = A(BX_0) = A0 = 0$$

and the $m \times m$ matrix AB is therefore singular, as $X_0 \neq 0$.

26. (i) Let B be a singular $n \times n$ matrix. Then $BX = 0$ for some non–zero column vector X. Then $(AB)X = A(BX) = A0 = 0$ and hence AB is also singular.

(ii) Suppose A is a singular $n \times n$ matrix. Then A^t is also singular and hence by (i) so is $B^t A^t = (AB)^t$. Consequently AB is also singular

Section 3.6

1. (a) Let S be the set of vectors $[x,\, y]$ satisfying $x = 2y$. Then S is a vector subspace of $\mathbb{R}^2$. For

(i) $[0,\, 0] \in S$ as $x = 2y$ holds with $x = 0$ and $y = 0$.

(ii) S is closed under addition. For let $[x_1,\, y_1]$ and $[x_2,\, y_2]$ belong to S. Then $x_1 = 2y_1$ and $x_2 = 2y_2$. Hence

$$x_1 + x_2 = 2y_1 + 2y_2 = 2(y_1 + y_2)$$

and hence

$$[x_1 + x_2,\, y_1 + y_2] = [x_1,\, y_1] + [x_2,\, y_2]$$

belongs to S.

(iii) S is closed under scalar multiplication. For let $[x,\, y] \in S$ and $t \in \mathbb{R}$. Then $x = 2y$ and hence $tx = 2(ty)$. Consequently

$$[tx,\, ty] = t[x,\, y] \in S.$$

(b) Let S be the set of vectors $[x,\, y]$ satisfying $x = 2y$ and $2x = y$. Then S is a subspace of $\mathbb{R}^2$. This can be proved in the same way as (a), or alternatively we see that $x = 2y$ and $2x = y$ imply $x = 4x$ and hence $x = 0 = y$. Hence $S = \{[0,\, 0]\}$, the set consisting of the zero vector. This is always a subspace.

(c) Let S be the set of vectors $[x,\, y]$ satisfying $x = 2y + 1$. Then S doesn't contain the zero vector and consequently fails to be a vector subspace.

(d) Let S be the set of vectors $[x,\, y]$ satisfying $xy = 0$. Then S is not closed under addition of vectors. For example $[1,\, 0] \in S$ and $[0,\, 1] \in S$, but $[1,\, 0] + [0,\, 1] = [1,\, 1] \notin S$.

(e) Let S be the set of vectors $[x,\, y]$ satisfying $x \geq 0$ and $y \geq 0$. Then S is not closed under scalar multiplication. For example $[1,\, 0] \in S$ and $-1 \in \mathbb{R}$, but $(-1)[1,\, 0] = [-1,\, 0] \notin S$.

2. Let X, Y, Z be vectors in $\mathbb{R}^n$. Then by Lemma 3.2.1

$$\langle X + Y,\, X + Z,\, Y + Z\rangle \subseteq \langle X,\, Y,\, Z\rangle,$$

as each of $X + Y$, $X + Z$, $Y + Z$ is a linear combination of X, Y, Z.

Also

$$\begin{aligned}
X &= \frac{1}{2}(X+Y) + \frac{1}{2}(X+Z) - \frac{1}{2}(Y+Z),\\
Y &= \frac{1}{2}(X+Y) - \frac{1}{2}(X+Z) + \frac{1}{2}(Y+Z),\\
Z &= \frac{-1}{2}(X+Y) + \frac{1}{2}(X+Z) + \frac{1}{2}(Y+Z),
\end{aligned}$$

so

$$\langle X, Y, Z\rangle \subseteq \langle X+Y, X+Z, Y+Z\rangle.$$

Hence

$$\langle X, Y, Z\rangle = \langle X+Y, X+Z, Y+Z\rangle.$$

3. Let $X_1 = \begin{bmatrix} 1\\0\\1\\2 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0\\1\\1\\2 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 1\\1\\1\\3 \end{bmatrix}$. We have to decide if X_1, X_2, X_3 are linearly independent, that is if the equation $xX_1 + yX_2 + zX_3 = 0$ has only the trivial solution. This equation is equivalent to the folowing homogeneous system

$$\begin{aligned}
x + 0y + z &= 0\\
0x + y + z &= 0\\
x + y + z &= 0\\
2x + 2y + 3z &= 0.
\end{aligned}$$

We reduce the coefficient matrix to reduced row–echelon form:

$$\begin{bmatrix} 1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 1\\ 2 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0 \end{bmatrix}$$

and consequently the system has only the trivial solution $x = 0$, $y = 0$, $z = 0$. Hence the given vectors are linearly independent.

4. The vectors

$$X_1 = \begin{bmatrix} \lambda\\ -1\\ -1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1\\ \lambda\\ -1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -1\\ -1\\ \lambda \end{bmatrix}$$

are linearly dependent for precisely those values of λ for which the equation $xX_1 + yX_2 + zX_3 = 0$ has a non–trivial solution. This equation is equivalent to the system of homogeneous equations

$$\begin{aligned} \lambda x - y - z &= 0 \\ -x + \lambda y - z &= 0 \\ -x - y + \lambda z &= 0. \end{aligned}$$

Now the coefficient determinant of this system is

$$\begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} = (\lambda+1)^2(\lambda-2).$$

So the values of λ which make X_1, X_2, X_3 linearly independent are those λ satisfying $\lambda \neq -1$ and $\lambda \neq 2$.

5. Let A be the following matrix of rationals:

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 5 & 0 & 3 \\ 0 & 0 & 0 & 1 & 3 \\ 8 & 11 & 19 & 0 & 11 \end{bmatrix}.$$

Then A has reduced row–echelon form

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

From B we read off the following:

(a) The rows of B form a basis for $R(A)$. (Consequently the rows of A also form a basis for $R(A)$.)

(b) The first four columns of A form a basis for $C(A)$.

(c) To find a basis for $N(A)$, we solve $AX = 0$ and equivalently $BX = 0$. From B we see that the solution is

$$\begin{aligned} x_1 &= x_5 \\ x_2 &= 0 \\ x_3 &= -x_5 \\ x_4 &= -3x_5, \end{aligned}$$

with x_5 arbitrary. Then

$$X = \begin{bmatrix} x_5 \\ 0 \\ -x_5 \\ -3x_5 \\ x_5 \end{bmatrix} = x_5 \begin{bmatrix} 1 \\ 0 \\ -1 \\ -3 \\ 1 \end{bmatrix},$$

so $[1,\ 0,\ -1,\ -3,\ 1]^t$ is a basis for $N(A)$.

6. In Section 1.6, problem 12, we found that the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

has reduced row–echelon form

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From B we read off the following:

(a) The three non–zero rows of B form a basis for $R(A)$.

(b) The first three columns of A form a basis for $C(A)$.

(c) To find a basis for $N(A)$, we solve $AX = 0$ and equivalently $BX = 0$. From B we see that the solution is

$$\begin{aligned} x_1 &= -x_4 - x_5 = x_4 + x_5 \\ x_2 &= -x_4 - x_5 = x_4 + x_5 \\ x_3 &= -x_4 = x_4, \end{aligned}$$

with x_4 and x_5 arbitrary elements of $\mathbb{Z}_2$. Hence

$$X = \begin{bmatrix} x_4 + x_5 \\ x_4 + x_5 \\ x_4 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence $[1,\ 1,\ 1,\ 1,\ 0]^t$ and $[1,\ 1,\ 0,\ 0,\ 1]^t$ form a basis for $N(A)$.

7. Let A be the following matrix over $\mathbb{Z}_5$:

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 & 3 \\ 2 & 1 & 4 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 3 & 0 & 2 & 4 & 3 & 2 \end{bmatrix}.$$

We find that A has reduced row–echelon form B:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 & 4 & 4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{bmatrix}.$$

From B we read off the following:

(a) The four rows of B form a basis for $R(A)$. (Consequently the rows of A also form a basis for $R(A)$.

(b) The first four columns of A form a basis for $C(A)$.

(c) To find a basis for $N(A)$, we solve $AX = 0$ and equivalently $BX = 0$. From B we see that the solution is

$$\begin{aligned} x_1 &= -2x_5 - 4x_6 = 3x_5 + x_6 \\ x_2 &= -4x_5 - 4x_6 = x_5 + x_6 \\ x_3 &= 0 \\ x_4 &= -3x_5 = 2x_5, \end{aligned}$$

where x_5 and x_6 are arbitrary elements of $\mathbb{Z}_5$. Hence

$$X = x_5 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

so $[3,\ 1,\ 0,\ 2,\ 1,\ 0]^t$ and $[1,\ 1,\ 0,\ 0,\ 0,\ 1]^t$ form a basis for $N(A)$.

8. Let $F = \{0,\ 1,\ a,\ b\}$ be a field and let A be the following matrix over F:

$$A = \begin{bmatrix} 1 & a & b & a \\ a & b & b & 1 \\ 1 & 1 & 1 & a \end{bmatrix}.$$

In Section 1.6, problem 17, we found that A had reduced row–echelon form

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

From B we read off the following:

(a) The rows of B form a basis for $R(A)$. (Consequently the rows of A also form a basis for $R(A)$.

(b) The first three columns of A form a basis for $C(A)$.

(c) To find a basis for $N(A)$, we solve $AX = 0$ and equivalently $BX = 0$. From B we see that the solution is

$$\begin{aligned} x_1 &= 0 \\ x_2 &= -bx_4 = bx_4 \\ x_3 &= -x_4 = x_4, \end{aligned}$$

where x_4 is an arbitrary element of F. Hence

$$X = x_4 \begin{bmatrix} 0 \\ b \\ 1 \\ 1 \end{bmatrix},$$

so $[0,\ b,\ 1,\ 1]^t$ is a basis for $N(A)$.

9. Suppose that $X_1, \ldots, X_m$ form a basis for a subspace S. We have to prove that

$$X_1,\ X_1 + X_2, \ldots, X_1 + \cdots + X_m$$

also form a basis for S.

First we prove the independence of the family: Suppose

$$x_1 X_1 + x_2(X_1 + X_2) + \cdots + x_m(X_1 + \cdots + X_m) = 0.$$

Then

$$(x_1 + x_2 + \cdots + x_m)X_1 + \cdots + x_m X_m = 0.$$

Then the linear independence of $X_1, \ldots, X_m$ gives

$$x_1 + x_2 + \cdots + x_m = 0, \ldots, x_m = 0,$$

form which we deduce that $x_1 = 0, \ldots, x_m = 0$.

Secondly we have to prove that every vector of S is expressible as a linear combination of $X_1,\ X_1 + X_2, \ldots, X_1 + \cdots + X_m$. Suppose $X \in S$. Then

$$X = a_1 X_1 + \cdots + a_m X_m.$$

We have to find $x_1, \ldots, x_m$ such that

$$\begin{aligned} X &= x_1 X_1 + x_2(X_1 + X_2) + \cdots + x_m(X_1 + \cdots + X_m) \\ &= (x_1 + x_2 + \cdots + x_m)X_1 + \cdots + x_m X_m. \end{aligned}$$

Then

$$a_1 X_1 + \cdots + a_m X_m = (x_1 + x_2 + \cdots + x_m)X_1 + \cdots + x_m X_m.$$

So if we can solve the system

$$x_1 + x_2 + \cdots + x_m = a_1, \ldots, x_m = a_m,$$

we are finished. Clearly these equations have the unique solution

$$x_1 = a_1 - a_2, \ldots, x_{m-1} = a_m - a_{m-1},\ x_m = a_m.$$

10. Let $A = \begin{bmatrix} a & b & c \\ 1 & 1 & 1 \end{bmatrix}$. If $[a,\ b,\ c]$ is a multiple of $[1,\ 1,\ 1]$, (that is, $a = b = c$), then $\operatorname{rank} A = 1$. For if

$$[a,\ b,\ c] = t[1,\ 1,\ 1],$$

then

$$R(A) = \langle [a,\ b,\ c],\ [1,\ 1,\ 1] \rangle = \langle t[1,\ 1,\ 1],\ [1,\ 1,\ 1] \rangle = \langle [1,\ 1,\ 1] \rangle,$$

so $[1,\ 1,\ 1]$ is a basis for $R(A)$.

However if $[a,\ b,\ c]$ is not a multiple of $[1,\ 1,\ 1]$, (that is at least two of $a,\ b,\ c$ are distinct), then the left–to–right test shows that $[a,\ b,\ c]$ and $[1,\ 1,\ 1]$ are linearly independent and hence form a basis for $R(A)$. Consequently $\operatorname{rank} A = 2$ in this case.

11. Let S be a subspace of F^n with $\dim S = m$. Also suppose that $X_1, \ldots, X_m$ are vectors in S such that $S = \langle X_1, \ldots, X_m \rangle$. We have to prove that $X_1, \ldots, X_m$ form a basis for S; in other words, we must prove that $X_1, \ldots, X_m$ are linearly independent.

However if $X_1, \ldots, X_m$ were linearly dependent, then one of these vectors would be a linear combination of the remaining vectors. Consequently S would be spanned by $m-1$ vectors. But there exist a family of m linearly independent vectors in S. Then by Theorem 3.3.2, we would have the contradiction $m \leq m-1$.

12. Let $[x,\ y,\ z]^t \in S$. Then $x + 2y + 3z = 0$. Hence $x = -2y - 3z$ and

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y-3z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Hence $[-2,\ 1,\ 0]^t$ and $[-3,\ 0,\ 1]^t$ form a basis for S.

Next $(-1) + 2(-1) + 3(1) = 0$, so $[-1,\ -1,\ 1]^t \in S$.

To find a basis for S which includes $[-1,\ -1,\ 1]^t$, we note that $[-2,\ 1,\ 0]^t$ is not a multiple of $[-1,\ -1,\ 1]^t$. Hence we have found a linearly independent family of two vectors in S, a subspace of dimension equal to 2. Consequently these two vectors form a basis for S.

13. Without loss of generality, suppose that $X_1 = X_2$. Then we have the non–trivial dependency relation:

$$1X_1 + (-1)X_2 + 0X_3 + \cdots + 0X_m = 0.$$

14. (a) Suppose that X_{m+1} is a linear combination of $X_1, \ldots, X_m$. Then

$$\langle X_1, \ldots, X_m,\ X_{m+1} \rangle = \langle X_1, \ldots, X_m \rangle$$

and hence

$$\dim \langle X_1, \ldots, X_m,\ X_{m+1} \rangle = \dim \langle X_1, \ldots, X_m \rangle.$$

(b) Suppose that X_{m+1} is not a linear combination of $X_1, \ldots, X_m$. If not all of $X_1, \ldots, X_m$ are zero, there will be a subfamily $X_{c_1}, \ldots, X_{c_r}$ which is a basis for $\langle X_1, \ldots, X_m \rangle$.

Then as X_{m+1} is not a linear combination of $X_{c_1}, \ldots, X_{c_r}$, it follows that $X_{c_1}, \ldots, X_{c_r},\ X_{m+1}$ are linearly independent. Also

$$\langle X_1, \ldots, X_m,\ X_{m+1} \rangle = \langle X_{c_1}, \ldots, X_{c_r},\ X_{m+1} \rangle.$$

Consequently

$$\dim \langle X_1, \ldots, X_m,\ X_{m+1} \rangle = r + 1 = \dim \langle X_1, \ldots, X_m \rangle + 1.$$

Our result can be rephrased in a form suitable for the second part of the problem:

$$\dim \langle X_1, \ldots, X_m,\, X_{m+1}\rangle = \dim \langle X_1, \ldots, X_m\rangle$$

if and only if X_{m+1} is a linear combination of $X_1, \ldots, X_m$.

If $X = [x_1, \ldots, x_n]^t$, then $AX = B$ is equivalent to

$$B = x_1 A_{*1} + \cdots + x_n A_{*n}.$$

So $AX = B$ is soluble for X if and only if B is a linear combination of the columns of A, that is $B \in C(A)$. However by the first part of this question, $B \in C(A)$ if and only if $\dim C([A|B]) = \dim C(A)$, that is, $\mathrm{rank}\,[A|B] = \mathrm{rank}\,A$.

15. Let $a_1, \ldots, a_n$ be elements of F, not all zero. Let S denote the set of vectors $[x_1, \ldots, x_n]^t$, where $x_1, \ldots, x_n$ satisfy

$$a_1x_1 + \cdots + a_nx_n = 0.$$

Then $S = N(A)$, where A is the row matrix $[a_1, \ldots, a_n]$. Now $\mathrm{rank}\,A = 1$ as $A \neq 0$. So by the "rank + nullity" theorem, noting that the number of columns of A equals n, we have

$$\dim N(A) = \text{ nullity}\,(A) = n - \mathrm{rank}\,A = n - 1.$$

16. (a) (Proof of Lemma 3.2.1) Suppose that each of $X_1, \ldots, X_r$ is a linear combination of $Y_1, \ldots, Y_s$. Then

$$X_i = \sum_{j=1}^{s} a_{ij}Y_j, \quad (1 \le i \le r).$$

Now let $X = \sum_{i=1}^{r} x_iX_i$ be a linear combination of $X_1, \ldots, X_r$. Then

$$\begin{array}{rcl} X & = & x_1(a_{11}Y_1 + \cdots + a_{1s}Y_s) \\ & + & \cdots \\ & + & x_r(a_{r1}Y_1 + \cdots + a_{rs}Y_s) \\ & = & y_1Y_1 + \cdots + y_sY_s, \end{array}$$

where $y_j = a_{1j}x_1 + \cdots + a_{rj}x_r$. Hence X is a linear combination of $Y_1, \ldots, Y_s$.

Another way of stating Lemma 3.2.1 is

$$\langle X_1, \ldots, X_r\rangle \subseteq \langle Y_1, \ldots, Y_s\rangle, \quad (1)$$

if each of $X_1, \ldots, X_r$ is a linear combination of $Y_1, \ldots, Y_s$.

(b) (Proof of Theorem 3.2.1) Suppose that each of $X_1, \ldots, X_r$ is a linear combination of $Y_1, \ldots, Y_s$ and that each of $Y_1, \ldots, Y_s$ is a linear combination of $X_1, \ldots, X_r$. Then by (a) equation (1) above

$$\langle X_1, \ldots, X_r \rangle \subseteq \langle Y_1, \ldots, Y_s \rangle$$

and

$$\langle Y_1, \ldots, Y_s \rangle \subseteq \langle X_1, \ldots, X_r \rangle.$$

Hence

$$\langle X_1, \ldots, X_r \rangle = \langle Y_1, \ldots, Y_s \rangle.$$

(c) (Proof of Corollary 3.2.1) Suppose that each of $Z_1, \ldots, Z_t$ is a linear combination of $X_1, \ldots, X_r$. Then each of $X_1, \ldots, X_r, Z_1, \ldots, Z_t$ is a linear combination of $X_1, \ldots, X_r$.

Also each of $X_1, \ldots, X_r$ is a linear combination of $X_1, \ldots, X_r, Z_1, \ldots, Z_t$, so by Theorem 3.2.1

$$\langle X_1, \ldots, X_r, Z_1, \ldots, Z_t \rangle = \langle X_1, \ldots, X_r \rangle.$$

(d) (Proof of Theorem 3.3.2) Let $Y_1, \ldots, Y_s$ be vectors in $\langle X_1, \ldots, X_r \rangle$ and assume that $s > r$. We have to prove that $Y_1, \ldots, Y_s$ are linearly dependent. So we consider the equation

$$x_1 Y_1 + \cdots + x_s Y_s = 0.$$

Now $Y_i = \sum_{j=1}^{r} a_{ij} X_j$, for $1 \leq i \leq s$. Hence

$$\begin{aligned} x_1 Y_1 + \cdots + x_s Y_s &= x_1(a_{11} X_1 + \cdots + a_{1r} X_r) \\ &+ \cdots \\ &+ x_r(a_{s1} X_1 + \cdots + a_{sr} X_r). \\ &= y_1 X_1 + \cdots + y_r X_r, \quad (1) \end{aligned}$$

where $y_j = a_{1j} x_1 + \cdots + a_{sj} x_s$. However the homogeneous system

$$y_1 = 0, \ \cdots, \ y_r = 0$$

has a non–trivial solution $x_1, \ldots, x_s$, as $s > r$ and from (1), this results in a non–trivial solution of the equation

$$x_1 Y_1 + \cdots + x_s Y_s = 0.$$

Hence $Y_1, \ldots, Y_s$ are linearly dependent.

17. Let R and S be subspaces of F^n, with $R \subseteq S$. We first prove

$$\dim R \leq \dim S.$$

Let $X_1, \ldots, X_r$ be a basis for R. Now by Theorem 3.5.2, because $X_1, \ldots, X_r$ form a linearly independent family lying in S, this family can be extended to a basis $X_1, \ldots, X_r, \ldots, X_s$ for S. Then

$$\dim S = s \geq r = \dim R.$$

Next suppose that $\dim R = \dim S$. Let $X_1, \ldots, X_r$ be a basis for R. Then because $X_1, \ldots, X_r$ form a linearly independent family in S and S is a subspace whose dimension is r, it follows from Theorem 3.4.3 that $X_1, \ldots, X_r$ form a basis for S. Then

$$S = \langle X_1, \ldots, X_r \rangle = R.$$

18. Suppose that R and S are subspaces of F^n with the property that $R \cup S$ is also a subspace of F^n. We have to prove that $R \subseteq S$ or $S \subseteq R$. We argue by contradiction: Suppose that $R \not\subseteq S$ and $S \not\subseteq R$. Then there exist vectors u and v such that

$$u \in R \text{ and } u \notin S, \quad v \in S \text{ and } v \notin R.$$

Consider the vector $u + v$. As we are assuming $R \cup S$ is a subspace, $R \cup S$ is closed under addition. Hence $u + v \in R \cup S$ and so $u + v \in R$ or $u + v \in S$. However if $u + v \in R$, then $v = (u + v) - u \in R$, which is a contradiction; similarly if $u + v \in S$.

Hence we have derived a contradiction on the asumption that $R \not\subseteq S$ and $S \not\subseteq R$. Consequently at least one of these must be false. In other words $R \subseteq S$ or $S \subseteq R$.

19. Let $X_1, \ldots, X_r$ be a basis for S.

(i) First let

$$\begin{aligned} Y_1 &= a_{11}X_1 + \cdots + a_{1r}X_r \\ &\vdots \\ Y_r &= a_{r1}X_1 + \cdots + a_{rr}X_r, \end{aligned} \tag{2}$$

where $A = [a_{ij}]$ is non–singular. Then the above system of equations can be solved for $X_1, \ldots, X_r$ in terms of $Y_1, \ldots, Y_r$. Consequently by Theorem 3.2.1

$$\langle Y_1, \ldots, Y_r \rangle = \langle X_1, \ldots, X_r \rangle = S.$$

It follows from problem 11 that $Y_1, \ldots, Y_r$ is a basis for S.

(ii) We show that *all* bases for S are given by equations 2. So suppose that $Y_1, \ldots, Y_r$ forms a basis for S. Then because $X_1, \ldots, X_r$ form a basis for S, we can express $Y_1, \ldots, Y_r$ in terms of $X_1, \ldots, X_r$ as in 2, for some matrix $A = [a_{ij}]$. We show A is non–singular by demonstrating that the linear independence of $Y_1, \ldots, Y_r$ implies that the rows of A are linearly independent.

So assume

$$x_1[a_{11}, \ldots, a_{1r}] + \cdots + x_r[a_{r1}, \ldots, a_{rr}] = [0, \ldots, 0].$$

Then on equating components, we have

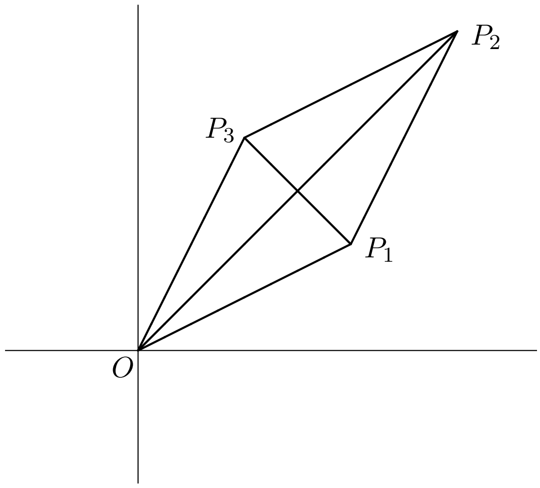
$$\begin{aligned} a_{11}x_1 + \cdots + a_{r1}x_r &= 0 \\ &\vdots \\ a_{1r}x_1 + \cdots + a_{rr}x_r &= 0. \end{aligned}$$

Hence

$$\begin{aligned} x_1Y_1 + \cdots + x_rY_r &= x_1(a_{11}X_1 + \cdots + a_{1r}X_r) + \cdots + x_r(a_{r1}X_1 + \cdots + a_{rr}X_r) \\ &= (a_{11}x_1 + \cdots + a_{r1}x_r)X_1 + \cdots + (a_{1r}x_1 + \cdots + a_{rr}x_r)X_r \\ &= 0X_1 + \cdots + 0X_r = 0. \end{aligned}$$

Then the linear independence of $Y_1, \ldots, Y_r$ implies $x_1 = 0, \ldots, x_r = 0$.

(We mention that the last argument is reversible and provides an alternative proof of part (i).)



Section 4.1

1. We first prove that the area of a triangle $P_1P_2P_3$, where the points are in anti–clockwise orientation, is given by the formula

$$\frac{1}{2}\left\{\left|\begin{array}{cc} x_1 & x_2 \\ y_1 & y_2 \end{array}\right| + \left|\begin{array}{cc} x_2 & x_3 \\ y_2 & y_3 \end{array}\right| + \left|\begin{array}{cc} x_3 & x_1 \\ y_3 & y_1 \end{array}\right|\right\}.$$

Referring to the above diagram, we have

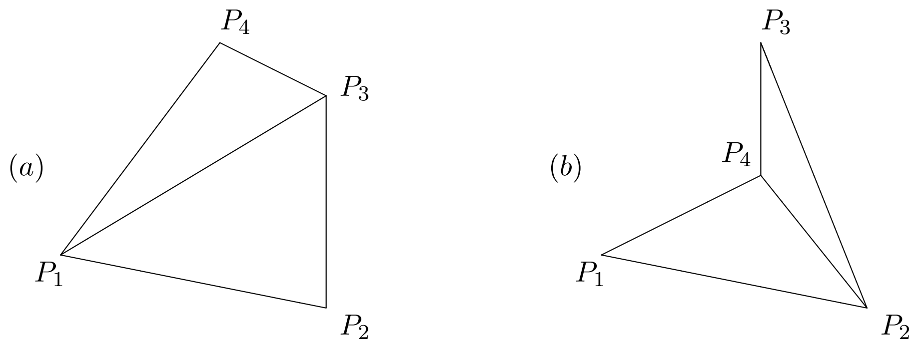
$$\begin{aligned} \text{Area}\, P_1P_2P_3 &= \text{Area}\, OP_1P_2 + \text{Area}\, OP_2P_3 - \text{Area}\, OP_1P_3 \\ &= \frac{1}{2}\left|\begin{array}{cc} x_1 & x_2 \\ y_1 & y_2 \end{array}\right| + \frac{1}{2}\left|\begin{array}{cc} x_2 & x_3 \\ y_2 & y_3 \end{array}\right| - \frac{1}{2}\left|\begin{array}{cc} x_1 & x_3 \\ y_1 & y_3 \end{array}\right|, \end{aligned}$$

which gives the desired formula.

We now turn to the area of a quadrilateral. One possible configuration occurs when the quadrilateral is convex as in figure (a) below. The interior diagonal breaks the quadrilateral into two triangles $P_1P_2P_3$ and $P_1P_3P_4$. Then

$$\text{Area}\, P_1P_2P_3P_4 = \text{Area}\, P_1P_2P_3 + \text{Area}\, P_1P_3P_4$$

$$= \frac{1}{2}\left\{\left|\begin{array}{cc} x_1 & x_2 \\ y_1 & y_2 \end{array}\right| + \left|\begin{array}{cc} x_2 & x_3 \\ y_2 & y_3 \end{array}\right| + \left|\begin{array}{cc} x_3 & x_1 \\ y_3 & y_1 \end{array}\right|\right\}$$



$$
\begin{aligned}
& + \frac{1}{2}\left\{ \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & x_4 \\ y_3 & y_4 \end{vmatrix} + \begin{vmatrix} x_4 & x_1 \\ y_4 & y_1 \end{vmatrix} \right\} \\
= \; & \frac{1}{2}\left\{ \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & x_4 \\ y_3 & y_4 \end{vmatrix} + \begin{vmatrix} x_4 & x_1 \\ y_4 & y_1 \end{vmatrix} \right\},
\end{aligned}
$$

after cancellation.

Another possible configuration for the quadrilateral occurs when it is not convex, as in figure (b). The interior diagonal P_2P_4 then gives two triangles $P_1P_2P_4$ and $P_2P_3P_4$ and we can proceed similarly as before.

2.

$$
\Delta = \begin{vmatrix} a+x & b+y & c+z \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} = \begin{vmatrix} a & b & c \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} + \begin{vmatrix} x & y & z \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix}.
$$

Now

$$
\begin{vmatrix} a & b & c \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ u+a & v+b & w+c \end{vmatrix} + \begin{vmatrix} a & b & c \\ u & v & w \\ u+a & v+b & w+c \end{vmatrix}
$$

$$
\begin{aligned}
&= \begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix} + \begin{vmatrix} a & b & c \\ x & y & z \\ a & b & c \end{vmatrix} + \begin{vmatrix} a & b & c \\ u & v & w \\ u & v & w \end{vmatrix} + \begin{vmatrix} a & b & c \\ u & v & w \\ a & b & c \end{vmatrix} \\
&= \begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix}.
\end{aligned}
$$

Similarly

$$
\begin{vmatrix} x & y & z \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} = \begin{vmatrix} x & y & z \\ u & v & w \\ a & b & c \end{vmatrix} = - \begin{vmatrix} x & y & z \\ a & b & c \\ u & v & w \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix}.
$$

Hence $\Delta = 2\begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix}$.

3. $$\begin{vmatrix} n^2 & (n+1)^2 & (n+2)^2 \\ (n+1)^2 & (n+2)^2 & (n+3)^2 \\ (n+2)^2 & (n+3)^2 & (n+4)^2 \end{vmatrix} \begin{array}{c} C_3 \to C_3 - C_2 \\ C_2 \to C_2 - C_1 \\ = \end{array} \begin{vmatrix} n^2 & 2n+1 & 2n+3 \\ (n+1)^2 & 2n+3 & 2n+5 \\ (n+2)^2 & 2n+5 & 2n+7 \end{vmatrix}$$

$$\begin{array}{c} C_3 \to C_3 - C_2 \\ = \end{array} \begin{vmatrix} n^2 & 2n+1 & 2 \\ (n+1)^2 & 2n+3 & 2 \\ (n+2)^2 & 2n+5 & 2 \end{vmatrix}$$

$$\begin{array}{c} R_3 \to R_3 - R_2 \\ R_2 \to R_2 - R_1 \\ = \end{array} \begin{vmatrix} n^2 & 2n+1 & 2 \\ 2n+1 & 2 & 0 \\ 2n+3 & 2 & 0 \end{vmatrix} = -8.$$

4. (a)

$$\begin{vmatrix} 246 & 427 & 327 \\ 1014 & 543 & 443 \\ -342 & 721 & 621 \end{vmatrix} = \begin{vmatrix} 246 & 100 & 327 \\ 1014 & 100 & 443 \\ -342 & 100 & 621 \end{vmatrix} = 100\begin{vmatrix} 246 & 1 & 327 \\ 1014 & 1 & 443 \\ -342 & 1 & 621 \end{vmatrix}$$

$$= \quad 100\begin{vmatrix} 246 & 1 & 327 \\ 768 & 0 & 116 \\ -588 & 0 & 294 \end{vmatrix} = 100(-1)\begin{vmatrix} 768 & 116 \\ -588 & 294 \end{vmatrix} = -29400000.$$

(b)

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & -4 & 3 \\ 3 & -4 & -1 & 2 \\ 4 & 3 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 2 & 11 \\ 0 & -10 & -10 & -10 \\ 0 & -5 & -14 & -17 \end{vmatrix}$$

$$= \begin{vmatrix} 5 & 2 & 11 \\ -10 & -10 & -10 \\ -5 & -14 & -17 \end{vmatrix} = -10\begin{vmatrix} 5 & 2 & 11 \\ 1 & 1 & 1 \\ -5 & -14 & -17 \end{vmatrix}$$

$$= \quad -10\begin{vmatrix} 5 & -3 & 6 \\ 1 & 0 & 0 \\ -5 & -9 & -12 \end{vmatrix} = -10(-1)\begin{vmatrix} -3 & 6 \\ -9 & -12 \end{vmatrix} = 900.$$

5. $$\det A = \begin{vmatrix} 1 & 0 & -2 \\ 3 & 1 & 4 \\ 5 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 10 \\ 5 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 10 \\ 2 & 7 \end{vmatrix} = -13.$$

Hence A is non–singular and

$$A^{-1} = \frac{1}{-13}\,\mathrm{adj}\,A = \frac{1}{-13}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \frac{1}{-13}\begin{bmatrix} -11 & -4 & 2 \\ 29 & 7 & -10 \\ 1 & -2 & 1 \end{bmatrix}.$$

6. (i)

$$\begin{vmatrix} 2a & 2b & b-c \\ 2b & 2a & a+c \\ a+b & a+b & b \end{vmatrix} \overset{R_1 \to R_1 + R_2}{=} \begin{vmatrix} 2a+2b & 2b+2a & b+a \\ 2b & 2a & a+c \\ a+b & a+b & b \end{vmatrix}$$

$$= (a{+}b)\begin{vmatrix} 2 & 2 & 1 \\ 2b & 2a & a+c \\ a+b & a+b & b \end{vmatrix} \overset{C_1 \to C_1 - C_2}{=} (a{+}b)\begin{vmatrix} 0 & 2 & 1 \\ 2(b-a) & 2a & a+c \\ 0 & a+b & b \end{vmatrix}$$

$$= 2(a+b)(a-b)\begin{vmatrix} 2 & 1 \\ a+b & b \end{vmatrix} = -2(a+b)(a-b)^2.$$

(ii)

$$\begin{vmatrix} b+c & b & c \\ c & c+a & a \\ b & a & a+b \end{vmatrix} \overset{C_1 \to C_1 - C_2}{=} \begin{vmatrix} c & b & c \\ -a & c+a & a \\ b-a & a & a+b \end{vmatrix}$$

$$\overset{C_3 \to C_3 - C_1}{=} \begin{vmatrix} c & b & 0 \\ -a & c+a & 2a \\ b-a & a & 2a \end{vmatrix} = 2a\begin{vmatrix} c & b & 0 \\ -a & c+a & 1 \\ b-a & a & 1 \end{vmatrix}$$

$$\overset{R_3 \to R_3 - R_2}{=} 2a\begin{vmatrix} c & b & 0 \\ -a & c+a & 1 \\ b & -c & 0 \end{vmatrix} = -2a\begin{vmatrix} c & b \\ b & -c \end{vmatrix} = 2a(c^2 + b^2).$$

7. Suppose that the curve $y = ax^2 + bx + c$ passes through the points $(x_1,\, y_1),\ (x_2,\, y_2),\ (x_3,\, y_3)$, where $x_i \neq x_j$ if $i \neq j$. Then

$$\begin{aligned} ax_1^2 + bx_1 + c &= y_1 \\ ax_2^2 + bx_2 + c &= y_2 \\ ax_3^2 + bx_3 + c &= y_3. \end{aligned}$$

The coefficient determinant is essentially a Vandermonde determinant:

$$\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = -(x_2{-}x_1)(x_3{-}x_1)(x_3{-}x_2).$$

Hence the coefficient determinant is non–zero and by Cramer's rule, there is a unique solution for a, b, c.

8. Let $\Delta = \det A = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & k \\ 1 & k & 3 \end{vmatrix}$. Then

$$\begin{aligned} \Delta = & \quad \begin{array}{c} C_3 \to C_3 + C_1 \\ C_2 \to C_2 - C_1 \end{array} \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & k+2 \\ 1 & k-1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & k+2 \\ k-1 & 4 \end{vmatrix} \\ = & \quad 4 - (k-1)(k+2) = -(k^2 - k - 6) = -(k+3)(k-2). \end{aligned}$$

Hence $\det A = 0$ if and only if $k = -3$ or $k = 2$.

Consequently if $k \neq -3$ and $k \neq 2$, then $\det A \neq 0$ and the given system

$$\begin{aligned} x + y - z &= 1 \\ 2x + 3y + kz &= 3 \\ x + ky + 3z &= 2 \end{aligned}$$

has a unique solution. We consider the cases $k = -3$ and $k = 2$ separately.

$k = -3$:

$$AM = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & -3 & 3 \\ 1 & -3 & 3 & 2 \end{bmatrix} \begin{array}{c} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -4 & 4 & 1 \end{bmatrix}$$

$$R_3 \to R_3 + 4R_2 \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix},$$

from which we read off inconsistency.

$k = 2$:

$$AM = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & 2 & 3 \\ 1 & 2 & 3 & 2 \end{bmatrix} \begin{array}{c} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 1 & 4 & 1 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We read off the complete solution $x = 5z$, $y = 1 - 4z$, where z is arbitrary.

Finally we have to determine the solution for which $x^2+y^2+z^2$ is least.

$$\begin{aligned}
x^2+y^2+z^2 &= (5z)^2+(1-4z)^2+z^2 = 42z^2-8z+1 \\
&= 42(z^2-\frac{4}{21}z+\frac{1}{42}) = 42\left\{\left(z-\frac{2}{21}\right)^2+\frac{1}{42}-\left(\frac{2}{21}\right)^2\right\} \\
&= 42\left\{\left(z-\frac{2}{21}\right)^2+\frac{13}{882}\right\}.
\end{aligned}$$

We see that the least value of $x^2+y^2+z^2$ is $42\times\frac{13}{882}=\frac{13}{21}$ and this occurs when $z=2/21$, with corresponding values $x=10/21$ and $y=1-4\times\frac{2}{21}=13/21$.

9. Let $\Delta = \begin{vmatrix} 1 & -2 & b \\ a & 0 & 2 \\ 5 & 2 & 0 \end{vmatrix}$ be the coefficient determinant of the given system. Then expanding along column 2 gives

$$\begin{aligned}
\Delta &= 2\begin{vmatrix} a & 2 \\ 5 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & b \\ a & 2 \end{vmatrix} = -20-2(2-ab) \\
&= 2ab-24 = 2(ab-12).
\end{aligned}$$

Hence $\Delta=0$ if and only if $ab=12$. Hence if $ab\neq 12$, the given system has a unique solution.

If $ab=12$ we must argue with care:

$$\begin{aligned}
AM &= \begin{bmatrix} 1 & -2 & b & 3 \\ a & 0 & 2 & 2 \\ 5 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & b & 3 \\ 0 & 2a & 2-ab & 2-3a \\ 0 & 12 & -5b & -14 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & -2 & b & 3 \\ 0 & 1 & \frac{-5b}{12} & \frac{-7}{6} \\ 0 & 2a & 2-ab & 2-3a \end{bmatrix} \to \begin{bmatrix} 1 & -2 & b & 3 \\ 0 & 1 & \frac{-5b}{12} & \frac{-7}{6} \\ 0 & 0 & \frac{12-ab}{6} & \frac{6-2a}{3} \end{bmatrix} \\
&= \begin{bmatrix} 1 & -2 & b & 3 \\ 0 & 1 & \frac{-5b}{12} & \frac{-7}{6} \\ 0 & 0 & 0 & \frac{6-2a}{3} \end{bmatrix} = B.
\end{aligned}$$

Hence if $6-2a\neq 0$, i.e. $a\neq 3$, the system has no solution.

If $a=3$ (and hence $b=4$), then

$$B = \begin{bmatrix} 1 & -2 & 4 & 3 \\ 0 & 1 & \frac{-5}{3} & \frac{-7}{6} \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2/3 & 2/3 \\ 0 & 1 & \frac{-5}{3} & \frac{-7}{6} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Consequently the complete solution of the system is $x = -\frac{2}{3} + \frac{2}{3}z, \; y = \frac{-7}{6} + \frac{5}{3}z$, where z is arbitrary. Hence there are infinitely many solutions.

10.

$$\begin{aligned}
\Delta &= \begin{vmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 2t+6 \\ 2 & 2 & 6-t & t \end{vmatrix} \begin{array}{c} R_4 \to R_4 - 2R_1 \\ R_3 \to R_3 - 2R_1 \\ R_2 \to R_2 - R_1 \\ = \end{array} \begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 2t+4 \\ 0 & 0 & 2-t & t-2 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 & 3 \\ 2 & 3 & 2t+4 \\ 0 & 2-t & t-2 \end{vmatrix} \begin{array}{c} R_2 \to R_2 - 2R_1 \\ = \end{array} \begin{vmatrix} 1 & 1 & 3 \\ 0 & 1 & 2t-2 \\ 0 & 2-t & t-2 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 2t-2 \\ 2-t & t-2 \end{vmatrix} = (t-2) \begin{vmatrix} 1 & 2t-2 \\ -1 & 1 \end{vmatrix} = (t-2)(2t-1).
\end{aligned}$$

Hence $\Delta = 0$ if and only if $t = 2$ or $t = \frac{1}{2}$. Consequently the given matrix B is non–singular if and only if $t \neq 2$ and $t \neq \frac{1}{2}$.

11. Let A be a 3×3 matrix with $\det A \neq 0$. Then

(i)

$$\begin{aligned}
A \operatorname{adj} A &= (\det A) I_3 \qquad (1) \\
(\det A) \det(\operatorname{adj} A) &= \det(\det A \cdot I_3) = (\det A)^3.
\end{aligned}$$

Hence, as $\det A \neq 0$, dividing out by $\det A$ in the last equation gives

$$\det(\operatorname{adj} A) = (\det A)^2.$$

(ii) . Also from equation (1)

$$\left(\frac{1}{\det A} A\right) \operatorname{adj} A = I_3,$$

so $\operatorname{adj} A$ is non–singular and

$$(\operatorname{adj} A)^{-1} = \frac{1}{\det A} A.$$

Finally

$$A^{-1} \operatorname{adj}(A^{-1}) = (\det A^{-1}) I_3$$

and multiplying both sides of the last equation by A gives

$$\operatorname{adj}(A^{-1}) = A(\det A^{-1}) I_3 = \frac{1}{\det A} A.$$

12. Let A be a real 3×3 matrix satisfying $A^tA = I_3$. Then

$$\begin{aligned}
\text{(i) } A^t(A - I_3) &= A^tA - A^t = I_3 - A^t \\
&= -(A^t - I_3) = -(A^t - I_3^t) = -(A - I_3)^t.
\end{aligned}$$

Taking determinants of both sides then gives

$$\begin{aligned}
\det A^t \det(A - I_3) &= \det(-(A - I_3)^t) \\
\det A \det(A - I_3) &= (-1)^3 \det(A - I_3)^t \\
&= -\det(A - I_3) \qquad (1).
\end{aligned}$$

(ii) Also $\det AA^t = \det I_3$, so

$$\det A^t \det A = 1 = (\det A)^2.$$

Hence $\det A = \pm 1$.
(iii) Suppose that $\det A = 1$. Then equation (1) gives

$$\det(A - I_3) = -\det(A - I_3),$$

so $(1 + 1)\det(A - I_3) = 0$ and hence $\det(A - I_3) = 0$.

13. Suppose that column 1 is a linear combination of the remaining columns:

$$A_{*1} = x_2A_{*2} + \cdots + x_nA_{*n}.$$

Then

$$\det A = \begin{vmatrix} x_2a_{12} + \cdots + x_na_{1n} & a_{12} & \cdots & a_{1n} \\ x_2a_{22} + \cdots + x_na_{2n} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_2a_{n2} + \cdots + x_na_{nn} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Now $\det A$ is unchanged in value if we perform the operation

$$C_1 \to C_1 - x_2C_2 - \cdots - x_nC_n :$$

$$\det A = \begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0.$$

Conversely, suppose that $\det A = 0$. Then the homogeneous system $AX = 0$ has a non–trivial solution $X = [x_1, \ldots, x_n]^t$. So

$$x_1 A_{*1} + \cdots + x_n A_{*n} = 0.$$

Suppose for example that $x_1 \neq 0$. Then

$$A_{*1} = \left(-\frac{x_2}{x_1}\right) + \cdots + \left(-\frac{x_n}{x_1}\right) A_{*n}$$

and the first column of A is a linear combination of the remaining columns.

14. Consider the system

$$\begin{aligned} -2x + 3y - z &= 1 \\ x + 2y - z &= 4 \\ -2x - y + z &= -3 \end{aligned}$$

Let $\Delta = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 7 & -3 \\ 1 & 2 & -1 \\ 0 & 3 & -1 \end{vmatrix} = -\begin{vmatrix} 7 & -3 \\ 3 & -1 \end{vmatrix} = -2 \neq 0.$

Hence the system has a unique solution which can be calculated using Cramer's rule:

$$x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta}, \quad z = \frac{\Delta_3}{\Delta},$$

where

$$\begin{aligned} \Delta_1 &= \begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix} = -4, \\ \Delta_2 &= \begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix} = -6, \\ \Delta_3 &= \begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix} = -8. \end{aligned}$$

Hence $x = \frac{-4}{-2} = 2,\ y = \frac{-6}{-2} = 3,\ z = \frac{-8}{-2} = 4.$

15. In Remark 4.0.4, take $A = I_n$. Then we deduce

(a) $\det E_{ij} = -1$;

(b) $\det E_i(t) = t$;

(c) $\det E_{ij}(t) = 1$.

Now suppose that B is a non–singular $n \times n$ matrix. Then we know that B is a product of elementary row matrices:

$$B = E_1 \cdots E_m.$$

Consequently we have to prove that

$$\det E_1 \cdots E_m A = \det E_1 \cdots E_m \det A.$$

We prove this by induction on m.

First the case $m = 1$. We have to prove $\det E_1 A = \det E_1 \det A$ if E_1 is an elementary row matrix. This follows form Remark 4.0.4:

(a) $\det E_{ij} A = -\det A = \det E_{ij} \det A$;

(b) $\det E_i(t) A = t \det A = \det E_i(t) \det A$;

(c) $\det E_{ij}(t) A = \det A = \det E_{ij}(t) \det A$.

Let $m \geq 1$ and assume the proposition holds for products of m elementary row matrices. Then

$$\begin{aligned}
\det E_1 \cdots E_m E_{m+1} A &= \det (E_1 \cdots E_m)(E_{m+1} A) \\
&= \det (E_1 \cdots E_m) \det (E_{m+1} A) \\
&= \det (E_1 \cdots E_m) \det E_{m+1} \det A \\
&= \det ((E_1 \cdots E_m) E_{m+1}) \det A
\end{aligned}$$

and the induction goes through.

Hence $\det BA = \det B \det A$ if B is non–singular.

If B is singular, problem 26, Chapter 2.7 tells us that BA is also singlular. However singular matrices have zero determinant, so

$$\det B = 0 \quad \det BA = 0,$$

so the equation $\det BA = \det B \det A$ holds trivially in this case.

16.

$$\begin{vmatrix} a+b+c & a+b & a & a \\ a+b & a+b+c & a & a \\ a & a & a+b+c & a+b \\ a & a & a+b & a+b+c \end{vmatrix}$$

$$\begin{array}{c} R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3 \\ R_3 \to R_3 - R_4 \\ = \end{array} \begin{vmatrix} c & -c & 0 & 0 \\ b & b+c & -b-c & -b \\ 0 & 0 & c & -c \\ a & a & a+b & a+b+c \end{vmatrix}$$

$$\begin{array}{c} C_2 \to C_2 + C_1 \\ = \end{array} \begin{vmatrix} c & 0 & 0 & 0 \\ b & 2b+c & -b-c & -b \\ 0 & 0 & c & -c \\ a & 2a & a+b & a+b+c \end{vmatrix} = c \begin{vmatrix} 2b+c & -b-c & -b \\ 0 & c & -c \\ 2a & a+b & a+b+c \end{vmatrix}$$

$$\begin{array}{c} C_3 \to C_3 + C_2 \\ = \end{array} c \begin{vmatrix} 2b+c & -b-c & -2b-c \\ 0 & c & 0 \\ 2a & a+b & 2a+2b+c \end{vmatrix} = c^2 \begin{vmatrix} 2b+c & -2b-c \\ 2a & 2a+2b+c \end{vmatrix}$$

$$= c^2(2b+c) \begin{vmatrix} 1 & -1 \\ 2a & 2a+2b+c \end{vmatrix} = c^2(2b+c)(4a+2b+c).$$

17. Let $\Delta = \begin{vmatrix} 1+u_1 & u_1 & u_1 & u_1 \\ u_2 & 1+u_2 & u_2 & u_2 \\ u_3 & u_3 & 1+u_3 & u_3 \\ u_4 & u_4 & u_4 & 1+u_4 \end{vmatrix}$. Then using the operation

$$R_1 \to R_1 + R_2 + R_3 + R_4$$

we have

$$\Delta = \begin{vmatrix} t & t & t & t \\ u_2 & 1+u_2 & u_2 & u_2 \\ u_3 & u_3 & 1+u_3 & u_3 \\ u_4 & u_4 & u_4 & 1+u_4 \end{vmatrix}$$

(where $t = 1 + u_1 + u_2 + u_3 + u_4$)

$$= (1 + u_1 + u_2 + u_3 + u_4) \begin{vmatrix} 1 & 1 & 1 & 1 \\ u_2 & 1+u_2 & u_2 & u_2 \\ u_3 & u_3 & 1+u_3 & u_3 \\ u_4 & u_4 & u_4 & 1+u_4 \end{vmatrix}$$

The last determinant equals

$$\begin{array}{c} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \\ C_4 \to C_4 - C_1 \end{array} \begin{vmatrix} 1 & 0 & 0 & 0 \\ u_2 & 1 & 0 & 0 \\ u_3 & 0 & 1 & 0 \\ u_4 & 0 & 0 & 1 \end{vmatrix} = 1.$$

18. Suppose that $A^t = -A$, that $A \in M_{n\times n}(F)$, where n is odd. Then

$$\begin{aligned} \det A^t &= \det(-A) \\ \det A &= (-1)^n \det A = -\det A. \end{aligned}$$

Hence $(1+1)\det A = 0$ and consequently $\det A = 0$ if $1+1 \neq 0$ in F.

19.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ r & 1 & 1 & 1 \\ r & r & 1 & 1 \\ r & r & r & 1 \end{vmatrix} = \begin{matrix} C_4 \to C_4 - C_3 \\ C_3 \to C_3 - C_2 \\ C_2 \to C_2 - C_1 \\ = \end{matrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ r & 1-r & 0 & 0 \\ r & 0 & 1-r & 0 \\ r & 0 & 0 & 1-r \end{vmatrix} = (1-r)^3.$$

20.

$$\begin{vmatrix} 1 & a^2 - bc & a^4 \\ 1 & b^2 - ca & b^4 \\ 1 & c^2 - ab & c^4 \end{vmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ = \end{matrix} \begin{vmatrix} 1 & a^2 - bc & a^4 \\ 0 & b^2 - ca - a^2 + bc & b^4 - a^4 \\ 0 & c^2 - ab - a^2 + bc & c^4 - a^4 \end{vmatrix}$$

$$\begin{aligned} &= \begin{vmatrix} b^2 - ca - a^2 + bc & b^4 - a^4 \\ c^2 - ab - a^2 + bc & c^4 - a^4 \end{vmatrix} \\ &= \begin{vmatrix} (b-a)(b+a) + c(b-a) & (b-a)(b+a)(b^2+a^2) \\ (c-a)(c+a) + b(c-a) & (c-a)(c+a)(c^2+a^2) \end{vmatrix} \\ &= \begin{vmatrix} (b-a)(b+a+c) & (b-a)(b+a)(b^2+a^2) \\ (c-a)(c+a+b) & (c-a)(c+a)(c^2+a^2) \end{vmatrix} \\ &= (b-a)(c-a)\begin{vmatrix} b+a+c & (b+a)(b^2+a^2) \\ c+a+b & (c+a)(c^2+a^2) \end{vmatrix} \\ &= (b-a)(c-a)(a+b+c)\begin{vmatrix} 1 & (b+a)(b^2+a^2) \\ 1 & (c+a)(c^2+a^2) \end{vmatrix}. \end{aligned}$$

Finally

$$\begin{aligned} \begin{vmatrix} 1 & (b+a)(b^2+a^2) \\ 1 & (c+a)(c^2+a^2) \end{vmatrix} &= (c^3 + ac^2 + ca^2 + a^3) - (b^3 + ab^2 + ba^2 + a^3) \\ &= (c^3 - b^3) + a(c^2 - b^2) + a^2(c - b) \\ &= (c-b)(c^2 + cb + b^2 + a(c+b) + a^2) \\ &= (c-b)(c^2 + cb + b^2 + ac + ab + a^2). \end{aligned}$$

Section 5.8

1.

$$\begin{aligned}
\text{(i) } (-3+i)(14-2i) &= (-3)(14-2i)+i(14-2i)\\
&= \{(-3)14-(-3)(2i)\}+i(14)-i(2i)\\
&= (-42+6i)+(14i+2)=-40+20i.
\end{aligned}$$

$$\begin{aligned}
\text{(ii) } \frac{2+3i}{1-4i} &= \frac{(2+3i)(1+4i)}{(1-4i)(1+4i)}\\
&= \frac{((2+3i)+(2+3i)(4i)}{1^2+4^2}\\
&= \frac{-10+11i}{17}=\frac{-10}{17}+\frac{11}{17}i.
\end{aligned}$$

$$\begin{aligned}
\text{(iii) } \frac{(1+2i)^2}{1-i} &= \frac{1+4i+(2i)^2}{1-i}\\
&= \frac{1+4i-4}{1-i}=\frac{-3+4i}{1-i}\\
&= \frac{(-3+4i)(1+i)}{2}=\frac{-7+i}{2}=-\frac{7}{2}+\frac{1}{2}i.
\end{aligned}$$

2. (i)

$$\begin{aligned}
iz+(2-10i)z=3z+2i \quad &\Leftrightarrow \quad z(i+2-10i-3)=2i\\
&= \Leftrightarrow \quad z(-1-9i)=2i \Leftrightarrow z=\frac{-2i}{1+9i}\\
&= \quad \frac{-2i(1-9i)}{1+81}=\frac{-18-2i}{82}=\frac{-9-i}{41}.
\end{aligned}$$

(ii) The coefficient determinant is

$$\begin{vmatrix} 1+i & 2-i \\ 1+2i & 3+i \end{vmatrix} = (1+i)(3+i)-(2-i)(1+2i)=-2+i\neq 0.$$

Hence Cramer's rule applies: there is a unique solution given by

$$\begin{aligned}
z &= \frac{\begin{vmatrix} -3i & 2-i \\ 2+2i & 3+i \end{vmatrix}}{-2+i}=\frac{-3-11i}{-2+i}=-1+5i\\
w &= \frac{\begin{vmatrix} 1+i & -3i \\ 1+2i & 2+2i \end{vmatrix}}{-2+i}=\frac{-6+7i}{-2+i}=\frac{19-8i}{5}.
\end{aligned}$$

3.

$$
\begin{aligned}
1+(1+i)+\cdots+(1+i)^{99} &= \frac{(1+i)^{100}-1}{(1+i)-1} \\
&= \frac{(1+i)^{100}-1}{i} = -i\left\{(1+i)^{100}-1\right\}.
\end{aligned}
$$

Now $(1+i)^2 = 2i$. Hence

$$(1+i)^{100} = (2i)^{50} = 2^{50}i^{50} = 2^{50}(-1)^{25} = -2^{50}.$$

Hence $-i\left\{(1+i)^{100}-1\right\} = -i(-2^{50}-1) = (2^{50}+1)i$.

4. (i) Let $z^2 = -8-6i$ and write z=x+iy, where x and y are real. Then

$$z^2 = x^2 - y^2 + 2xyi = -8-6i,$$

so $x^2 - y^2 = -8$ and $2xy = -6$. Hence

$$y = -3/x, \quad x^2 - \left(\frac{-3}{x}\right)^2 = -8,$$

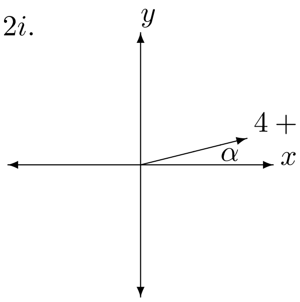
so $x^4 + 8x^2 - 9 = 0$. This is a quadratic in x^2. Hence $x^2 = 1$ or -9 and consequently $x^2 = 1$. Hence $x = 1,\ y = -3$ or $x = -1$ and $y = 3$. Hence $z = 1-3i$ or $z = -1+3i$.

(ii) $z^2 - (3+i)z + 4 + 3i = 0$ has the solutions $z = (3+i\pm d)/2$, where d is any complex number satisfying

$$d^2 = (3+i)^2 - 4(4+3i) = -8-6i.$$

Hence by part (i) we can take $d = 1-3i$. Consequently

$$z = \frac{3+i\pm(1-3i)}{2} = 2-i \quad \text{or} \quad 1+2i.$$



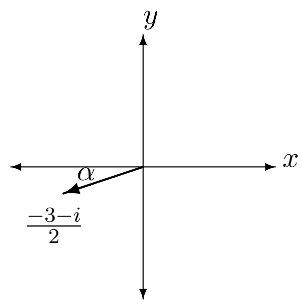
(i) The number lies in the first quadrant of the complex plane.

$$|4+i| = \sqrt{4^2+1^2} = \sqrt{17}.$$

Also $\mathrm{Arg}\,(4+i) = \alpha$, where $\tan\alpha = 1/4$ and $0 < \alpha < \pi/2$. Hence $\alpha = \tan^{-1}(1/4)$.

(ii) The number lies in the third quadrant of the complex plane.

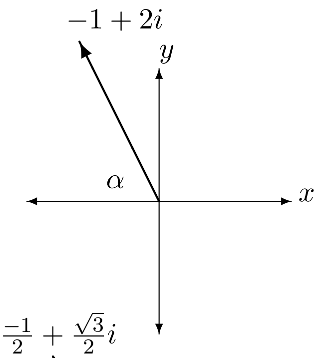
$$\left|\frac{-3-i}{2}\right| = \frac{|-3-i|}{2}$$
$$= \frac{1}{2}\sqrt{(-3)^2+(-1)^2} = \frac{1}{2}\sqrt{9+1} = \frac{\sqrt{10}}{2}.$$



Also $\mathrm{Arg}\,(\frac{-3-i}{2}) = -\pi+\alpha$, where $\tan\alpha = \frac{1}{2}/\frac{3}{2} = 1/3$ and $0<\alpha<\pi/2$. Hence $\alpha = \tan^{-1}(1/3)$.

(iii) The number lies in the second quadrant of the complex plane.

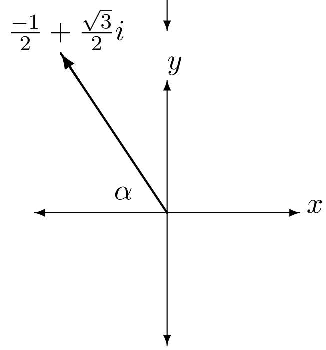
$$|-1+2i| = \sqrt{(-1)^2+2^2} = \sqrt{5}.$$



Also $\mathrm{Arg}\,(-1+2i) = \pi-\alpha$, where $\tan\alpha = 2$ and $0<\alpha<\pi/2$. Hence $\alpha = \tan^{-1}2$.

(iv) The number lies in the second quadrant of the complex plane.

$$\left|\frac{-1+i\sqrt{3}}{2}\right| = \frac{|-1+i\sqrt{3}|}{2}$$
$$= \frac{1}{2}\sqrt{(-1)^2+(\sqrt{3})^2} = \frac{1}{2}\sqrt{1+3} = 1.$$



Also $\mathrm{Arg}\,(\frac{-1}{2}+\frac{\sqrt{3}}{2}i) = \pi-\alpha$, where $\tan\alpha = \frac{\sqrt{3}}{2}/\frac{1}{2} = \sqrt{3}$ and $0<\alpha<\pi/2$. Hence $\alpha = \pi/3$.

6. (i) Let $z = (1+i)(1+\sqrt{3}i)(\sqrt{3}-i)$. Then

$$\begin{aligned} |z| &= |1+i||1+\sqrt{3}i||\sqrt{3}-i| \\ &= \sqrt{1^2+1^2}\sqrt{1^2+(\sqrt{3})^2}\sqrt{(\sqrt{3})^2+(-1)^2} \\ &= \sqrt{2}\sqrt{4}\sqrt{4} = 4\sqrt{2}. \end{aligned}$$

$$\mathrm{Arg}\,z \equiv \mathrm{Arg}\,(1+i) + \mathrm{Arg}\,(1+\sqrt{3}) + \mathrm{Arg}\,(\sqrt{3}-i) \pmod{2\pi}$$

$$\equiv \quad \frac{\pi}{4}+\frac{\pi}{3}-\frac{\pi}{6} \equiv \frac{5}{12}.$$

Hence $\operatorname{Arg} z = \frac{5}{12}$ and the polar decomposition of z is

$$z = 4\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right).$$

(ii) Let $z = \frac{(1+i)^5(1-i\sqrt{3})^5}{(\sqrt{3}+i)^4}$. Then

$$|z| = \frac{|(1+i)|^5|(1-i\sqrt{3})|^5}{|(\sqrt{3}+i)|^4} = \frac{\left(\sqrt{2}\right)^5 2^5}{2^4} = 2^{7/2}.$$

$$\begin{aligned}
\operatorname{Arg} z &\equiv \operatorname{Arg}(1+i)^5 + \operatorname{Arg}(1-\sqrt{3}i)^5 - \operatorname{Arg}(\sqrt{3}+i)^4 \pmod{2\pi} \\
&\equiv 5\operatorname{Arg}(1+i) + 5\operatorname{Arg}(1-\sqrt{3}i) - 4\operatorname{Arg}(\sqrt{3}+i) \\
&\equiv 5\frac{\pi}{4} + 5\left(\frac{-\pi}{3}\right) - 4\frac{\pi}{6} \equiv \frac{-13\pi}{12} \equiv \frac{11\pi}{12}.
\end{aligned}$$

Hence $\operatorname{Arg} z = \frac{11\pi}{12}$ and the polar decomposition of z is

$$z = 2^{7/2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right).$$

7. (i) Let $z = 2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ and $w = 3(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$. (Both of these numbers are already in polar form.)

(a) $zw = 6(\cos(\frac{\pi}{4}+\frac{\pi}{6}) + i\sin(\frac{\pi}{4}+\frac{\pi}{6}))$
$= 6(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})$.

(b) $\frac{z}{w} = \frac{2}{3}(\cos(\frac{\pi}{4}-\frac{\pi}{6}) + i\sin(\frac{\pi}{4}-\frac{\pi}{6}))$
$= \frac{2}{3}(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$.

(c) $\frac{w}{z} = \frac{3}{2}(\cos(\frac{\pi}{6}-\frac{\pi}{4}) + i\sin(\frac{\pi}{6}-\frac{\pi}{4}))$
$= \frac{3}{2}(\cos(\frac{-\pi}{12}) + i\sin(\frac{-\pi}{12}))$.

(d) $\frac{z^5}{w^2} = \frac{2^5}{3^2}(\cos(\frac{5\pi}{4}-\frac{2\pi}{6}) + i\sin(\frac{5\pi}{4}-\frac{2\pi}{6}))$
$= \frac{32}{9}(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12})$.

(a) $(1+i)^2 = 2i$, so

$$(1+i)^{12} = (2i)^6 = 2^6 i^6 = 64(i^2)^3 = 64(-1)^3 = -64.$$

(b) $(\frac{1-i}{\sqrt{2}})^2 = -i$, so

$$\begin{aligned}\left(\frac{1-i}{\sqrt{2}}\right)^{-6} &= \left(\left(\frac{1-i}{\sqrt{2}}\right)^2\right)^{-3}\\ &= (-i)^{-3} = \frac{-1}{i^3} = \frac{-1}{-i} = \frac{1}{i} = -i.\end{aligned}$$

8. (i) To solve the equation $z^2 = 1+\sqrt{3}i$, we write $1+\sqrt{3}i$ in modulus–argument form:

$$1+\sqrt{3}i = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}).$$

Then the solutions are

$$z_k = \sqrt{2}\left(\cos\left(\frac{\frac{\pi}{3}+2k\pi}{2}\right) + i\sin\left(\frac{\frac{\pi}{3}+2k\pi}{2}\right)\right), \quad k=0,\ 1.$$

Now $k=0$ gives the solution

$$z_0 = \sqrt{2}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = \frac{\sqrt{3}}{\sqrt{2}} + \frac{i}{\sqrt{2}}.$$

Clearly $z_1 = -z_0$.

(ii) To solve the equation $z^4 = i$, we write i in modulus–argument form:

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}.$$

Then the solutions are

$$z_k = \cos\left(\frac{\frac{\pi}{2}+2k\pi}{4}\right) + i\sin\left(\frac{\frac{\pi}{2}+2k\pi}{4}\right), \quad k=0,\ 1,\ 2,\ 3.$$

Now $\cos\left(\frac{\frac{\pi}{2}+2k\pi}{4}\right) = \cos\left(\frac{\pi}{8}+\frac{k\pi}{2}\right)$, so

$$\begin{aligned}z_k &= \cos\left(\frac{\pi}{8}+\frac{k\pi}{2}\right) + \sin\left(\frac{\pi}{8}+\frac{k\pi}{2}\right)\\ &= \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^k (\cos\frac{\pi}{8} + i\sin\frac{\pi}{8})\\ &= i^k(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}).\end{aligned}$$

Geometrically, the solutions lie equi–spaced on the unit circle at arguments

$$\frac{\pi}{8},\ \frac{\pi}{8}+\frac{\pi}{2}=\frac{5\pi}{8},\ \frac{\pi}{8}+\pi=\frac{9\pi}{8},\ \frac{\pi}{8}+3\frac{\pi}{2}=\frac{13\pi}{8}.$$

Also $z_2=-z_0$ and $z_3=-z_1$.

(iii) To solve the equation $z^3=-8i$, we rewrite the equation as

$$\left(\frac{z}{-2i}\right)^3=1.$$

Then

$$\left(\frac{z}{-2i}\right)=1,\quad \frac{-1+\sqrt{3}i}{2},\quad \text{or}\quad \frac{-1-\sqrt{3}i}{2}.$$

Hence $z=-2i,\ \sqrt{3}+i$ or $-\sqrt{3}+i$.

Geometrically, the solutions lie equi–spaced on the circle $|z|=2$, at arguments

$$\frac{\pi}{6},\ \frac{\pi}{6}+\frac{2\pi}{3}=\frac{5\pi}{6},\ \frac{\pi}{6}+2\frac{2\pi}{3}=\frac{3\pi}{2}.$$

(iv) To solve $z^4=2-2i$, we write $2-2i$ in modulus–argument form:

$$2-2i=2^{3/2}\left(\cos\frac{-\pi}{4}+i\sin\frac{-\pi}{4}\right).$$

Hence the solutions are

$$z_k=2^{3/8}\cos\left(\frac{\frac{-\pi}{4}+2k\pi}{4}\right)+i\sin\left(\frac{\frac{-\pi}{4}+2k\pi}{4}\right),\quad k=0,\,1,\,2,\,3.$$

We see the solutions can also be written as

$$\begin{aligned} z_k &= 2^{3/8}i^k\left(\cos\frac{-\pi}{16}+i\sin\frac{-\pi}{16}\right)\\ &= 2^{3/8}i^k\left(\cos\frac{\pi}{16}-i\sin\frac{\pi}{16}\right). \end{aligned}$$

Geometrically, the solutions lie equi–spaced on the circle $|z|=2^{3/8}$, at arguments

$$\frac{-\pi}{16},\ \frac{-\pi}{16}+\frac{\pi}{2}=\frac{7\pi}{16},\ \frac{-\pi}{16}+2\frac{\pi}{2}=\frac{15\pi}{16},\ \frac{-\pi}{16}+3\frac{\pi}{2}=\frac{23\pi}{16}.$$

Also $z_2=-z_0$ and $z_3=-z_1$.

9.

$$\left[\begin{array}{ccc} 2+i & -1+2i & 2 \\ 1+i & -1+i & 1 \\ 1+2i & -2+i & 1+i \end{array}\right] \begin{array}{c} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{array} \left[\begin{array}{ccc} 1 & i & 1 \\ 1+i & -1+i & 1 \\ i & -1 & i \end{array}\right]$$

$$\begin{array}{c} R_2 \to R_2 - (1+i)R_1 \\ R_3 \to R_3 - iR_1 \end{array} \left[\begin{array}{ccc} 1 & i & 1 \\ 0 & 0 & -i \\ 0 & 0 & 0 \end{array}\right] \; R_2 \to iR_2 \left[\begin{array}{ccc} 1 & i & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$R_1 \to R_1 - R_2 \left[\begin{array}{ccc} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

The last matrix is in reduced row–echelon form.

10. (i) Let $p = l + im$ and $z = x + iy$. Then

$$\begin{aligned} \overline{p}z + p\overline{z} &= (l - im)(x + iy) + (l + im)(x - iy) \\ &= (lx + liy - imx + my) + (lx - liy + imx + my) \\ &= 2(lx + my). \end{aligned}$$

Hence $\overline{p}z + p\overline{z} = 2n \Leftrightarrow lx + my = n$.

(ii) Let w be the complex number which results from reflecting the complex number z in the line $lx + my = n$. Then because p is perpendicular to the given line, we have

$$w - z = tp, \quad t \in \mathbb{R}. \qquad (a)$$

Also the midpoint $\frac{w+z}{2}$ of the segment joining w and z lies on the given line, so

$$\begin{aligned} \overline{p}\left(\frac{w+z}{2}\right) + p\left(\overline{\frac{w+z}{2}}\right) &= n, \\ \overline{p}\left(\frac{w+z}{2}\right) + p\left(\frac{\overline{w}+\overline{z}}{2}\right) &= n. \qquad (b) \end{aligned}$$

Taking conjugates of equation (a) gives

$$\overline{w} - \overline{z} = t\overline{p}. \qquad (c)$$

Then substituting in (b), using (a) and (c), gives

$$\overline{p}\left(\frac{2w - tp}{2}\right) + p\left(\frac{2\overline{z} + t\overline{p}}{2}\right) = n$$

and hence

$$\overline{p}w + p\overline{z} = n.$$

(iii) Let $p = b - a$ and $n = |b|^2 - |a|^2$. Then

$$\begin{aligned}
|z-a| = |z-b| &\Leftrightarrow& |z-a|^2 = |z-b|^2 \\
\Leftrightarrow (z-a)(\overline{z-a}) &=& (z-b)(\overline{z-b}) \\
\Leftrightarrow (z-a)(\overline{z}-\overline{a}) &=& (z-b)(\overline{z}-\overline{b}) \\
\Leftrightarrow z\overline{z} - a\overline{z} - z\overline{a} + a\overline{a} &=& z\overline{z} - b\overline{z} - z\overline{b} + b\overline{b} \\
\Leftrightarrow (\overline{b}-\overline{a})z + (b-a)\overline{z} &=& |b|^2 - |a|^2 \\
\Leftrightarrow \overline{p}z + p\overline{z} &=& n.
\end{aligned}$$

Suppose z lies on the circle $\left|\frac{z-a}{z-b}\right|$ and let w be the reflection of z in the line $\overline{p}z + p\overline{z} = n$. Then by part (ii)

$$\overline{p}w + p\overline{z} = n.$$

Taking conjugates gives $p\overline{w} + \overline{p}z = n$ and hence

$$z = \frac{n - p\overline{w}}{\overline{p}} \qquad (a)$$

Substituting for z in the circle equation, using (a) gives

$$\lambda = \left|\frac{\frac{n-p\overline{w}}{\overline{p}} - a}{\frac{n-p\overline{w}}{\overline{p}} - b}\right| = \left|\frac{n - p\overline{w} - \overline{p}a}{n - p\overline{w} - \overline{p}b}\right|. \qquad (b)$$

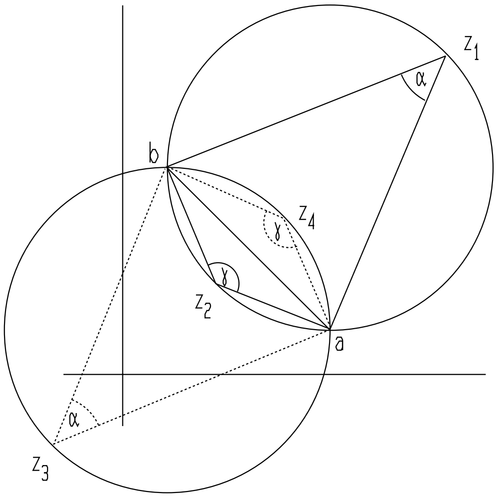
However

$$\begin{aligned}
n - \overline{p}a &= |b|^2 - |a|^2 - (\overline{b}-\overline{a})a \\
&= \overline{b}b - \overline{a}a - \overline{b}a + \overline{a}a \\
&= \overline{b}(b-a) = \overline{b}p.
\end{aligned}$$

Similarly $n - \overline{p}b = \overline{a}p$. Consequently (b) simplifies to

$$\lambda = \left|\frac{\overline{b}p - p\overline{w}}{\overline{a}p - p\overline{w}}\right| = \left|\frac{\overline{b} - \overline{w}}{\overline{a} - \overline{w}}\right| = \left|\frac{w-b}{w-a}\right|,$$

which gives $\left|\frac{w-a}{w-b}\right| = \frac{1}{\lambda}$.



11. Let a and b be distinct complex numbers and $0 < \alpha < \pi$.

(i) When z_1 lies on the circular arc shown, it subtends a constant angle α. This angle is given by $\operatorname{Arg}(z_1 - a) - \operatorname{Arg}(z_1 - b)$. However

$$\begin{aligned}\operatorname{Arg}\left(\frac{z_1 - a}{z_1 - b}\right) &= \operatorname{Arg}(z_1 - a) - \operatorname{Arg}(z_1 - b) + 2k\pi \\ &= \alpha + 2k\pi.\end{aligned}$$

It follows that $k = 0$, as $0 < \alpha < \pi$ and $-\pi < \operatorname{Arg}\theta \le \pi$. Hence

$$\operatorname{Arg}\left(\frac{z_1 - a}{z_1 - b}\right) = \alpha.$$

Similarly if z_2 lies on the circular arc shown, then

$$\operatorname{Arg}\left(\frac{z_2 - a}{z_2 - b}\right) = -\gamma = -(\pi - \alpha) = \alpha - \pi.$$

Replacing α by $\pi - \alpha$, we deduce that if z_4 lies on the circular arc shown, then

$$\operatorname{Arg}\left(\frac{z_4 - a}{z_4 - b}\right) = \pi - \alpha,$$

while if z_3 lies on the circular arc shown, then

$$\operatorname{Arg}\left(\frac{z_3 - a}{z_3 - b}\right) = -\alpha.$$

The straight line through a and b has the equation

$$z = (1 - t)a + tb,$$

where t is real. Then $0 < t < 1$ describes the segment ab. Also

$$\frac{z-a}{z-b} = \frac{t}{t-1}.$$

Hence $\frac{z-a}{z-b}$ is real and negative if z is on the segment a, but is real and positive if z is on the remaining part of the line, with corresponding values

$$\text{Arg}\left(\frac{z-a}{z-b}\right) = \pi,\ 0,$$

respectively.

(ii) Case (a) Suppose $z_1,\ z_2$ and z_3 are not collinear. Then these points determine a circle. Now z_1 and z_2 partition this circle into two arcs. If z_3 and z_4 lie on the same arc, then

$$\text{Arg}\left(\frac{z_3-z_1}{z_3-z_2}\right) = \text{Arg}\left(\frac{z_4-z_1}{z_4-z_2}\right);$$

whereas if z_3 and z_4 lie on opposite arcs, then

$$\text{Arg}\left(\frac{z_3-z_1}{z_3-z_2}\right) = \alpha$$

and

$$\text{Arg}\left(\frac{z_4-z_1}{z_4-z_2}\right) = \alpha - \pi.$$

Hence in both cases

$$\begin{aligned}\text{Arg}\left(\frac{z_3-z_1}{z_3-z_2}\Big/\frac{z_4-z_1}{z_4-z_2}\right) &\equiv \text{Arg}\left(\frac{z_3-z_1}{z_3-z_2}\right) - \text{Arg}\left(\frac{z_4-z_1}{z_4-z_2}\right) \pmod{2\pi}\\ &\equiv 0 \text{ or } \pi.\end{aligned}$$

In other words, the cross–ratio

$$\frac{z_3-z_1}{z_3-z_2}\Big/\frac{z_4-z_1}{z_4-z_2}$$

is real.

(b) If $z_1,\ z_2$ and z_3 are collinear, then again the cross–ratio is real.

The argument is reversible.

(iii) Assume that A, B, C, D are distinct points such that the cross–ratio

$$r = \frac{z_3-z_1}{z_3-z_2}\Big/\frac{z_4-z_1}{z_4-z_2}$$

is real. Now r cannot be 0 or 1. Then there are three cases:

(i) $0 < r < 1$;

(ii) $r < 0$;

(iii) $r > 1$.

Case (i). Here $|r| + |1 - r| = 1$. So

$$\left|\frac{z_4 - z_1}{z_4 - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1}\right| + \left|1 - \left(\frac{z_4 - z_1}{z_4 - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1}\right)\right| = 1.$$

Multiplying both sides by the denominator $|z_4 - z_2||z_3 - z_1|$ gives after simplification

$$|z_4 - z_1||z_3 - z_2| + |z_2 - z_1||z_4 - z_3| = |z_4 - z_2||z_3 - z_1|,$$

or

$$\text{(a)} \quad AD \cdot BC + AB \cdot CD = BD \cdot AC.$$

Case (ii). Here $1 + |r| = |1 - r|$. This leads to the equation

$$\text{(b)} \quad BD \cdot AC + AD \cdot BC + = AB \cdot CD.$$

Case (iii). Here $1 + |1 - r| = |r|$. This leads to the equation

$$\text{(c)} \quad BD \cdot AC + AB \cdot CD = AD \cdot BC.$$

Conversely if (a), (b) or (c) hold, then we can reverse the argument to deduce that r is a complex number satisfying one of the equations

$$|r| + |1 - r| = 1, \quad 1 + |r| = |1 - r|, \quad 1 + |1 - r| = |r|,$$

from which we deduce that r is real.

Section 6.3

1. Let $A = \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$. Then A has characteristic equation $\lambda^2 - 4\lambda + 3 = 0$ or $(\lambda - 3)(\lambda - 1) = 0$. Hence the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 1$.

$\lambda_1 = 3$. The corresponding eigenvectors satisfy $(A - \lambda_1 I_2)X = 0$, or

$$\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or equivalently $x - 3y = 0$. Hence

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3y \\ y \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and we take $X_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Similarly for $\lambda_2 = 1$ we find the eigenvector $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Hence if $P = [X_1|X_2] = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$, then P is non–singular and

$$P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence

$$A = P \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} P^{-1}$$

and consequently

$$\begin{aligned}
A^n &= P \begin{bmatrix} 3^n & 0 \\ 0 & 1^n \end{bmatrix} P^{-1} \\
&= \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^n & 0 \\ 0 & 1^n \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 3^{n+1} & 1 \\ 3^n & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 3^{n+1} - 1 & -3^{n+1} + 3 \\ 3^n - 1 & -3^n + 3 \end{bmatrix} \\
&= \frac{3^n - 1}{2} A + \frac{3 - 3^n}{2} I_2.
\end{aligned}$$

2. Let $A = \begin{bmatrix} 3/5 & 4/5 \\ 2/5 & 1/5 \end{bmatrix}$. Then we find that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1/5$, with corresponding eigenvectors

$$X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Then if $P = [X_1|X_2]$, P is non–singular and

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1/5 \end{bmatrix} \quad \text{and} \quad A = P\begin{bmatrix} 1 & 0 \\ 0 & -1/5 \end{bmatrix}P^{-1}.$$

Hence

$$\begin{aligned} A^n &= P\begin{bmatrix} 1 & 0 \\ 0 & (-1/5)^n \end{bmatrix}P^{-1} \\ &\to P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}P^{-1} \\ &= \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix}. \end{aligned}$$

3. The given system of differential equations is equivalent to $\dot{X} = AX$, where

$$A = \begin{bmatrix} 3 & -2 \\ 5 & -4 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The matrix $P = \begin{bmatrix} 2 & 1 \\ 5 & 1 \end{bmatrix}$ is a non-singular matrix of eigenvectors corresponding to eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 1$. Then

$$P^{-1}AP = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}.$$

The substitution $X = PY$, where $Y = [x_1,\, y_1]^t$, gives

$$\dot{Y} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}Y,$$

or equivalently $\dot{x_1} = -2x_1$ and $\dot{y_1} = y_1$.

Hence $x_1 = x_1(0)e^{-2t}$ and $y_1 = y_1(0)e^t$. To determine $x_1(0)$ and $y_1(0)$, we note that

$$\begin{bmatrix} x_1(0) \\ y_1(0) \end{bmatrix} = P^{-1}\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 1 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 13 \\ 22 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

Hence $x_1 = 3e^{-2t}$ and $y_1 = 7e^t$. Consequently

$$x = 2x_1 + y_1 = 6e^{-2t} + 7e^t \quad \text{and} \quad y = 5x_1 + y_1 = 15e^{-2t} + 7e^t.$$

4. Introducing the vector $X_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$, the system of recurrence relations

$$\begin{aligned} x_{n+1} &= 3x_n - y_n \\ y_{n+1} &= -x_n + 3y_n, \end{aligned}$$

becomes $X_{n+1} = AX_n$, where $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$. Hence $X_n = A^n X_0$, where $X_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

To find A^n we can use the eigenvalue method. We get

$$A^n = \frac{1}{2}\begin{bmatrix} 2^n + 4^n & 2^n - 4^n \\ 2^n - 4^n & 2^n + 4^n \end{bmatrix}.$$

Hence

$$\begin{aligned} X_n &= \frac{1}{2}\begin{bmatrix} 2^n + 4^n & 2^n - 4^n \\ 2^n - 4^n & 2^n + 4^n \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 2^n + 4^n + 2(2^n - 4^n) \\ 2^n - 4^n + 2(2^n + 4^n) \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 3 \times 2^n - 4^n \\ 3 \times 2^n + 4^n \end{bmatrix} = \begin{bmatrix} (3 \times 2^n - 4^n)/2 \\ (3 \times 2^n + 4^n)/2 \end{bmatrix}. \end{aligned}$$

Hence $x_n = \frac{1}{2}(3 \times 2^n - 4^n)$ and $y_n = \frac{1}{2}(3 \times 2^n + 4^n)$.

5. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a real or complex matrix with distinct eigenvalues λ_1, λ_2 and corresponding eigenvectors X_1, X_2. Also let $P = [X_1|X_2]$.

(a) The system of recurrence relations

$$\begin{aligned} x_{n+1} &= ax_n + by_n \\ y_{n+1} &= cx_n + dy_n \end{aligned}$$

has the solution

$$\begin{aligned}
\begin{bmatrix} x_n \\ y_n \end{bmatrix} &= A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \left(P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}\right)^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\
&= P \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} P^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\
&= [X_1|X_2] \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\
&= [X_1|X_2] \begin{bmatrix} \lambda_1^n \alpha \\ \lambda_2^n \beta \end{bmatrix} = \lambda_1^n \alpha X_1 + \lambda_2^n \beta X_2,
\end{aligned}$$

where

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = P^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

(b) In matrix form, the system is $\dot{X} = AX$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$. We substitute $X = PY$, where $Y = [x_1, y_1]^t$. Then

$$\dot{X} = P\dot{Y} = AX = A(PY),$$

so

$$\dot{Y} = (P^{-1}AP)Y = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Hence $\dot{x_1} = \lambda_1 x_1$ and $\dot{y_1} = \lambda_2 y_1$. Then

$$x_1 = x_1(0)e^{\lambda_1 t} \quad \text{and} \quad y_1 = y_1(0)e^{\lambda_2 t}.$$

But

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = P \begin{bmatrix} x_1(0) \\ y_1(0) \end{bmatrix},$$

so

$$\begin{bmatrix} x_1(0) \\ y_1(0) \end{bmatrix} = P^{-1} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Consequently $x_1(0) = \alpha$ and $y_1(0) = \beta$ and

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &= P \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = [X_1|X_2] \begin{bmatrix} \alpha e^{\lambda_1 t} \\ \beta e^{\lambda_2 t} \end{bmatrix} \\
&= \alpha e^{\lambda_1 t} X_1 + \beta e^{\lambda_2 t} X_2.
\end{aligned}$$

6. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a real matrix with non–real eigenvalues $\lambda = a + ib$ and $\overline{\lambda} = a - ib$, with corresponding eigenvectors $X = U + iV$ and $\overline{X} = U - iV$, where U and V are real vectors. Also let P be the real matrix defined by $P = [U|V]$. Finally let $a + ib = re^{i\theta}$, where $r > 0$ and θ is real.

(a) As X is an eigenvector corresponding to the eigenvalue λ, we have $AX = \lambda X$ and hence

$$\begin{aligned} A(U + iV) &= (a + ib)(U + iV) \\ AU + iAV &= aU - bV + i(bU + aV). \end{aligned}$$

Equating real and imaginary parts then gives

$$\begin{aligned} AU &= aU - bV \\ AV &= bU + aV. \end{aligned}$$

(b)

$$AP = A[U|V] = [AU|AV] = [aU-bV|bU+aV] = [U|V]\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = P\begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

Hence, as P can be shown to be non–singular,

$$P^{-1}AP = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

(The fact that P is non–singular is easily proved by showing the columns of P are linearly independent: Assume $xU + yV = 0$, where x and y are real. Then we find

$$(x + iy)(U - iV) + (x - iy)(U + iV) = 0.$$

Consequently $x + iy = 0$ as $U - iV$ and $U + iV$ are eigenvectors corresponding to distinct eigenvalues $a - ib$ and $a + ib$ and are hence linearly independent. Hence $x = 0$ and $y = 0$.)

(c) The system of recurrence relations

$$\begin{aligned} x_{n+1} &= ax_n + by_n \\ y_{n+1} &= cx_n + dy_n \end{aligned}$$

has solution

$$
\begin{aligned}
\begin{bmatrix} x_n \\ y_n \end{bmatrix} &= A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\
&= P \begin{bmatrix} a & b \\ -b & a \end{bmatrix}^n P^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\
&= P \begin{bmatrix} r\cos\theta & r\sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}^n \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\
&= Pr^n \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^n \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\
&= r^n[U|V] \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\
&= r^n[U|V] \begin{bmatrix} \alpha\cos n\theta + \beta\sin n\theta \\ -\alpha\sin n\theta + \beta\cos n\theta \end{bmatrix} \\
&= r^n \left\{(\alpha\cos n\theta + \beta\sin n\theta)U + (-\alpha\sin n\theta + \beta\cos n\theta)V\right\} \\
&= r^n \left\{(\cos n\theta)(\alpha U + \beta V) + (\sin n\theta)(\beta U - \alpha V)\right\}.
\end{aligned}
$$

(d) The system of differential equations

$$
\begin{aligned}
\frac{dx}{dt} &= ax + by \\
\frac{dy}{dt} &= cx + dy
\end{aligned}
$$

is attacked using the substitution $X = PY$, where $Y = [x_1,\, y_1]^t$. Then

$$\dot{Y} = (P^{-1}AP)Y,$$

so

$$\begin{bmatrix} \dot{x_1} \\ \dot{y_1} \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Equating components gives

$$
\begin{aligned}
\dot{x_1} &= ax_1 + by_1 \\
\dot{y_1} &= -bx_1 + ay_1.
\end{aligned}
$$

Now let $z = x_1 + iy_1$. Then

$$
\begin{aligned}
\dot{z} = \dot{x_1} + i\dot{y_1} &= (ax_1 + by_1) + i(-bx_1 + ay_1) \\
&= (a - ib)(x_1 + iy_1) = (a - ib)z.
\end{aligned}
$$

Hence

$$\begin{aligned} z &= z(0)e^{(a-ib)t} \\ x_1 + iy_1 &= (x_1(0) + iy_1(0))e^{at}(\cos bt - i\sin bt). \end{aligned}$$

Equating real and imaginary parts gives

$$\begin{aligned} x_1 &= e^{at}\left\{x_1(0)\cos bt + y_1(0)\sin bt\right\} \\ y_1 &= e^{at}\left\{y_1(0)\cos bt - x_1(0)\sin bt\right\}. \end{aligned}$$

Now if we define α and β by

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = P^{-1}\begin{bmatrix} x(0) \\ y(0) \end{bmatrix},$$

we see that $\alpha = x_1(0)$ and $\beta = y_1(0)$. Then

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= P\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \\ &= [U|V]\begin{bmatrix} e^{at}(\alpha\cos bt + \beta\sin bt) \\ e^{at}(\beta\cos bt - \alpha\sin bt) \end{bmatrix} \\ &= e^{at}\{(\alpha\cos bt + \beta\sin bt)U + (\beta\cos bt - \alpha\sin bt)V\} \\ &= e^{at}\{\cos bt(\alpha U + \beta V) + \sin bt(\beta U - \alpha V)\}. \end{aligned}$$

7. (The case of repeated eigenvalues.) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and suppose that the characteristic polynomial of A, $\lambda^2 - (a+d)\lambda + (ad - bc)$, has a repeated root α. Also assume that $A \neq \alpha I_2$.

(i)

$$\begin{aligned} \lambda^2 - (a+d)\lambda + (ad-bc) &= (\lambda - \alpha)^2 \\ &= \lambda^2 - 2\alpha\lambda + \alpha^2. \end{aligned}$$

Hence $a + d = 2\alpha$ and $ad - bc = \alpha^2$ and

$$\begin{aligned} (a+d)^2 &= 4(ad-bc), \\ a^2 + 2ad + d^2 &= 4ad - 4bc, \\ a^2 - 2ad + d^2 + 4bc &= 0, \\ (a-d)^2 + 4bc &= 0. \end{aligned}$$

(ii) Let $B - A - \alpha I_2$. Then

$$\begin{aligned} B^2 = (A - \alpha I_2)^2 &= A^2 - 2\alpha A + \alpha^2 I_2 \\ &= A^2 - (a+d)A + (ad - bc)I_2, \end{aligned}$$

But by problem 3, chapter 2.4, $A^2 - (a+d)A + (ad - bc)I_2 = 0$, so $B^2 = 0$.

(iii) Now suppose that $B \neq 0$. Then $BE_1 \neq 0$ or $BE_2 \neq 0$, as BE_i is the i–th column of B. Hence $BX_2 \neq 0$, where $X_2 = E_1$ or $X_2 = E_2$.

(iv) Let $X_1 = BX_2$ and $P = [X_1|X_2]$. We prove P is non–singular by demonstrating that X_1 and X_2 are linearly independent.

Assume $xX_1 + yX_2 = 0$. Then

$$\begin{aligned} xBX_2 + yX_2 &= 0 \\ B(xBX_2 + yX_2) &= B0 = 0 \\ xB^2X_2 + yBX_2 &= 0 \\ x0X_2 + yBX_2 &= 0 \\ yBX_2 &= 0. \end{aligned}$$

Hence $y = 0$ as $BX_2 \neq 0$. Hence $xBX_2 = 0$ and so $x = 0$.

Finally, $BX_1 = B(BX_2) = B^2X_2 = 0$, so $(A - \alpha I_2)X_1 = 0$ and

$$AX_1 = \alpha X_1. \qquad (2)$$

Also

$$X_1 = BX_2 = (A - \alpha I_2)X_2 = AX_2 - \alpha X_2.$$

Hence

$$AX_2 = X_1 + \alpha X_2. \qquad (3)$$

Then, using (2) and (3), we have

$$\begin{aligned} AP = A[X_1|X_2] &= [AX_1|AX_2] \\ &= [\alpha X_1|X_1 + \alpha X_2] \\ &= [X_1|X_2]\begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}. \end{aligned}$$

Hence

$$AP = P\begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}$$

and hence

$$P^{-1}AP = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}.$$

8. The system of differential equations is equivalent to the single matrix equation $\dot{X} = AX$, where $A = \begin{bmatrix} 4 & -1 \\ 4 & 8 \end{bmatrix}$.

The characteristic polynomial of A is $\lambda^2 - 12\lambda + 36 = (\lambda - 6)^2$, so we can use the previous question with $\alpha = 6$. Let

$$B = A - 6I_2 = \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}.$$

Then $BX_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, if $X_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Also let $X_1 = BX_2$. Then if $P = [X_1|X_2]$, we have

$$P^{-1}AP = \begin{bmatrix} 6 & 1 \\ 0 & 6 \end{bmatrix}.$$

Now make the change of variables $X = PY$, where $Y = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$. Then

$$\dot{Y} = (P^{-1}AP)Y = \begin{bmatrix} 6 & 1 \\ 0 & 6 \end{bmatrix} Y,$$

or equivalently $\dot{x_1} = 6x_1 + y_1$ and $\dot{y_1} = 6y_1$.

Solving for y_1 gives $y_1 = y_1(0)e^{6t}$. Consequently

$$\dot{x_1} = 6x_1 + y_1(0)e^{6t}.$$

Multiplying both side of this equation by e^{-6t} gives

$$\begin{aligned} \frac{d}{dt}(e^{-6t}x_1) &= e^{-6t}\dot{x_1} - 6e^{-6t}x_1 = y_1(0) \\ e^{-6t}x_1 &= y_1(0)t + c, \end{aligned}$$

where c is a constant. Substituting $t = 0$ gives $c = x_1(0)$. Hence

$$e^{-6t}x_1 = y_1(0)t + x_1(0)$$

and hence

$$x_1 = e^{6t}(y_1(0)t + x_1(0)).$$

However, since we are assuming $x(0) = 1 = y(0)$, we have

$$\begin{aligned}
\begin{bmatrix} x_1(0) \\ y_1(0) \end{bmatrix} &= P^{-1} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} \\
&= \frac{1}{-4} \begin{bmatrix} 0 & -1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 \\ -6 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 3/2 \end{bmatrix}.
\end{aligned}$$

Hence $x_1 = e^{6t}(\frac{3}{2}t + \frac{1}{4})$ and $y_1 = \frac{3}{2}e^{6t}$.

Finally, solving for x and y,

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -2 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} e^{6t}(\frac{3}{2}t + \frac{1}{4}) \\ \\ \frac{3}{2}e^{6t} \end{bmatrix} \\
&= \begin{bmatrix} (-2)e^{6t}(\frac{3}{2}t + \frac{1}{4}) + \frac{3}{2}e^{6t} \\ \\ 4e^{6t}(\frac{3}{2}t + \frac{1}{4}) \end{bmatrix} \\
&= \begin{bmatrix} e^{6t}(1 - 3t) \\ e^{6t}(6t + 1) \end{bmatrix}.
\end{aligned}$$

Hence $x = e^{6t}(1 - 3t)$ and $y = e^{6t}(6t + 1)$.

9. Let

$$A = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}.$$

(a) We first determine the characteristic polynomial $\mathrm{ch}_A(\lambda)$.

$$\begin{aligned}
\mathrm{ch}_A(\lambda) &= \det(\lambda I_3 - A) = \begin{vmatrix} \lambda - 1/2 & -1/2 & 0 \\ -1/4 & \lambda - 1/4 & -1/2 \\ -1/4 & -1/4 & \lambda - 1/2 \end{vmatrix} \\
&= \left(\lambda - \frac{1}{2}\right) \begin{vmatrix} \lambda - 1/4 & -1/2 \\ -1/4 & \lambda - 1/2 \end{vmatrix} + \frac{1}{2} \begin{vmatrix} -1/4 & -1/2 \\ -1/4 & \lambda - 1/2 \end{vmatrix} \\
&= \left(\lambda - \frac{1}{2}\right) \left\{ \left(\lambda - \frac{1}{4}\right) \left(\lambda - \frac{1}{2}\right) - \frac{1}{8} \right\} + \frac{1}{2} \left\{ \frac{-1}{4} \left(\lambda - \frac{1}{2}\right) - \frac{1}{8} \right\} \\
&= \left(\lambda - \frac{1}{2}\right) \left(\lambda^2 - \frac{3\lambda}{4}\right) - \frac{\lambda}{8} \\
&= \lambda \left\{ \left(\lambda - \frac{1}{2}\right) \left(\lambda - \frac{3}{4}\right) - \frac{1}{8} \right\}
\end{aligned}$$

$$\begin{aligned} &= \lambda\left(\lambda^2 - \frac{5\lambda}{4} + \frac{1}{4}\right) \\ &= \lambda(\lambda-1)\left(\lambda - \frac{1}{4}\right). \end{aligned}$$

(b) Hence the characteristic polynomial has no repeated roots and we can use Theorem 6.2.2 to find a non–singular matrix P such that

$$P^{-1}AP = \mathrm{diag}(1,\, 0,\, \frac{1}{4}).$$

We take $P = [X_1|X_2|X_3]$, where $X_1,\ X_2,\ X_3$ are eigenvectors corresponding to the respective eigenvalues $1,\ 0,\ \frac{1}{4}$.
Finding X_1: We have to solve $(A - I_3)X = 0$. we have

$$A - I_3 = \begin{bmatrix} -1/2 & 1/2 & 0 \\ 1/4 & -3/4 & 1/2 \\ 1/4 & 1/4 & -1/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence the eigenspace consists of vectors $X = [x,\ y,\ z]^t$ satisfying $x = z$ and $y = z$, with z arbitrary. Hence

$$X = \begin{bmatrix} z \\ z \\ z \end{bmatrix} = z\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and we can take $X_1 = [1,\ 1,\ 1]^t$.
Finding X_2: We solve $AX = 0$. We have

$$A = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence the eigenspace consists of vectors $X = [x,\ y,\ z]^t$ satisfying $x = -y$ and $z = 0$, with y arbitrary. Hence

$$X = \begin{bmatrix} -y \\ y \\ 0 \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and we can take $X_2 = [-1,\ 1,\ 0]^t$.

Finding X_3: We solve $(A - \frac{1}{4}I_3)X = 0$. We have

$$A - \frac{1}{4}I_3 = \begin{bmatrix} 1/4 & 1/2 & 0 \\ 1/4 & 0 & 1/2 \\ 1/4 & 1/4 & 1/4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence the eigenspace consists of vectors $X = [x,\, y,\, z]^t$ satisfying $x = -2z$ and $y = z$, with z arbitrary. Hence

$$X = \begin{bmatrix} -2z \\ z \\ 0 \end{bmatrix} = z \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

and we can take $X_3 = [-2,\, 1,\, 1]^t$.

Hence we can take $P = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

(c) $A = P\mathrm{diag}(1,\, 0,\, \frac{1}{4})P^{-1}$ so $A^n = P\mathrm{diag}(1,\, 0,\, \frac{1}{4^n})P^{-1}$.
Hence

$$\begin{aligned} A^n &= \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4^n} \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -3 \\ -1 & -1 & 2 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 & 0 & -\frac{2}{4^n} \\ 1 & 0 & \frac{1}{4^n} \\ 1 & 0 & \frac{1}{4^n} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -3 \\ -1 & -1 & 2 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1+\frac{2}{4^n} & 1+\frac{2}{4^n} & 1-\frac{4}{4^n} \\ 1-\frac{1}{4^n} & 1-\frac{1}{4^n} & 1+\frac{2}{4^n} \\ 1-\frac{1}{4^n} & 1-\frac{1}{4^n} & 1+\frac{2}{4^n} \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{1}{3 \cdot 4^n} \begin{bmatrix} 2 & 2 & -4 \\ -1 & -1 & 2 \\ -1 & -1 & 2 \end{bmatrix}. \end{aligned}$$

10. Let

$$A = \begin{bmatrix} 5 & 2 & -2 \\ 2 & 5 & -2 \\ -2 & -2 & 5 \end{bmatrix}.$$

(a) We first determine the characteristic polynomial $\mathrm{ch}_A(\lambda)$.

$$\begin{aligned} \mathrm{ch}_A(\lambda) &= \left| \begin{matrix} \lambda-5 & -2 & 2 \\ -2 & \lambda-5 & 2 \\ 2 & 2 & \lambda-5 \end{matrix} \right| \overset{R_3 \to R_3 + R_2}{=} \left| \begin{matrix} \lambda-5 & -2 & 2 \\ -2 & \lambda-5 & 2 \\ 0 & \lambda-3 & \lambda-3 \end{matrix} \right| \\ &= (\lambda-3) \left| \begin{matrix} \lambda-5 & -2 & 2 \\ -2 & \lambda-5 & 2 \\ 0 & 1 & 1 \end{matrix} \right| \end{aligned}$$

$$
\begin{aligned}
C_3 \to C_3 - C_2 \quad &= (\lambda-3)\begin{vmatrix} \lambda-5 & -2 & 4 \\ -2 & \lambda-5 & -\lambda+7 \\ 0 & 1 & 0 \end{vmatrix} \\
&= -(\lambda-3)\begin{vmatrix} \lambda-5 & 4 \\ -2 & -\lambda+7 \end{vmatrix} \\
&= -(\lambda-3)\{(\lambda-5)(-\lambda+7)+8\} \\
&= -(\lambda-3)(-\lambda^2+5\lambda+7\lambda-35+8) \\
&= -(\lambda-3)(-\lambda^2+12\lambda-27) \\
&= -(\lambda-3)(-1)(\lambda-3)(\lambda-9) \\
&= (\lambda-3)^2(\lambda-9).
\end{aligned}
$$

We have to find bases for each of the eigenspaces $N(A-9I_3)$ and $N(A-3I_3)$.

First we solve $(A-3I_3)X=0$. We have

$$
A-3I_3 = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

Hence the eigenspace consists of vectors $X = [x,\ y,\ z]^t$ satisfying $x = -y+z$, with y and z arbitrary. Hence

$$
X = \begin{bmatrix} -y+z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},
$$

so $X_1 = [-1,\ 1,\ 0]^t$ and $X_2 = [1,\ 0,\ 1]^t$ form a basis for the eigenspace corresponding to the eigenvalue 3.

Next we solve $(A-9I_3)X=0$. We have

$$
A-9I_3 = \begin{bmatrix} -4 & 2 & -2 \\ 2 & -4 & -2 \\ -2 & -2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.
$$

Hence the eigenspace consists of vectors $X = [x,\ y,\ z]^t$ satisfying $x=-z$ and $y=-z$, with z arbitrary. Hence

$$
X = \begin{bmatrix} -z \\ -z \\ z \end{bmatrix} = z\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}
$$

and we can take $X_3 = [-1,\ -1,\ 1]^t$ as a basis for the eigenspace corresponding to the eigenvalue 9.

Then Theorem 6.2.3 assures us that $P = [X_1|X_2|X_3]$ is non–singular and

$$P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

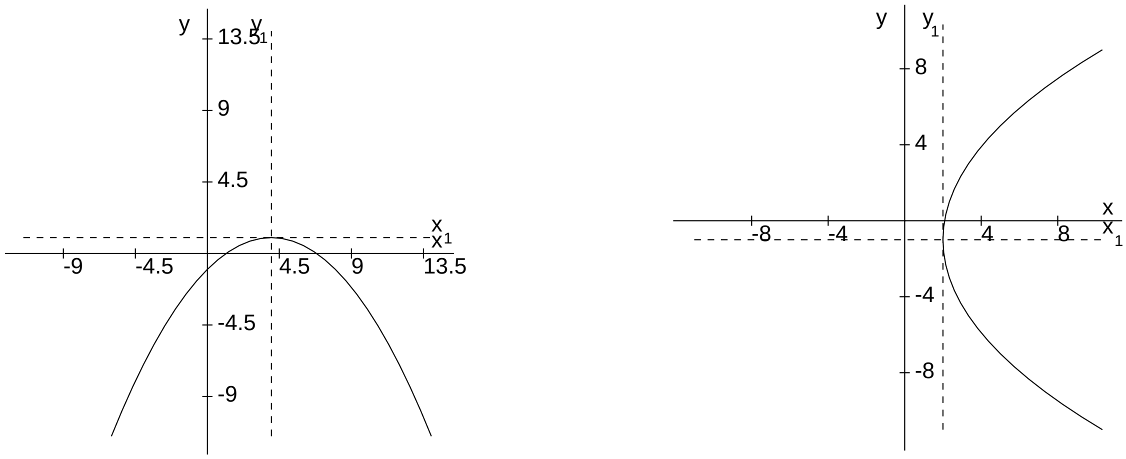


Figure 1: (a): $x^2 - 8x + 8y + 8 = 0$; (b): $y^2 - 12x + 2y + 25 = 0$

Section 7.3

1. (i) $x^2-8x+8y+8 = (x-4)^2+8(y-1)$. So the equation $x^2-8x+8y+8 = 0$ becomes

$$x_1^2 + 8y_1 = 0 \qquad (1)$$

if we make a translation of axes $x - 4 = x_1,\ y - 1 = y_1$.

However equation (1) can be written as a standard form

$$y_1 = -\frac{1}{8}x_1^2,$$

which represents a parabola with vertex at (4, 1). (See Figure 1(a).)

(ii) $y^2 - 12x + 2y + 25 = (y + 1)^2 - 12(x - 2)$. Hence $y^2 - 12x + 2y + 25 = 0$ becomes

$$y_1^2 - 12x_1 = 0 \qquad (2)$$

if we make a translation of axes $x - 2 = x_1,\ y + 1 = y_1$.

However equation (2) can be written as a standard form

$$y_1^2 = 12x_1,$$

which represents a parabola with vertex at (2, -1). (See Figure 1(b).)

2. $4xy - 3y^2 = X^tAX$, where $A = \begin{bmatrix} 0 & 2 \\ 2 & -3 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$. The eigenvalues of A are the roots of $\lambda^2 + 3\lambda - 4 = 0$, namely $\lambda_1 = -4$ and $\lambda_2 = 1$.

The eigenvectors corresponding to an eigenvalue λ are the non–zero vectors $[x,\ y]^t$ satisfying

$$\begin{bmatrix} 0-\lambda & 2 \\ 2 & -3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\lambda_1 = -4$ gives equations

$$\begin{aligned} 4x + 2y &= 0 \\ 2x + y &= 0 \end{aligned}$$

which has the solution $y = -2x$. Hence

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -2x \end{bmatrix} = x \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

A corresponding unit eigenvector is $[1/\sqrt{5},\ -2/\sqrt{5}]^t$.

$\lambda_2 = 1$ gives equations

$$\begin{aligned} -x + 2y &= 0 \\ 2x - 4y &= 0 \end{aligned}$$

which has the solution $x = 2y$. Hence

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ y \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

A corresponding unit eigenvector is $[2/\sqrt{5},\ 1/\sqrt{5}]^t$.

Hence if

$$P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix},$$

then P is an orthogonal matrix. Also as $\det P = 1$, P is a proper orthogonal matrix and the equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

represents a rotation to new x_1, y_1 axes whose positive directions are given by the respective columns of P. Also

$$P^tAP = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $X^tAX = -4x_1^2 + y_1^2$ and the original equation $4xy - 3y^2 = 8$ becomes $-4x_1^2 + y_1^2 = 8$, or the standard form

$$\frac{-x_1^2}{2} + \frac{y_1^2}{8} = 1,$$

which represents an hyperbola.

The asymptotes assist in drawing the curve. They are given by the equations

$$\frac{-x_1^2}{2} + \frac{y_1^2}{8} = 0, \quad \text{or} \quad y_1 = \pm 2x_1.$$

Now

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = P^t \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

so

$$x_1 = \frac{x - 2y}{\sqrt{5}}, \quad y_1 = \frac{2x + y}{\sqrt{5}}.$$

Hence the asymptotes are

$$\frac{2x + y}{\sqrt{5}} = \pm 2 \left(\frac{x - 2y}{\sqrt{5}} \right),$$

which reduces to $y = 0$ and $y = 4x/3$. (See Figure 2(a).)

3. $8x^2 - 4xy + 5y^2 = X^tAX$, where $A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$. The eigenvalues of A are the roots of $\lambda^2 - 13\lambda + 36 = 0$, namely $\lambda_1 = 4$ and $\lambda_2 = 9$. Corresponding unit eigenvectors turn out to be $[1/\sqrt{5}, 2/\sqrt{5}]^t$ and $[-2/\sqrt{5}, 1/\sqrt{5}]^t$. Hence if

$$P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix},$$

then P is an orthogonal matrix. Also as $\det P = 1$, P is a proper orthogonal matrix and the equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

represents a rotation to new x_1, y_1 axes whose positive directions are given by the respective columns of P. Also

$$P^tAP = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}.$$

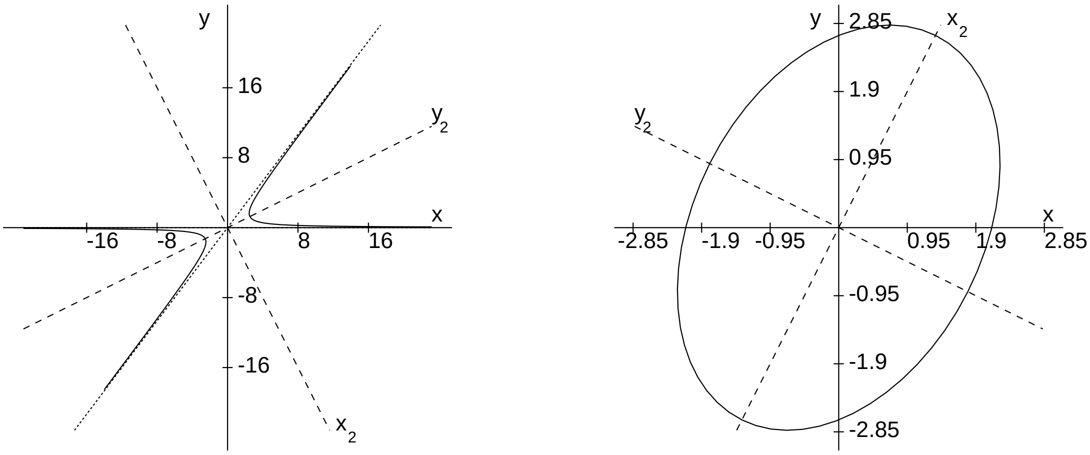


Figure 2: (a): $4xy - 3y^2 = 8$; (b): $8x^2 - 4xy + 5y^2 = 36$

Then $X^tAX = 4x_1^2 + 9y_1^2$ and the original equation $8x^2 - 4xy + 5y^2 = 36$ becomes $4x_1^2 + 9y_1^2 = 36$, or the standard form

$$\frac{x_1^2}{9} + \frac{y_1^2}{4} = 1,$$

which represents an ellipse as in Figure 2(b).

The axes of symmetry turn out to be $y = 2x$ and $x = -2y$.

4. We give the sketch only for parts (i), (iii) and (iv). We give the working for (ii) only. See Figures 3(a) and 4(a) and 4(b), respectively.

(ii) We have to investigate the equation

$$5x^2 - 4xy + 8y^2 + 4\sqrt{5}x - 16\sqrt{5}y + 4 = 0. \qquad (3)$$

Here $5x^2 - 4xy + 8y^2 = X^tAX$, where $A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$. The eigenvalues of A are the roots of $\lambda^2 - 13\lambda + 36 = 0$, namely $\lambda_1 = 9$ and $\lambda_2 = 4$. Corresponding unit eigenvectors turn out to be $[1/\sqrt{5}, -2/\sqrt{5}]^t$ and $[2/\sqrt{5}, 1/\sqrt{5}]^t$. Hence if

$$P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix},$$

then P is an orthogonal matrix. Also as $\det P = 1$, P is a proper orthogonal matrix and the equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

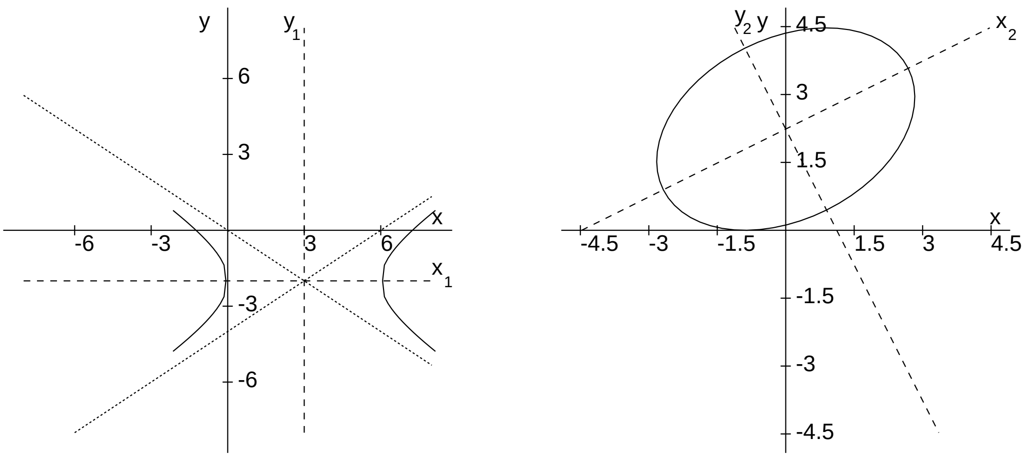


Figure 3: (a): $4x^2 - 9y^2 - 24x - 36y - 36 = 0$; (b): $5x^2 - 4xy + 8y^2 + \sqrt{5}x - 16\sqrt{5}y + 4 = 0$

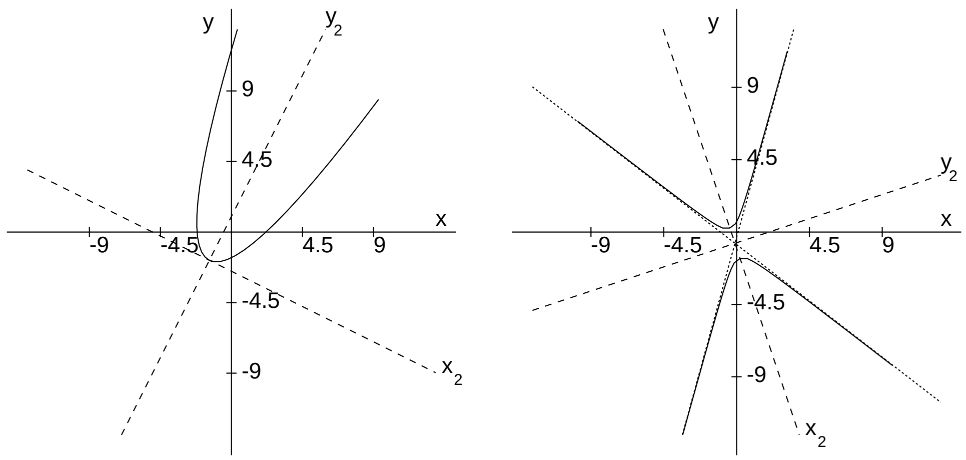


Figure 4: (a): $4x^2 + y^2 - 4xy - 10y - 19 = 0$; (b): $77x^2 + 78xy - 27y^2 + 70x - 30y + 29 = 0$

represents a rotation to new x_1, y_1 axes whose positive directions are given by the respective columns of P. Also

$$P^tAP = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}.$$

Moreover

$$5x^2 - 4xy + 8y^2 = 9x_1^2 + 4y_1^2.$$

To get the coefficients of x_1 and y_1 in the transformed form of equation (3), we have to use the rotation equations

$$x = \frac{1}{\sqrt{5}}(x_1 + 2y_1), \quad y = \frac{1}{\sqrt{5}}(-2x_1 + y_1).$$

Then equation (3) transforms to

$$9x_1^2 + 4y_1^2 + 36x_1 - 8y_1 + 4 = 0,$$

or, on completing the square,

$$9(x_1 + 2)^2 + 4(y_1 - 1)^2 = 36,$$

or in standard form

$$\frac{x_2^2}{4} + \frac{y_2^2}{9} = 1,$$

where $x_2 = x_1 + 2$ and $y_2 = y_1 - 1$. Thus we have an ellipse, centre $(x_2, y_2) = (0, 0)$, or $(x_1, y_1) = (-2, 1)$, or $(x, y) = (0, \sqrt{5})$.

The axes of symmetry are given by $x_2 = 0$ and $y_2 = 0$, or $x_1 + 2 = 0$ and $y_1 - 1 = 0$, or

$$\frac{1}{\sqrt{5}}(x - 2y) + 2 = 0 \quad \text{and} \quad \frac{1}{\sqrt{5}}(2x + y) - 1 = 0,$$

which reduce to $x - 2y + 2\sqrt{5} = 0$ and $2x + y - \sqrt{5} = 0$. See Figure 3(b).

5. (i) Consider the equation

$$2x^2 + y^2 + 3xy - 5x - 4y + 3 = 0. \qquad (4)$$

$$\Delta = \begin{vmatrix} 2 & 3/2 & -5/2 \\ 3/2 & 1 & -2 \\ -5/2 & -2 & 3 \end{vmatrix} = 8\begin{vmatrix} 4 & 3 & -5 \\ 3 & 2 & -4 \\ -5 & -4 & 6 \end{vmatrix} = 8\begin{vmatrix} 1 & 1 & -1 \\ 3 & 2 & -4 \\ -2 & -2 & 2 \end{vmatrix} = 0.$$

Let $x = x_1 + \alpha$, $y = y_1 + \beta$ and substitute in equation (4) to get

$$2(x_1+\alpha)^2+(y_1+\beta)^2+3(x_1+\alpha)(y_1+\beta)-5(x_1+\alpha)-4(y_1+\beta)+3=0 \quad (5).$$

Then equating the coefficients of x_1 and y_1 to 0 gives

$$\begin{aligned} 4\alpha + 3\beta - 5 &= 0 \\ 3\alpha + 2\beta - 4 &= 0, \end{aligned}$$

which has the unique solution $\alpha = 2$, $\beta = -1$. Then equation (5) simplifies to

$$2x_1^2 + y_1^2 + 3x_1y_1 = 0 = (2x_1 + y_1)(x_1 + y_1).$$

So relative to the x_1, y_1 coordinates, equation (4) describes two lines: $2x_1 + y_1 = 0$ and $x_1 + y_1 = 0$. In terms of the original x, y coordinates, these lines become $2(x-2)+(y+1)=0$ and $(x-2)+(y+1)=0$, i.e. $2x+y-3=0$ and $x+y-1=0$, which intersect in the point

$$(x,\, y) = (\alpha,\, \beta) = (2,\, -1).$$

(ii) Consider the equation

$$9x^2 + y^2 - 6xy + 6x - 2y + 1 = 0. \quad (6)$$

Here

$$\Delta = \begin{vmatrix} 9 & -3 & 3 \\ 3 & 1 & -1 \\ 3 & -1 & 1 \end{vmatrix} = 0,$$

as column 3 = − column 2.

Let $x = x_1 + \alpha$, $y = y_1 + \beta$ and substitute in equation (6) to get

$$9(x_1+\alpha)^2 + (y_1+\beta)^2 - 6(x_1+\alpha)(y_1+\beta) + 6(x_1+\alpha) - 2(y_1+\beta) + 1 = 0.$$

Then equating the coefficients of x_1 and y_1 to 0 gives

$$\begin{aligned} 18\alpha - 6\beta + 6 &= 0 \\ -6\alpha + 2\beta - 2 &= 0, \end{aligned}$$

or equivalently $-3\alpha + \beta - 1 = 0$. Take $\alpha = 0$ and $\beta = 1$. Then equation (6) simplifies to

$$9x_1^2 + y_1^2 - 6x_1y_1 = 0 = (3x_1 - y_1)^2. \quad (7)$$

In terms of x, y coordinates, equation (7) becomes

$$(3x - (y - 1))^2 = 0, \text{ or } 3x - y + 1 = 0.$$

(iii) Consider the equation

$$x^2 + 4xy + 4y^2 - x - 2y - 2 = 0. \qquad (8)$$

Arguing as in the previous examples, we find that any translation

$$x = x_1 + \alpha, \quad y = y_1 + \beta$$

where $2\alpha + 4\beta - 1 = 0$ has the property that the coefficients of x_1 and y_1 will be zero in the transformed version of equation (8). Take $\beta = 0$ and $\alpha = 1/2$. Then (8) reduces to

$$x_1^2 + 4x_1y_1 + 4y_1^2 - \frac{9}{4} = 0,$$

or $(x_1 + 2y_1)^2 = 3/2$. Hence $x_1 + 2y_1 = \pm 3/2$, with corresponding equations

$$x + 2y = 2 \quad \text{and} \quad x + 2y = -1.$$

Section 8.8

1. The given line has equations

$$\begin{aligned} x &= 3 + t(13-3) = 3 + 10t, \\ y &= -2 + t(3+2) = -2 + 5t, \\ z &= 7 + t(-8-7) = 7 - 15t. \end{aligned}$$

The line meets the plane $y = 0$ in the point $(x, 0, z)$, where $0 = -2 + 5t$, or $t = 2/5$. The corresponding values for x and z are 7 and 1, respectively.

2. $\mathbf{E} = \frac{1}{2}(\mathbf{B} + \mathbf{C})$, $\mathbf{F} = (1-t)\mathbf{A} + t\mathbf{E}$, where

$$t = \frac{AF}{AE} = \frac{AF}{AF + FE} = \frac{AF/FE}{(AF/FE) + 1} = \frac{2}{3}.$$

Hence

$$\begin{aligned} \mathbf{F} &= \frac{1}{3}\mathbf{A} + \frac{2}{3}\left(\frac{1}{2}(\mathbf{B} + \mathbf{C})\right) \\ &= \frac{1}{3}\mathbf{A} + \frac{1}{3}(\mathbf{B} + \mathbf{C}) \\ &= \frac{1}{3}(\mathbf{A} + \mathbf{B} + \mathbf{C}). \end{aligned}$$

3. Let $A = (2, 1, 4)$, $B = (1, -1, 2)$, $C = (3, 3, 6)$. Then we prove $\overrightarrow{AC} = t\,\overrightarrow{AB}$ for some real t. We have

$$\overrightarrow{AC} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \overrightarrow{AB} = \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}.$$

Hence $\overrightarrow{AC} = (-1)\,\overrightarrow{AB}$ and consequently C is on the line AB. In fact A is between C and B, with $AC = AB$.

4. The points P on the line AB which satisfy $AP = \frac{2}{5}PB$ are given by $\mathbf{P} = \mathbf{A} + t\,\overrightarrow{AB}$, where $|t/(1-t)| = 2/5$. Hence $t/(1-t) = \pm 2/5$.

The equation $t/(1-t) = 2/5$ gives $t = 2/7$ and hence

$$\mathbf{P} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + \frac{2}{7}\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 16/7 \\ 29/7 \\ 3/7 \end{bmatrix}.$$

Hence $P = (16/7,\ 29/7,\ 3/7)$.

The equation $t/(1-t) = -2/5$ gives $t = -2/3$ and hence

$$\mathbf{P} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 1/3 \\ -13/3 \end{bmatrix}.$$

Hence $P = (4/3,\ 1/3,\ -13/3)$.

5. An equation for $\mathcal{M}$ is $\mathbf{P} = \mathbf{A} + t\,\overrightarrow{BC}$, which reduces to

$$\begin{aligned} x &= 1 + 6t \\ y &= 2 - 3t \\ z &= 3 + 7t. \end{aligned}$$

An equation for $\mathcal{N}$ is $\mathbf{Q} = \mathbf{E} + s\,\overrightarrow{EF}$, which reduces to

$$\begin{aligned} x &= 1 + 9s \\ y &= -1 \\ z &= 8 + 3s. \end{aligned}$$

To find if and where $\mathcal{M}$ and $\mathcal{N}$ intersect, we set $P = Q$ and attempt to solve for s and t. We find the unique solution $t = 1,\ s = 2/3$, proving that the lines meet in the point

$$(x,\ y,\ z) = (1+6,\ 2-3,\ 3+7) = (7,\ -1,\ 10).$$

6. Let $A = (-3,\ 5,\ 6),\ B = (-2,\ 7,\ 9),\ C = (2,\ 1,\ 7)$. Then

(i)

$$\cos\angle ABC = (\overrightarrow{BA} \cdot \overrightarrow{BC})/(BA \cdot BC),$$

where $\overrightarrow{BA} = [-1,\ -2,\ -3]^t$ and $\overrightarrow{BC} = [4,\ -6,\ -2]^t$. Hence

$$\cos\angle ABC = \frac{-4+12+6}{\sqrt{14}\sqrt{56}} = \frac{14}{\sqrt{14}\sqrt{56}} = \frac{1}{2}.$$

Hence $\angle ABC = \pi/3$ radians or 60°.

(ii)

$$\cos \angle BAC = (\overrightarrow{AB} \cdot \overrightarrow{AC})/(AB \cdot AC),$$

where $\overrightarrow{AB} = [1,\ 2,\ 3]^t$ and $\overrightarrow{AC} = [5,\ -4,\ 1]^t$. Hence

$$\cos \angle BAC = \frac{5-8+3}{\sqrt{14}\sqrt{42}} = 0.$$

Hence $\angle ABC = \pi/2$ radians or 90°.

(iii)

$$\cos \angle ACB = (\overrightarrow{CA} \cdot \overrightarrow{CB})/(CA \cdot CB),$$

where $\overrightarrow{CA} = [-5,\ 4,\ -1]^t$ and $\overrightarrow{CB} = [-4,\ 6,\ 2]^t$. Hence

$$\cos \angle ACB = \frac{20+24-2}{\sqrt{42}\sqrt{56}} = \frac{42}{\sqrt{42}\sqrt{56}} = \frac{\sqrt{42}}{\sqrt{56}} = \frac{\sqrt{3}}{2}.$$

Hence $\angle ACB = \pi/6$ radians or 30°.

7. By Theorem 8.5.2, the closest point P on the line AB to the origin O is given by $\mathbf{P} = \mathbf{A} + t\,\overrightarrow{AB}$, where

$$t = \frac{\overrightarrow{AO} \cdot \overrightarrow{AB}}{AB^2} = \frac{-\mathbf{A} \cdot \overrightarrow{AB}}{AB^2}.$$

Now

$$\mathbf{A} \cdot \overrightarrow{AB} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = -2.$$

Hence $t = 2/11$ and

$$\mathbf{P} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} + \frac{2}{11}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -16/11 \\ 13/11 \\ 35/11 \end{bmatrix}$$

and $P = (-16/11,\ 13/11, 35/11)$.

Consequently the shortest distance OP is given by

$$\sqrt{\left(\frac{-16}{11}\right)^2 + \left(\frac{13}{11}\right)^2 + \left(\frac{35}{11}\right)^2} = \frac{\sqrt{1650}}{11} = \frac{\sqrt{15 \times 11 \times 10}}{11} = \frac{\sqrt{150}}{\sqrt{11}}.$$

Alternatively, we can calculate the distance OP^2, where P is an arbitrary point on the line AB and then minimize OP^2:

$$\mathbf{P} = \mathbf{A} + t\,\overrightarrow{AB} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+3t \\ 1+t \\ 3+t \end{bmatrix}.$$

Hence

$$\begin{aligned} OP^2 &= (-2+3t)^2 + (1+t)^2 + (3+t)^2 \\ &= 11t^2 - 4t + 14 \\ &= 11\left(t^2 - \frac{4}{11}t + \frac{14}{11}\right) \\ &= 11\left(\left\{t - \frac{2}{11}\right\}^2 + \frac{14}{11} - \frac{4}{121}\right) \\ &= 11\left(\left\{t - \frac{2}{11}\right\}^2 + \frac{150}{121}\right). \end{aligned}$$

Consequently

$$OP^2 \geq 11 \times \frac{150}{121}$$

for all t; moreover

$$OP^2 = 11 \times \frac{150}{121}$$

when $t = 2/11$.

8. We first find parametric equations for $\mathcal{N}$ by solving the equations

$$\begin{aligned} x + y - 2z &= 1 \\ x + 3y - z &= 4. \end{aligned}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 1 & 3 & -1 & 4 \end{bmatrix},$$

which reduces to

$$\begin{bmatrix} 1 & 0 & -5/2 & -1/2 \\ 0 & 1 & 1/2 & 3/2 \end{bmatrix}.$$

Hence $x = -\frac{1}{2} + \frac{5}{2}z$, $y = \frac{3}{2} - \frac{z}{2}$, with z arbitrary. Taking $z = 0$ gives a point $A = (-\frac{1}{2}, \frac{3}{2}, 0)$, while $z = 1$ gives a point $B = (2, 1, 1)$.

Hence if $C = (1, 0, 1)$, then the closest point on $\mathcal{N}$ to C is given by $\mathbf{P} = \mathbf{A} + t\,\overrightarrow{AB}$, where $t = (\overrightarrow{AC} \cdot \overrightarrow{AB})/AB^2$.

Now

$$\overrightarrow{AC} = \begin{bmatrix} 3/2 \\ -3/2 \\ 1 \end{bmatrix} \quad \text{and} \quad \overrightarrow{AB} = \begin{bmatrix} 5/2 \\ -1/2 \\ 1 \end{bmatrix},$$

so

$$t = \frac{\frac{3}{2} \times \frac{5}{2} + \frac{-3}{2} \times \frac{-1}{2} + 1 \times 1}{\left(\frac{5}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 + 1^2} = \frac{11}{15}.$$

Hence

$$\mathbf{P} = \begin{bmatrix} -1/2 \\ 3/2 \\ 0 \end{bmatrix} + \frac{11}{15} \begin{bmatrix} 5/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 17/15 \\ 11/15 \end{bmatrix},$$

so $P = (4/3, 17/15, 11/15)$.

Also the shortest distance PC is given by

$$PC = \sqrt{\left(1 - \frac{4}{3}\right)^2 + \left(0 - \frac{17}{15}\right)^2 + \left(1 - \frac{11}{15}\right)^2} = \frac{\sqrt{330}}{15}.$$

9. The intersection of the planes $x + y - 2z = 4$ and $3x - 2y + z = 1$ is the line given by the equations

$$x = \frac{9}{5} + \frac{3}{5}z, \; y = \frac{11}{5} + \frac{7}{5}z,$$

where z is arbitrary. Hence the line $\mathcal{L}$ has a direction vector $[3/5, 7/5, 1]^t$ or the simpler $[3, 7, 5]^t$. Then any plane of the form $3x + 7y + 5z = d$ will be perpendicualr to $\mathcal{L}$. The required plane has to pass through the point $(6, 0, 2)$, so this determines d:

$$3 \times 6 + 7 \times 0 + 5 \times 2 = d = 28.$$

10. The length of the projection of the segment AB onto the line CD is given by the formula

$$\frac{|\,\overrightarrow{CD} \cdot \overrightarrow{AB}\,|}{CD}.$$

Here $\overrightarrow{CD} = [-8, 4, -1]^t$ and $\overrightarrow{AB} = [4, -4, 3]^t$, so

$$\begin{aligned} \frac{|\,\overrightarrow{CD} \cdot \overrightarrow{AB}\,|}{CD} &= \frac{|(-8) \times 4 + 4 \times (-4) + (-1) \times 3|}{\sqrt{(-8)^2 + 4^2 + (-1)^2}} \\ &= \frac{|-51|}{\sqrt{81}} = \frac{51}{9} = \frac{17}{3}. \end{aligned}$$

11. A direction vector for $\mathcal{L}$ is given by $\overrightarrow{BC}= [-5,\ -2,\ 3]^t$. Hence the plane through A perpendicular to $\mathcal{L}$ is given by

$$-5x - 2y + 3z = (-5) \times 3 + (-2) \times (-1) + 3 \times 2 = -7.$$

The position vector $\mathbf{P}$ of an arbitrary point P on $\mathcal{L}$ is given by $\mathbf{P} = \mathbf{B}+t\ \overrightarrow{BC}$, or

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} -5 \\ -2 \\ 3 \end{bmatrix},$$

or equivalently $x = 2 - 5t$, $y = 1 - 2t$, $z = 4 + 3t$.

To find the intersection of line $\mathcal{L}$ and the given plane, we substitute the expressions for x, y, z found in terms of t into the plane equation and solve the resulting linear equation for t:

$$-5(2 - 5t) - 2(1 - 2t) + 3(4 + 3t) = -7,$$

which gives $t = -7/38$. Hence $P = \left(\frac{111}{38}, \frac{52}{38}, \frac{131}{38}\right)$ and

$$\begin{aligned} AP &= \sqrt{\left(3 - \frac{111}{38}\right)^2 + \left(-1 - \frac{52}{38}\right)^2 + \left(2 - \frac{131}{38}\right)^2} \\ &= \frac{\sqrt{11134}}{38} = \frac{\sqrt{293 \times 38}}{38} = \frac{\sqrt{293}}{\sqrt{38}}. \end{aligned}$$

12. Let P be a point inside the triangle ABC. Then the line through P and parallel to AC will meet the segments AB and BC in D and E, respectively. Then

$$\begin{aligned} \mathbf{P} &= (1 - r)\mathbf{D} + r\mathbf{E}, \quad 0 < r < 1; \\ \mathbf{D} &= (1 - s)\mathbf{B} + s\mathbf{A}, \quad 0 < s < 1; \\ \mathbf{E} &= (1 - t)\mathbf{B} + t\mathbf{C}, \quad 0 < t < 1. \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{P} &= (1 - r)\left\{(1 - s)\mathbf{B} + s\mathbf{A}\right\} + r\left\{(1 - t)\mathbf{B} + t\mathbf{C}\right\} \\ &= (1 - r)s\mathbf{A} + \left\{(1 - r)(1 - s) + r(1 - t)\right\}\mathbf{B} + rt\mathbf{C} \\ &= \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}, \end{aligned}$$

where

$$\alpha = (1-r)s, \quad \beta = (1-r)(1-s) + r(1-t), \quad \gamma = rt.$$

Then $0 < \alpha < 1,\ 0 < \gamma < 1,\ 0 < \beta < (1-r) + r = 1$. Also

$$\alpha + \beta + \gamma = (1-r)s + (1-r)(1-s) + r(1-t) + rt = 1.$$

13. The line AB is given by $\mathbf{P} = \mathbf{A} + t[3,\, 4,\, 5]^t$, or

$$x = 6 + 3t, \quad y = -1 + 4t, \quad z = 11 + 5t.$$

Then B is found by substituting these expressions in the plane equation

$$3x + 4y + 5z = 10.$$

We find $t = -59/50$ and consequently

$$B = \left(6 - \frac{177}{50},\, -1 - \frac{236}{50},\, 11 - \frac{295}{50}\right) = \left(\frac{123}{50},\, \frac{-286}{50},\, \frac{255}{50}\right).$$

Then

$$\begin{aligned} AB &= ||\,\overrightarrow{AB}\,|| = ||t \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}|| \\ &= |t|\sqrt{3^2 + 4^2 + 5^2} = \frac{59}{50} \times \sqrt{50} = \frac{59}{\sqrt{50}}. \end{aligned}$$

14. Let $A = (-3,\, 0,\, 2),\ B = (6,\, 1,\, 4),\ C = (-5,\, 1,\, 0)$. Then the area of triangle ABC is $\frac{1}{2}||\,\overrightarrow{AB} \times \overrightarrow{AC}\,||$. Now

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} 9 \\ 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 14 \\ 11 \end{bmatrix}.$$

Hence $||\,\overrightarrow{AB} \times \overrightarrow{AC}\,|| = \sqrt{333}$.

15. Let $A_1 = (2,\, 1,\, 4),\ A_2 = (1,\, -1,\, 2),\ A_3 = (4,\, -1,\, 1)$. Then the point $P = (x,\, y,\, z)$ lies on the plane $A_1A_2A_3$ if and only if

$$\overrightarrow{A_1P} \cdot (\overrightarrow{A_1A_2} \times \overrightarrow{A_1A_3}) = 0,$$

or

$$\begin{vmatrix} x-2 & y-1 & z-4 \\ -1 & -2 & -2 \\ 2 & -2 & -3 \end{vmatrix} = 2x - 7y + 6z - 21 = 0.$$

16. Non–parallel lines $\mathcal{L}$ and $\mathcal{M}$ in three dimensional space are given by equations

$$\mathbf{P} = \mathbf{A} + sX, \quad \mathbf{Q} = \mathbf{B} + tY.$$

(i) Suppose $\overrightarrow{PQ}$ is orthogonal to both X and Y. Now

$$\overrightarrow{PQ} = \mathbf{Q} - \mathbf{P} = (\mathbf{B} + tY) - (\mathbf{A} + sX) = \overrightarrow{AB} + tY - sX.$$

Hence

$$\begin{aligned} (\overrightarrow{AB} + tY + sX) \cdot X &= 0 \\ (\overrightarrow{AB} + tY + sX) \cdot Y &= 0. \end{aligned}$$

More explicitly

$$\begin{aligned} t(Y \cdot X) - s(X \cdot X) &= -\overrightarrow{AB} \cdot X \\ t(Y \cdot Y) - s(X \cdot Y) &= -\overrightarrow{AB} \cdot Y. \end{aligned}$$

However the coefficient determinant of this system of linear equations in t and s is equal to

$$\begin{aligned} \begin{vmatrix} Y \cdot X & -X \cdot X \\ Y \cdot Y & -X \cdot Y \end{vmatrix} &= -(X \cdot Y)^2 + (X \cdot X)(Y \cdot Y) \\ &= ||X \times Y||^2 \neq 0, \end{aligned}$$

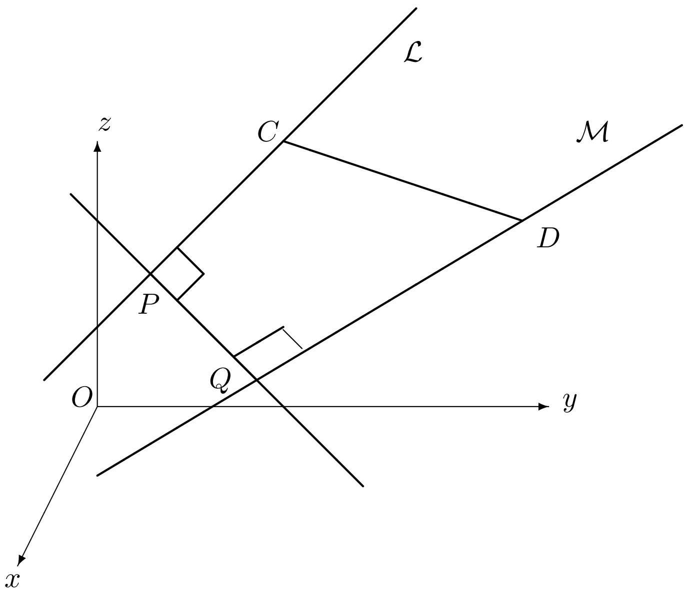
as $X \neq 0,\ Y \neq 0$ and X and Y are not proportional ($\mathcal{L}$ and $\mathcal{M}$ are not parallel).

(ii) P and Q can be viewed as the projections of C and D onto the line PQ, where C and D are arbitrary points on the lines $\mathcal{L}$ and $\mathcal{M}$, respectively. Hence by equation (8.14) of Theorem 8.5.3, we have

$$PQ \leq CD.$$

Finally we derive a useful formula for PQ. Again by Theorem 8.5.3

$$PQ = \frac{|\ \overrightarrow{AB} \cdot \overrightarrow{PQ}\ |}{PQ} = |\ \overrightarrow{AB} \cdot \hat{n}|,$$



where $\hat{n} = \frac{1}{PQ} \overrightarrow{PQ}$ is a unit vector which is orthogonal to X and Y. Hence

$$\hat{n} = t(X \times Y),$$

where $t = \pm 1/||X \times Y||$. Hence

$$PQ = \frac{|\overrightarrow{AB} \cdot (X \times Y)|}{||X \times Y||}.$$

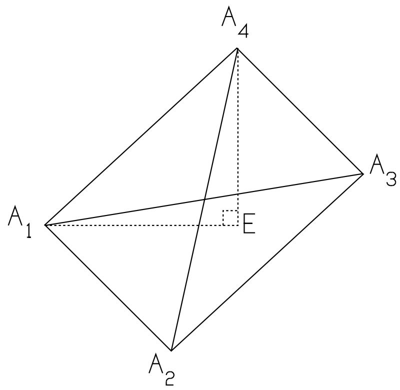
17. We use the formula of the previous question.
 Line $\mathcal{L}$ has the equation $\mathbf{P} = \mathbf{A} + sX$, where

$$X = \overrightarrow{AC} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}.$$

Line $\mathcal{M}$ has the equation $\mathbf{Q} = \mathbf{B} + tY$, where

$$Y = \overrightarrow{BD} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Hence $X \times Y = [-6,\, 1,\, 5]^t$ and $||X \times Y|| = \sqrt{62}$.



Hence the shortest distance between lines AC and BD is equal to

$$\frac{|\overrightarrow{AB}\cdot(X\times Y)|}{||X\times Y||}=\frac{\left|\begin{bmatrix}0\\-2\\1\end{bmatrix}\cdot\begin{bmatrix}-6\\1\\5\end{bmatrix}\right|}{\sqrt{62}}=\frac{3}{\sqrt{62}}.$$

18. Let E be the foot of the perpendicular from A_4 to the plane $A_1A_2A_3$. Then

$$\mathrm{vol}\,A_1A_2A_3A_4=\frac{1}{3}(\text{ area}\,\Delta A_1A_2A_3)\cdot A_4E.$$

Now

$$\text{area}\,\Delta A_1A_2A_3=\frac{1}{2}||\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}||.$$

Also A_4E is the length of the projection of A_1A_4 onto the line A_4E. (See figure above.)

Hence $A_4E=|\overrightarrow{A_1A_4}\cdot X|$, where X is a unit direction vector for the line A_4E. We can take

$$X=\frac{\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}}{||\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}||}.$$

Hence

$$\begin{aligned}\mathrm{vol}\,A_1A_2A_3A_4 &= \frac{1}{6}||\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}||\frac{|\overrightarrow{A_1A_4}\cdot(\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3})|}{||\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3}||}\\ &= \frac{1}{6}|\overrightarrow{A_1A_4}\cdot(\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_3})|\end{aligned}$$

$$= \frac{1}{6}|(\overrightarrow{A_1A_2} \times \overrightarrow{A_1A_3}) \cdot \overrightarrow{A_1A_4}\,|.$$

19. We have $\overrightarrow{CB} = [1,\ 4,\ -1]^t$, $\overrightarrow{CD} = [-3,\ 3,\ 0]^t$, $\overrightarrow{AD} = [3,\ 0,\ 3]^t$. Hence

$$\overrightarrow{CB} \times \overrightarrow{CD} = 3\mathbf{i} + 3\mathbf{j} + 15\mathbf{k},$$

so the vector $\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ is perpendicular to the plane BCD.

Now the plane BCD has equation $x + y + 5z = 9$, as $B = (2,\ 2,\ 1)$ is on the plane.

Also the line through A normal to plane BCD has equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} = (1+t)\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}.$$

Hence $x = 1 + t,\ y = 1 + t,\ z = 5(1 + t)$.

[We remark that this line meets plane BCD in a point E which is given by a value of t found by solving

$$(1 + t) + (1 + t) + 5(5 + 5t) = 9.$$

So $t = -2/3$ and $E = (1/3,\ 1/3,\ 5/3)$.]

The distance from A to plane BCD is

$$\frac{|1 \times 1 + 1 \times 1 + 5 \times 5 - 9|}{1^2 + 1^2 + 5^2} = \frac{18}{\sqrt{27}} = 2\sqrt{3}.$$

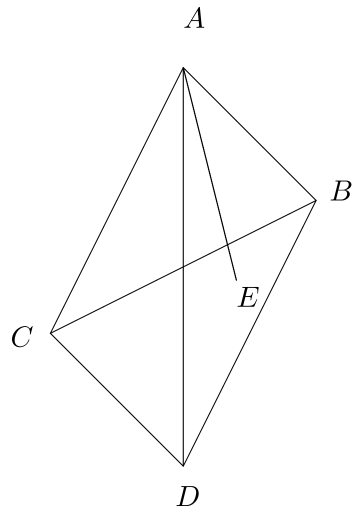
To find the distance between lines AD and BC, we first note that

(a) The equation of AD is

$$\mathbf{P} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 + 3t \\ 1 \\ 5 + 3t \end{bmatrix};$$

(b) The equation of BC is

$$\mathbf{Q} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 + s \\ 2 + 4s \\ 1 - s \end{bmatrix}.$$



Then $\overrightarrow{PQ}= [1+s-3t,\ 1+4s,\ -4-s-3t]^t$ and we find s and t by solving the equations $\overrightarrow{PQ}\cdot\overrightarrow{AD}= 0$ and $\overrightarrow{PQ}\cdot\overrightarrow{BC}= 0$, or

$$\begin{aligned}(1+s-3t)3+(1+4s)0+(-4-s-3t)3 &= 0\\ (1+s-3t)+4(1+4s)-(-4-s-3t) &= 0.\end{aligned}$$

Hence $t = -1/2 = s$.

Correspondingly, $P = (-1/2,\ 1,\ 7/2)$ and $Q = (3/2,\ 0,\ 3/2)$.

Thus we have found the closest points P and Q on the respective lines AD and BC. Finally the shortest distance between the lines is

$$PQ = ||\ \overrightarrow{PQ}\ || = 3.$$