Section 2.4

2. Suppose $B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ and that $AB = I_2$. Then

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -a+e & -b+f \\ c+e & d+f \end{bmatrix}.$$

Hence

$$\begin{matrix} -a+e=1 \\ c+e=0 \end{matrix}, \quad \begin{matrix} -b+f=0 \\ d+f=1 \end{matrix};$$

$$\begin{matrix} e=a+1 \\ c=-e=-(a+1) \end{matrix}, \quad \begin{matrix} f=b \\ d=1-f=1-b \end{matrix};$$

$$B = \begin{bmatrix} a & b \\ -a-1 & 1-b \\ a+1 & b \end{bmatrix}.$$

Next,

$$(BA)^2B = (BA)(BA)B = B(AB)(AB) = BI_2I_2 = BI_2 = B$$

4. Let p_n denote the statement

$$A^n = \tfrac{(3^n-1)}{2}A + \tfrac{(3-3^n)}{2}I_2.$$

Then p_1 asserts that $A = \frac{(3-1)}{2}A + \frac{(3-3)}{2}I_2$, which is true. So let $n \geq 1$ and assume p_n. Then from (1),

$$\begin{aligned} A^{n+1} &= A \cdot A^n = A\left\{\tfrac{(3^n-1)}{2}A + \tfrac{(3-3^n)}{2}I_2\right\} = \tfrac{(3^n-1)}{2}A^2 + \tfrac{(3-3^n)}{2}A \\ &= \tfrac{(3^n-1)}{2}(4A-3I_2) + \tfrac{(3-3^n)}{2}A = \tfrac{(3^n-1)4+(3-3^n)}{2}A + \tfrac{(3^n-1)(-3)}{2}I_2 \\ &= \tfrac{(4 \cdot 3^n-3^n)-1}{2}A + \tfrac{(3-3^{n+1})}{2}I_2 \\ &= \tfrac{(3^{n+1}-1)}{2}A + \tfrac{(3-3^{n+1})}{2}I_2. \end{aligned}$$

Hence p_{n+1} is true and the induction proceeds.

5. The equation $x_{n+1} = ax_n + bx_{n-1}$ is seen to be equivalent to

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

or

$$X_n = AX_{n-1},$$

where $X_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$. Then

$$X_n = A^n X_0$$

if $n \geq 1$. Hence by Question 3,

$$\begin{aligned}
\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} &= \left\{ \frac{(3^n - 1)}{2} A + \frac{(3 - 3^n)}{2} I_2 \right\} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} \\
&= \left\{ \frac{(3^n - 1)}{2} \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} \frac{3-3^n}{2} & 0 \\ 0 & \frac{3-3^n}{2} \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} \\
&= \begin{bmatrix} (3^n - 1)2 + \frac{3-3^n}{2} & \frac{(3^n-1)(-3)}{2} \\ \frac{3^n-1}{2} & \frac{3-3^n}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}
\end{aligned}$$

Hence, equating the $(2, 1)$ elements gives

$$x_n = \frac{(3^n - 1)}{2} x_1 + \frac{(3 - 3^n)}{2} x_0 \qquad \text{if } n \geq 1$$

7. Note: $\lambda_1 + \lambda_2 = a + d$ and $\lambda_1 \lambda_2 = ad - bc$.
Then

$$\begin{aligned}
(\lambda_1 + \lambda_2)k_n - \lambda_1\lambda_2 k_{n-1} &= (\lambda_1 + \lambda_2)(\lambda_1^{n-1} + \lambda_1^{n-2}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-2} + \lambda_2^{n-1}) \\
&\quad -\lambda_1\lambda_2(\lambda_1^{n-2} + \lambda_1^{n-3}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-3} + \lambda_2^{n-2})
\end{aligned}$$

$$\begin{aligned}
&= (\lambda_1^n + \lambda_1^{n-1}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-1}) \\
&\quad +(\lambda_1^{n-1}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-1} + \lambda_2^n) \\
&\quad -(\lambda_1^{n-1}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-1}) \\
&= \lambda_1^n + \lambda_1^{n-1}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-1} + \lambda_2^n = k_{n+1}
\end{aligned}$$

If $\lambda_1 = \lambda_2$, we see

$$\begin{aligned}
k_n &= \lambda_1^{n-1} + \lambda_1^{n-2}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-2} + \lambda_2^{n-1} \\
&= \lambda_1^{n-1} + \lambda_1^{n-2}\lambda_1 + \cdots + \lambda_1\lambda_1^{n-2} + \lambda_1^{n-1} \\
&= n\lambda_1^{n-1}
\end{aligned}$$

If $\lambda_1 \neq \lambda_2$, we see that

$$\begin{aligned}
(\lambda_1 - \lambda_2)k_n &= (\lambda_1 - \lambda_2)(\lambda_1^{n-1} + \lambda_1^{n-2}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-2} + \lambda_2^{n-1}) \\
&= \lambda_1^n + \lambda_1^{n-1}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-1} \\
&\quad -(\lambda_1^{n-1}\lambda_2 + \cdots + \lambda_1\lambda_2^{n-1} + \lambda_2^n) \\
&= \lambda_1^n - \lambda_2^n.
\end{aligned}$$

Hence $k_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$.

We have to prove

$$A^n = k_n A - \lambda_1\lambda_2 k_{n-1} I_2. \quad *$$

n=1:

$$\begin{aligned}
A^1 = A;\ \text{also } k_1A - \lambda_1\lambda_2 k_0 I_2 &= k_1 A - \lambda_1\lambda_2 0 I_2 \\
&= A.
\end{aligned}$$

Let $n \geq 1$ and assume equation $*$ holds. Then

$$\begin{aligned}
A^{n+1} = A^n \cdot A &= (k_n A - \lambda_1\lambda_2 k_{n-1} I_2)A \\
&= k_n A^2 - \lambda_1\lambda_2 k_{n-1} A.
\end{aligned}$$

Now $A^2 = (a+d)A - (ad-bc)I_2 = (\lambda_1 + \lambda_2)A - \lambda_1\lambda_2 I_2$. Hence

$$\begin{aligned}
A^{n+1} &= k_n(\lambda_1 + \lambda_2)A - \lambda_1\lambda_2 I_2 - \lambda_1\lambda_2 k_{n-1} A \\
&= \{k_n(\lambda_1 + \lambda_2) - \lambda_1\lambda_2 k_{n-1}\}A - \lambda_1\lambda_2 k_n I_2 \\
&= k_{n+1}A - \lambda_1\lambda_2 k_n I_2,
\end{aligned}$$

and the induction goes through.

8. Here λ_1, λ_2 are the roots of the polynomial $x^2 - 2x - 3 = (x-3)(x+1)$. So we can take $\lambda_1 = 3, \lambda_2 = -1$. Then

$$k_n = \frac{3^n - (-1)^n}{3 - (-1)} = \frac{3^n + (-1)^{n+1}}{4}.$$

Hence

$$\begin{aligned}
A^n &= \left\{\frac{3^n + (-1)^{n+1}}{4}\right\} A - (-3)\left\{\frac{3^{n-1} + (-1)^n}{4}\right\} I_2 \\
&= \frac{3^n + (-1)^{n+1}}{4}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + 3\left\{\frac{3^{n-1} + (-1)^n}{4}\right\}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\end{aligned}$$

which is equivalent to the stated result.

9. In terms of matrices, we have

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} \text{ for } n \geq 1.$$

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Now λ_1, λ_2 are the roots of the polynomial $x^2 - x - 1$ here.
Hence $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$ and

$$\begin{aligned} k_n &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\frac{1+\sqrt{5}}{2} - \left(\frac{1-\sqrt{5}}{2}\right)} \\ &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}}. \end{aligned}$$

Hence

$$\begin{aligned} A^n &= k_n A - \lambda_1 \lambda_2 k_{n-1} I_2 \\ &= k_n A + k_{n-1} I_2 \end{aligned}$$

So

$$\begin{aligned} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} &= (k_n A + k_{n-1} I_2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= k_n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + k_{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} k_n + k_{n-1} \\ k_n \end{bmatrix}. \end{aligned}$$

Hence

$$F_n = k_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}}.$$

10. From Question 5, we know that

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & r \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} a \\ b \end{bmatrix}.$$

Now by Question 7, with $A = \begin{bmatrix} 1 & r \\ 1 & 1 \end{bmatrix}$,

$$\begin{aligned} A^n &= k_n A - \lambda_1 \lambda_2 k_{n-1} I_2 \\ &= k_n A - (1-r) k_{n-1} I_2, \end{aligned}$$

where $\lambda_1 = 1 + \sqrt{r}$ and $\lambda_2 = 1 - \sqrt{r}$ are the roots of the polynomial $x^2 - 2x + (1-r)$ and

$$k_n = \frac{\lambda_1^n - \lambda_2^n}{2\sqrt{r}}.$$

Hence

$$\begin{aligned} \begin{bmatrix} x_n \\ y_n \end{bmatrix} &= (k_n A - (1-r)k_{n-1} I_2) \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \left(\begin{bmatrix} k_n & k_n r \\ k_n & k_n \end{bmatrix} - \begin{bmatrix} (1-r)k_{n-1} & 0 \\ 0 & (1-r)k_{n-1} \end{bmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} k_n - (1-r)k_{n-1} & k_n r \\ k_n & k_n - (1-r)k_{n-1} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} a(k_n - (1-r)k_{n-1}) + b k_n r \\ a k_n + b(k_n - (1-r)k_{n-1}) \end{bmatrix}. \end{aligned}$$

Hence, in view of the fact that

$$\frac{k_n}{k_{n-1}} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1^{n-1} - \lambda_2^{n-1}} = \frac{\lambda_1^n (1 - \{\frac{\lambda_2}{\lambda_1}\}^n)}{\lambda_1^{n-1}(1 - \{\frac{\lambda_2}{\lambda_1}\}^{n-1})} \to \lambda_1, \quad \text{as } n \to \infty,$$

we have

$$\begin{aligned} \begin{bmatrix} x_n \\ y_n \end{bmatrix} &= \frac{a(k_n - (1-r)k_{n-1}) + b k_n r}{a k_n + b(k_n - (1-r)k_{n-1})} \\ &= \frac{a(\frac{k_n}{k_{n-1}} - (1-r)) + b \frac{k_n}{k_{n-1}} r}{a \frac{k_n}{k_{n-1}} + b(\frac{k_n}{k_{n-1}} - (1-r))} \\ &\to \frac{a(\lambda_1 - (1-r)) + b\lambda_1 r}{a\lambda_1 + b(\lambda_1 - (1-r))} \\ &= \frac{a(\sqrt{r} + r) + b(1+\sqrt{r})r}{a(1+\sqrt{r}) + b(\sqrt{r}+r)} \\ &= \frac{\sqrt{r}\{a(1+\sqrt{r}) + b(1+\sqrt{r})\sqrt{r}\}}{a(1+\sqrt{r}) + b(\sqrt{r}+r)} \\ &= \sqrt{r}. \end{aligned}$$