Nonlinear interactions in organ flue pipes

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The ideas introduced by Coltman and Benade are developed into a quantitative formalism for treating the oscillatory behavior of an air column in nonlinear interaction with an air jet, as in an organ flue pipe or a flute. Explicit solutions are given for the case when the pipe has only two resonances and the nonlinearity is described up to cubic terms. The results of illustrative calculations are discussed.

Subject Classification: 75.60.

INTRODUCTION

The scaling of ranks of organ flue pipes to produce a coherent, characteristic, and tonally balanced ensemble is a problem which has confronted organ builders for many centuries. The solutions which they developed were initially empirical and the more satisfactory ones were then refined into scaling laws of which several exist.[1] The precise relationships between attack, intensity, and harmonic development of the pipes which are necessary to constitute a satisfactory rank design are still not well understood, except in a qualitative way, and a great deal depends upon the experience and skill of the designer and voicer.

When we try to attack this problem we find that many things about the behavior of a single flue pipe are not well understood either. Even leaving aside the musically most important attack transient, the way in which the spectrum of the sound radiated by a pipe depends upon its scale (length to diameter ratio) and upon the geometry of mouth and jet is known only in qualitative outline.

Much of the reason for this lack of detailed understanding comes from the fact that the problem is essentially nonlinear. The linear theory has been explored by many workers, particularly for the related cases of the flute[2,3] and the clarinet.[4-6] (References 5 and 6, which are books, give copious additional references.) Unfortunately, the linear theory gives information primarily about the frequency of the fundamental and tells us very little about the relative amplitudes of the upper partials.

Almost the only studies of nonlinear effects in pipes have been those of Benade.[7,8] He proposed a treatment in which the normal resonance modes of the air column within the pipe are coupled together through nonlinear interaction with the air jet (or reed in the case of reed pipes). In principle, provided the interaction law and the separate characteristics of the pipe and jet are known, this approach allows calculation of the amplitudes of all the harmonics of the oscillation. Even without such a calculation, Benade has shown that these considerations allow general predictions of the change with amplitude of harmonic content and of frequency to be made.

The purpose of the present paper is to develop in quantitative form the nonlinear interaction theory put forward by Benade. In subsequent papers, we hope to present the results of an experimental study of some of the predictions of the theory and finally to relate our conclusions to the initial problem of the tonal design of a rank of flue pipes. We shall see that the results also have relevance to tone development in the flute.

I. THE JET AS AN EXCITATION MECHANISM

The interaction between sound waves and an air jet issuing from an orifice has been studied in some detail by Brown.[9] He showed that interaction takes place almost exclusively at the point where the jet emerges from the orifice and that the interaction induces the formation of vortices which progress along the jet with a velocity rather less than that of the fluid.

Coltman[3] has carried out detailed measurements of the acoustic impedance of such a jet when it travels a relatively short distance and then impinges upon an edge which is part of the embouchure hole of a flute. In this study, he was able to measure the acoustic impedance at a given frequency as a function of jet velocity and found a behavior of the sort illustrated in Fig. 1. The real part of the impedance was negative (corresponding to a supply of acoustic energy to the tube resonator) over several discrete ranges of airjet velocity. He pointed out that only the outermost loop of the spiral is in fact used in practical flute playing and that, unless the velocity is such as to make the impedance exactly real, its reactive part will cause a correction to the natural tube resonance, making the sounding tone either sharp or flat.

For our present purposes, we require an approximate treatment of the acoustic impedance of such a jet (in this case for an organ pipe rather than a flute) as a function of frequency. It is, unfortunately, out of the question to attempt any sort of complete solution. Rayleigh[10] long ago gave attention to this problem and it has also been studied more recently by Cremer and Ising[11] and by Powell.[12] The behavior of the jet is both nonlinear and dispersive, but we shall concentrate, for the moment, on a linear approximation and simplify the discussion as much as we can.

Suppose that inside the organ pipe there is an acoustic disturbance with angular frequency ω so that at time t the acoustic particle velocity out of the pipe mouth near the jet orifice is $v\exp(i\omega t)$. Following the work of Brown[9] and of Coltman,[3] we know that this acoustic ve-

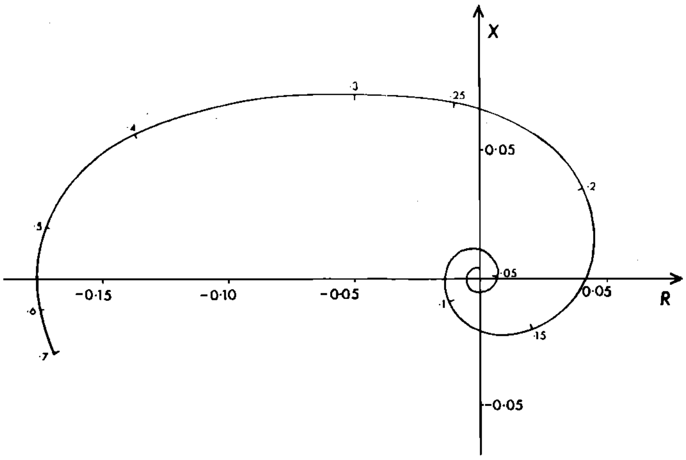

FIG. 1. Acoustic impedance $Z = R + iX$ of the blowing jet of a flute as measured by Coltman[3] for a particular jet configuration. The impedance is given in units of the characteristic impedance of the tube, the parameter on the curve is the blowing pressure in inches of water, and the sounding frequency is 440 Hz.

locity interacts with the jet and provokes a disturbance near the orifice which is essentially wave-like in character and which travels along the jet with a velocity u. Coltman's measurements show that u is about 0.4 times the fluid velocity in the jet but it also varies somewhat with frequency. In addition, over a reasonably large frequency range, the amplitude of the jet disturbance increases with distance from the orifice.[9,11] Taking all these effects into account, if the distance from the jet mouth to the pipe lip is d, then the jet disturbance reaches the lip after a time d/u. At the lip it generates a pressure fluctuation, due to its blowing either into or out of the tube, whose magnitude by Bernoulli's theorem is proportional to the square of the jet velocity and hence to the pressure P in the blowing reservoir.

In the linear approximation which we are using, the amplitude of the jet deflection is proportional to the acoustic disturbance $v\exp(i\omega t)$ which causes it, though we cannot immediately say whether or not any phase change is involved in the interaction, nor exactly what the coupling constant is. We can, however, absorb both these uncertainties into an effective interaction constant γ to write[13] the pressure disturbance p generated when the jet strikes the pipe lip as

$$p = -\gamma P v \exp[i\omega(t - d/u)] \ . \tag{1}$$

The interaction constant γ depends in detail upon the geometry of the jet, the pipe mouth, and the pipe lip and is in general complex, of the form $\gamma_0 \exp(-i\delta)$.

The acoustic impedance Z of the jet is now obtained by dividing this pressure fluctuation p by the acoustic volume velocity $Av\exp(i\omega t)$ out of the pipe mouth, where A is the effective area of the mouth opening. Thus,

$$Z = -(\gamma_0 P/A)\exp[-i(\omega d/u + \delta)] \ . \tag{2}$$

Measurements by Coltman[3] on flute jets led him to the conclusion that Z is real and negative, so that the flute tube sounds at its normal resonance frequency when the path length d of the jet is about half a wavelength. This implies a value close to π for the phase lag δ under the particular conditions chosen. The jet photographs of Cremer and Ising[11] suggest a similar result for the organ pipe jet which they studied. Finally, measurements by the present author, to be reported in detail in a later paper, indicate quite directly that $\delta \simeq \pi$ for both the fundamental and the next mode of a small organ pipe.

It is fairly easy to identify the approximate origin of this phase shift. The interaction between the jet and the acoustic current occurs at the jet aperture and the jet, if we assume it to be mass controlled, acquires a transverse velocity component which lags behind the forcing acoustic velocity by $\pi/2$. In the interaction between the jet and the pipe lip, there is a phase lag because the pressure is generated by a volume flow and the cross section of the jet is very much smaller than that of the pipe. In the limit of a very small jet and a very large pipe, the pressure should tend to follow the integral of the jet flux into the pipe and therefore lag behind the jet deflection by $\pi/2$. For a larger jet and smaller pipe, this phase lag should be smaller. The total phase lag δ, being the sum of these two contributions, should therefore approach π for a typical pipe geometry. The real situation is, of course, much more complex than this in its details.

If we adopt the value π for δ, then the impedance Z, given as a function of blowing pressure by Eq. 2, is as shown in Fig. 2. Because of the unknown magnitudes in γ_0 and the dependence of A upon the particular physical system involved, the absolute magnitude of the complex impedance Z is undetermined and we have simply drawn Fig. 2 to the same size as Fig. 1 so that a shape comparison can be made. The parameter on the curve giving blowing pressure (in inches of water for comparison with Coltman's data) is however calculated directly on the assumption of a jet length of 7 mm, a sounding fre-

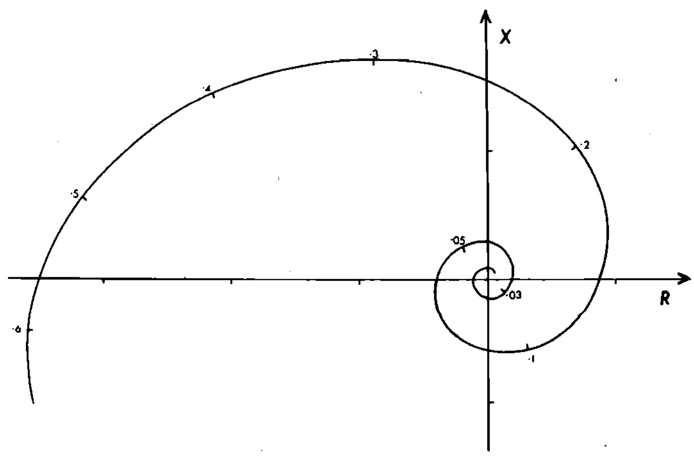

FIG. 2. Acoustic impedance $Z = R + iX$ of the blowing jet of an organ pipe as described by Eq. 2. As discussed in the text, the impedance units are undetermined but the figure has been drawn the same size as Fig. 1 for qualitative comparison. The parameter on the curve is the blowing pressure in inches of water assuming a jet length of 7 mm, a sounding frequency of 440 Hz, and a wave-propagation velocity along the jet of 0.4 times the jet air velocity, again for comparison with Fig. 1.

quency of 440 Hz, and a wave-propagation velocity along the jet of 0.4 times the airstream velocity, so that these figures are directly comparable with those of Fig. 1.

From a comparison of the two figures it is clear that, apart from the unknown scale factor which can be accommodated in the parameter γ_0, the semiquantitative agreement between theory and experiment is good. There are of course some minor discrepancies and there remains the possibility that ultimate experimental determination of γ_0 might not give the expected agreement in magnitude, but with these reservations we can go on to use the more general form of Eq. 2, which includes frequency dependence, as a basis for our further development of the theory.

We shall not discuss the linear theory further except to point out that the condition for maintenance of a pipe oscillation near its fundamental resonance frequency ω_1 is that the jet impedance $Z(\omega_1)$ at this frequency should have a negative real part of sufficient magnitude to overcome the pipe losses. If the blowing pressure is increased or the jet length decreased, the real part of the jet impedance can be made negative for the second pipe resonance at frequency ω_2, but positive for the first resonance, and the pipe will overblow. This has been discussed in more detail by Coltman.[3]

II. NONLINEAR JET EXCITATION

The theory we have set out above is linear: the pressure disturbance p generated by the jet and given by Eq. 1 is simply related to the velocity disturbance $v\exp(i\omega t)$ through the impedance Z given by Eq. 2. In reality, the situation is much more complicated than this and, in particular, there is a maximum pressure that can be generated by the jet (when it is blowing entirely inside the lip), as well as a minimum pressure (when it is blowing entirely outside the lip). Thus, the static (p, v) relation for the jet looks rather like the full curve in Fig. 3, instead of being simply a straight line.

A detailed experimental or theoretical study could, in principle, elucidate the form of this curve, but we shall not attempt this here. Instead we shall assume a general static characteristic of the form

$$p=\sum_{n=0}^{\infty} a_n' v^n \,, \tag{3}$$

where the zeroth-order term a_0' represents a static pressure produced by the jet and the linear term $a_1' v$ was written in Eq. 1 as $-\gamma P v$. Higher terms describe the saturation behavior shown in Fig. 3.

If we take all the coefficients a_n' to be real and follow a development similar to that leading to Eq. 1, then we find, for a velocity disturbance $v\exp(i\omega t)$, the jet-generated pressure disturbance

$$p=\sum_{n=0}^{\infty} a_n' v^n \exp[in\omega(t-d/u)-i\delta]\,, \tag{4}$$

where the jet phase shift δ is now incorporated explicitly in the argument. Any possible dispersive behavior of wave-propagation velocity along the jet is immaterial at this stage since the higher harmonics are generated only in the interaction of the jet with the lip edge, rather than propagating along the jet. This statement does not hold true, however, for the next step in the development.

We can now extend this analysis further to suppose that the original velocity disturbance has a Fourier series spectrum

$$v(t)=\sum_{m=-\infty}^{\infty} c_m \exp(im\omega t)\,. \tag{5}$$

In this expression, the coefficients c_m satisfy[13]

$$c_{-m}=c_m^*\,, \tag{6}$$

and ω is positive. The resulting pressure disturbance is then

$$p(t)=\sum_{n=0}^{\infty} a_n' \exp(-i\delta)\left\{\sum_{m=-\infty}^{\infty} c_m \exp[im\omega(t-d/u)]\right\}^n\,. \tag{7}$$

In Eq. 7, in distinction from Eq. 4, the individual pipe modes whose amplitudes are c_m are interacting with the jet at its orifice and the wave-like disturbance of frequency $m\omega$ then propagates along the jet and interacts with the lip. If there is an appreciable dispersion in wave velocities along the jet as a function of frequency, then the velocities u appearing in Eq. 7 should be subscripted to u_m and the velocity appropriate to frequency $m\omega$ used in each case. As in Eq. 4, there is no further complication of this type introduced by the nonunity exponents n and we only have to keep track algebraically of the individual velocities u_m involved. This is perfectly feasible if u is known as a function of frequency and the number of modes considered is not too large. For our present purposes, and in absence of detailed knowledge of dispersion in the propagation velocity, we shall assume u to be independent of frequency.

It now no longer makes sense to try to define an effective impedance for the jet because the nonlinearity leads to a great deal of frequency conversion. Rather, let us consider in a more detailed way the interaction between the jet and the resonant modes of the pipe.

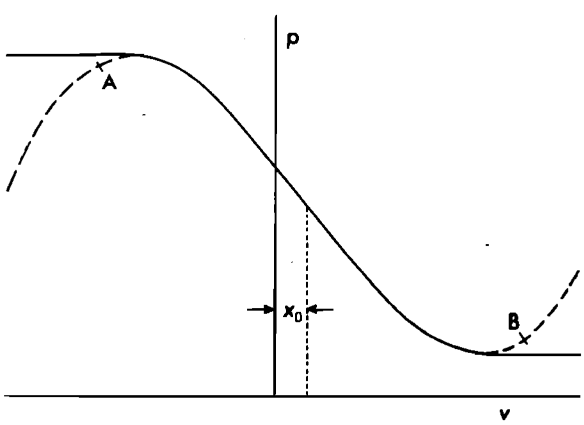

FIG. 3. Static interaction curve for a jet exciting an organ pipe. An acoustic particle velocity v out of the pipe mouth causes a pressure p due to interaction of the jet with the pipe lip. x_0 is the static offset in the position of the jet relative to the lip, as defined in Sec. 6. The broken curve is a cubic approximation to the interaction curve, valid within the range A to B.

III. THE JET-PIPE INTERACTION

The oscillating air column in an organ pipe is a continuous system, but for our present purposes it is most convenient to consider its behavior as resolved into an infinite sequence of normal modes. Because of the damping associated with each mode and arising from viscous, thermal, and radiation losses, the resonances are not sharp but exhibit finite Q values. If ψ_n is the acoustic displacement associated with the nth normal mode and B is a constant equal to the cross-sectional area of the pipe divided by the effective vibrating mass of the air column, then the equation of motion has the form

$$\frac{d^2\psi_n}{dt^2}+k_n\frac{d\psi_n}{dt}+\omega_n^2\psi_n=-p(t)B\,, \tag{8}$$

where ω_n is its resonance frequency and k_n its characteristic damping. The forcing term $p(t)$ arises from the pressure fluctuations produced by the jet and its sign is negative because displacements have been taken as positive in a direction out of the pipe mouth.

To solve the set of Eqs. 8 is a formidable task, but we can simplify them considerably if we assume the pipe resonances to be sufficiently sharp that they are essentially nonoverlapping. This is a good approximation for the first few resonances of all reasonably shaped pipes and is valid up to at least $n\simeq 10$ for pipes with reasonably narrow scales like diapasons. For cylindrical pipes, the resonance frequencies ω_n form an approximately harmonic series so that they each select from the pressure spectrum of Eq. 7 a single harmonic with which to interact. If we denote by ω the angular frequency of the fundamental component of $p(t)$, which approximately corresponds with the pipe fundamental ω_1, then Eq. 8 can be written as

$$\frac{d^2\psi_n}{dt^2}+k_n\frac{d\psi_n}{dt}+\omega_n^2\psi_n=-p_nB\exp(in\omega t)\,, \tag{9}$$

where p_n is the complex amplitude associated with the nth harmonic of $p(t)$. The fundamental mode is not necessarily the strongest, or even excited at all, but, because of the nearly harmonic relation between the ω_n, Eq. 9 is still a convenient formulation in most cases.

From Eq. 7, defining $a_n=Ba_n'$, we can write formally for this pressure amplitude:

$$Bp_n=\sum_m\sum_{s=1}^{\infty}a_s c_{m(1)}c_{m(2)}\cdots c_{m(s)}\exp(-in\omega d/u-i\delta)\,, \tag{10}$$

where $\sum_m$ implies a sum over all possible sets of $m(i)$ satisfying

$$m(1)+m(2)+\cdots+m(s)=n\,, \tag{11}$$

and, of course, the individual $m(i)$ may be either positive or negative integers.

Now the velocity disturbance $v(t)$ of Eq. 5 which interacts with the jet is simply the time derivative of the displacement ψ. But from Eq. 9, the form of the forcing term is such that the mode ψ_n is constrained to vibrate with the forcing frequency $n\omega$, so that

$$\psi_n=(c_n/in\omega)\exp(in\omega t)\,. \tag{12}$$

If we use this self-consistency condition in Eq. 9, then we find immediately the solution

$$c_n=\frac{-in\omega p_nB}{\omega_n^2-(n\omega)^2+in\omega k_n}\,. \tag{13}$$

Equations 10–13, when taken only to some finite value of s, constitute with their real and imaginary parts a set of $2s$ equations for the s complex velocity amplitudes $c_1,\ldots,c_s$ and the angular frequency ω. We can, however, without loss of generality, choose the origin of time so that the phase of the fundamental c_1 is zero, so that the $2s+1$ unknowns reduce to $2s$ and an explicit solution becomes possible.

IV. PIPE WITH A SINGLE RESONANCE

To gain some feeling for the behavior to be expected, let us first examine the behavior of a jet coupled to a pipe with only a single resonant mode of angular frequency ω_1. If the pipe resonance is sufficiently narrow, we can concentrate our attention on the behavior of the fundamental pipe mode and neglect the amplitude of higher modes in the pipe, even though they may be present in the jet.

For our later development, we shall be forced to truncate the power series expansion of Eq. 3 for the jet characteristic and it is important to find what limits this places on the range of validity of the solution. The analysis is simplified by noting that neither a_0 nor a_2 enter into the solution. The linear term a_1 is insufficient to yield a finite solution, so we investigate the behavior when Eq. 3 is truncated after the cubic term a_3v^3. The broken curve in Fig. 3 shows this cubic approximation and it is clear that, while the approximation is good for small amplitudes v and the saturation behavior is well reproduced, the curve becomes entirely incorrect for amplitudes outside the range AB. We note that, necessarily, $a_1<0$ and $a_3>0$.

Proceeding with the formal solution of Eq. 13 and 10 and writing

$$\theta\equiv\omega d/u\,, \tag{14}$$

we find

$$(\omega_1^2-\omega^2+i\omega k_1)\,c_1=-i\omega(a_1c_1+3a_3c_1^3)\,e^{-i(\theta+\delta)}\,, \tag{15}$$

the factor 3 arising in the cubic term since there are three terms in the summation on m in Eq. 10. The imaginary part of this equation gives

$$c_1^2=-[a_1\cos(\theta+\delta)+k_1]/3a_3\cos(\theta+\delta)\,, \tag{16}$$

so that oscillations occur provided $\cos(\theta+\delta)$ is positive and $-a_1\cos(\theta+\delta)>k_1$. This solution lies in the range of validity AB provided that $|a_1|\lesssim 2k_1$. The real part of Eq. 15 gives the sounding frequency and, neglecting smaller terms, we find

$$\omega\simeq\omega_1-\tfrac{1}{2}k_1\tan(\theta+\delta)\,. \tag{17}$$

In the "center" of the range of blowing pressures producing this fundamental, $\theta+\delta=2\pi$ and $\omega=\omega_1$. As the blowing pressure is increased, u rises and θ decreases, so that the sounding frequency ω increases above ω_1.

Conversely, for pressures less than that for the center of the range, ω becomes progressively less than ω_1.

The solutions of Eqs. 16 and 17 are mathematically and physically correct provided that the condition

$$|a_1| \lesssim 2k_1 \tag{18}$$

is met. This allows for an excursion of the jet from entirely inside to entirely outside the pipe lip, as well as all smaller excursions, and therefore represents a physically interesting range of situations. We must be carefuly, however, not to apply the solutions blindly to strongly blown pipes with very small damping coefficients k_1 or we may exceed their range of validity. We shall examine this criterion again after considering the more general case.

It is also worthwhile noting two other apparent solutions which arise from Eq. 15. The first is the quiescent situation with $c_1 = 0$. This is physically legitimate but unstable and any small perturbation of the jet will cause a transition to the situation described by Eqs. 16 and 17. The second is an artifact which arises from Eq. 16 if $\cos(\theta+\delta)$ is negative, and does not correspond to a real physical situation. It is easily eliminated by the requirement that $\cos(\theta+\delta)$ is positive.

V. PIPE WITH MANY RESONANCES

In this section we proceed as far as possible without restricting the generality of the problem. Finally then, we shall make some more explicit assumptions to treat a particular case as an example.

The Eqs. 10–13 can be simplified to some extent by writing the complex amplitudes c_n in the form

$$c_n = d_n \exp(i\phi_n), \tag{19}$$

where the d_n are real positive quantities and, by Eq. 7,

$$d_{-n} = d_n; \quad \phi_{-n} = -\phi_n. \tag{20}$$

Using this form and substituting from Eqs. 10, 12, and 14 into Eqs. 13, we deduce, after a little manipulation, the equations

$$\begin{aligned} -d_n(\omega_n^2 - n^2\omega^2)/n\omega &= a_1 d_n \sin(n\theta+\delta) \\ &+ a_2 \sum_m d_m d_{n-m} \sin(n\theta + \phi_n - \phi_m - \phi_{n-m} + \delta) \\ &+ a_3 \sum_l \sum_m d_l\, d_m\, d_{n-l-m} \sin(n\theta + \phi_n - \phi_l - \phi_m \\ &\qquad - \phi_{n-l-m} + \delta) + \cdots \end{aligned} \tag{21}$$

and

$$\begin{aligned} -k_n d_n &= a_1 d_n \cos(n\theta+\delta) \\ &+ a_2 \sum_m d_m d_{n-m} \cos(n\theta + \phi_n - \phi_m - \phi_{n-m} + \delta) \\ &+ a_3 \sum_l \sum_m d_l d_m d_{n-l-m} \cos(n\theta + \phi_n - \phi_m \\ &\qquad - \phi_{n-l-m} + \delta) + \cdots. \end{aligned} \tag{22}$$

Straightforward solution of these equations for the d_n, ϕ_n, and ω, given that $\phi_1 = 0$, now provides a solution to our problem. Usually one or other of the first two or three harmonics will have the dominant amplitude and the $p(v)$ series, subject to the criterion of Eq. 18, can be approximated by its first few terms. We can therefore proceed by truncating the problem to a relatively small order and, after solving this, proceed to the full solution by means of successive approximations.

To this end, let us represent the interaction $p(v)$ by the first three terms of its series expansion of Eq. 3, as discussed in the previous section. Let us also suppose that the physical situation (jet velocity and tube resonances) is such that either the first or the second harmonic is dominant. We therefore proceed to a solution for these two components only, neglecting all d_n for $n > 2$.

After a good deal of tedious algebra, we find the solutions

$$d_1^2 = -F_2 H, \tag{23}$$

$$d_2^2 = F_1 H, \tag{24}$$

$$\sin\phi_2 = (1/2\omega a_2)(F_1/H)^{1/2}, \tag{25}$$

$$\cos\phi_2 = (1/2\omega a_2)(F_1 H)^{-1/2}[G_1 + \omega a_1 + \omega a_3 H(3F_1 - F_2)], \tag{26}$$

where

$$F_1 = (\omega_1^2 - \omega^2)\cos(\theta+\delta) - \omega k_1 \sin(\theta+\delta), \tag{27}$$

$$F_2 = (\omega_2^2 - 4\omega^2)\cos(2\theta+\delta) - 2\omega k_2 \sin(2\theta+\delta), \tag{28}$$

$$G_1 = (\omega_1^2 - \omega^2)\sin(\theta+\delta) + \omega k_1 \cos(\theta+\delta), \tag{29}$$

$$G_2 = (\omega_2^2 - 4\omega^2)\sin(2\theta+\delta) + 2\omega k_2 \cos(2\theta+\delta), \tag{30}$$

$$H = \frac{\omega a_1(2F_1 + F_2) + F_2 G_1 + F_1 G_2}{\omega a_3(F_2^2 + 3F_1F_2 - 2F_1^2)}. \tag{31}$$

One of the Eqs. 25 and 26 would appear to be redundant, but in fact the allowed values of ω are determined by the consistency condition

$$\cos^2\phi_2 + \sin^2\phi_2 = 1. \tag{32}$$

These Eqs. 23–31 thus determine the coefficients c_1 and c_2 in the Fourier series spectrum of Eq. 5 of the pipe oscillation. With these two components known, in the form (d_n, ϕ_n) and provided that our original assumption that one or the other of them is the dominant mode is justified, we can determine c_3 as (d_3, ϕ_3) from Eqs. 21 and 22. We can proceed in this way to find all the Fourier components c_n.

This procedure effectively takes into account the coupling of a given mode c_n with all modes c_m for which $m < n$. Coupling with modes for which $m > n$ is neglected. It is, however, now possible to refine the result obtained above by using these first approximations in Eqs. 21 and 22. If the initial result or the refinement suggests that the dominant mode is not c_1 or c_2 but some other c_n, then the initial solution to find the oscillation frequency should be carried out for this c_n and the next most intense mode. For a stopped pipe, for example, c_1 and c_3 should be used and, for an overblown open pipe, c_2 and c_3 or c_4.

VI. FURTHER CONSIDERATION OF JET INTERACTION

To apply this theory to a real situation we must first define a reasonable set of parameters to describe the jet and its interaction with the acoustic particle velocity v and with the lip of the pipe. This we now proceed to do.

As shown in Fig. 4, the jet emerges from a reservoir under pressure P through a slit of length L and width $2W$ at the mouth of the pipe. It is a reasonable approximation to describe the velocity distribution across the jet in the direction of its thickness W by an expression of the form

$$V = V_0 \exp[-(x-x_0)^2/W^2] . \quad (33)$$

From Bernoulli's theorem we expect

$$P = \tfrac{1}{2}\rho V_0^2 , \quad (34)$$

where ρ is the density of the air. We do not interpret the velocity distribution as implying a distribution in the disturbance velocity u (although this may happen), but rather simply as a blurring of the edges of the jet.

In a real situation, the jet will spread and its velocity decrease as it moves across the mouth to the lip, but for simplicity, we shall ignore this complication and assume it to remain as a sheet of thickness W. The jet travels across the mouth a distance d and then impinges on the sharp lip of the pipe on the plane $x=0$ at a transverse distance x_0 (which we shall call the offset) from the plane of symmetry of the jet. We now further assume that, when a steady airflow of particle velocity v is flowing out of the pipe mouth, the jet is deflected outwards by an amount αv to be intercepted by the pipe lip at a distance $x_0+\alpha v$ from the jet symmetry plane. The coefficient α, with the addition of a phase shift as discussed before, then measures the sensitivity of the jet to acoustic disturbance.

For the purposes of our present calculation it is helpful to dissect this parameter α a little further to display in approximate form its dependence on other physical parameters of the system. We should expect, for example, that α should vary inversely with the mass of the jet per unit length and directly with the travel time to the lip (i.e., directly with θ/ω_1). It is therefore a reasonable approximation to write

$$\alpha = \beta\theta/\omega_1 W , \quad (35)$$

where β is a more fundamental parameter measuring the interaction strength and depending upon quantities like the Reynold's number of the jet.

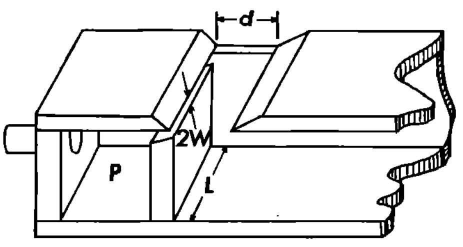

FIG. 4. Schematic diagram of the mouth of an organ flue pipe showing the jet thickness $2W$, jet width L, and lip cut-up d.

TABLE I. Assumed values of parameters.

$\omega_1 = 1000\ \text{sec}^{-1}$	$\omega_2 = 2100\ \text{sec}^{-1}$
$k_1 = 50\ \text{sec}^{-1}$	$k_2 = 50\ \text{sec}^{-1}$
$W = 0.1$ cm	$L = 3$ cm
$S = 10\ \text{cm}^2$	$x_0/W = 0.1$
$P = 10$ mbar	$\theta = \pi$
$\beta = 0.4$	$\delta = \pi$

Coltman[2] has analyzed the pressure produced by a jet of cross section s and velocity V blowing into the end of a pipe of cross sections S and finds, for $s \ll S$, that the pressure generated is

$$p = \rho V^2 s/S . \quad (36)$$

We can apply this equation to the portion of the jet entering the pipe when the intercept position is $x_0+\alpha v$ as above and, after some algebra, we find, for the first three coefficients in the expansion of Eq. 3, the values

$$\begin{aligned} a_1 &= -C\alpha , \\ a_2 &= 2C\alpha^2 W^{-2} x_0 , \\ a_3 &= \tfrac{2}{3} C\alpha^3 W^{-2}(1-4x_0^2 W^{-2}) , \\ C &= 2PLS^{-1}\exp(-2x_0^2 W^{-2}) . \end{aligned} \quad (37)$$

This description of the jet and the nonlinear character of its interaction with the air column of the pipe is manifestly incomplete and oversimplified and we make no claim for its general validity except as a heuristic approximation suitable for our present purposes. A more detailed description must clearly take into account some of the points raised, for example, by Powell.[12]

VII. CALCULATION FOR PIPE WITH TWO RESONANCES

In Sec. V we set out the formal solution of our problem for a pipe with two resonances and showed how this could be extended to treat more general systems. To examine the predictions of the theory we now solve Eqs. 19–32 together with Eqs. 35 and 37 for a representative system with two pipe resonances. The assumed values of the physical parameters are those set out in Table I, these parameters being varied one at a time to observe their effect on the solutions. In all cases, the overtones present are true harmonics of the fundamental and the condition in Eq. 18 is satisfied for the whole range of the variables displayed, so that truncation of the nonlinear expansion is a valid approximation, as discussed in Sec. IV.

The phenomenon of overblowing requires some separate comment. A pipe is said to be overblown when the amplitude at the pipe fundamental frequency falls to zero. The behavior of the second mode is then given by Eqs. 16 and 17 with $(c_2, \omega_2, k_2, 2\theta)$ in place of $(c_1, \omega_1, k_1, \theta)$. This oscillation state can occur when θ, as defined in Eq. 14, lies between $\pi/4$ and $3\pi/4$, provided now that $-a_1\cos(2\theta+\delta) > k_2$, and the solution is quantitatively

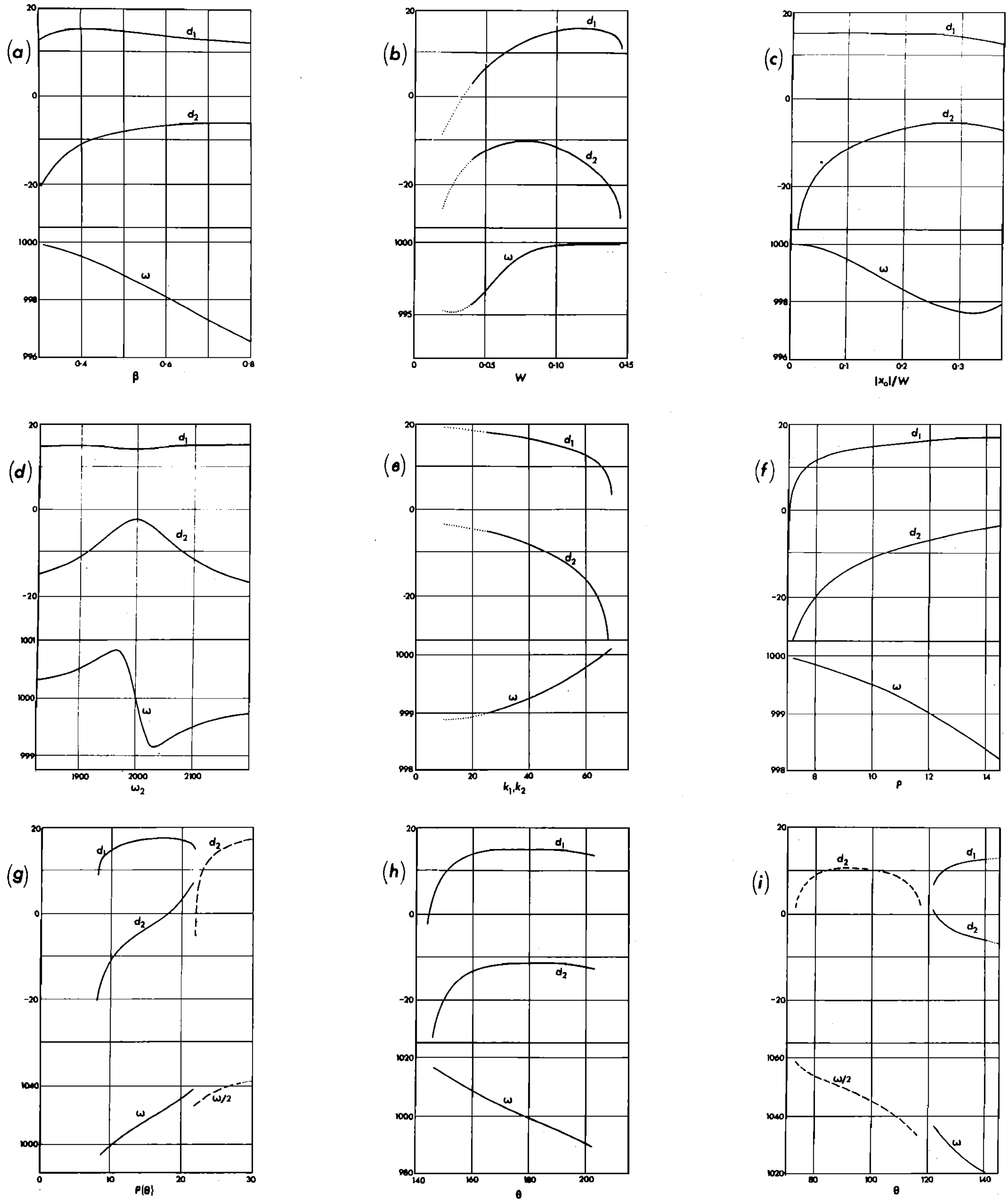

FIG. 5. (a)–(f) Effects on the oscillation behavior of a two-mode pipe caused by varying individual physical parameters while keeping the others fixed at the values given in Table I. (g) Behavior of the pipe of Table I as a function of blowing pressure with the lip cut-up held fixed at the value giving $\theta = \pi$ for $P = 10$ mbar. (h) Behavior of the pipe of Table I as a function of lip cut-up distance as reflected in the phase parameter θ. (i) As for (h), but with the coupling parameter β doubled to 0.8. In each part of the figure, a broken line indicates an overblown condition and a dotted line indicates a region where the truncation criterion of Eq. 18 is no longer valid. (P = blowing pressure in millibars, θ = phase shift in degrees for the fundamental as determined by propagation time across the lip cut-up, $2W$ = jet thickness and x_0 = jet offset both in centimetres, ω = sounding frequency in radians per second, k_1 and k_2 = damping coefficients for first and second resonances in seconds^{-1}, d_1 and d_2 are velocity levels of first and second harmonics of the pipe oscillation, in decibels, relative to 1 cm sec^{-1}.)

valid if $|a_1| < 2k_2$.

The results of these calculations are shown in Fig. 5, which is largely self explanatory. In Fig. 5(g), the phase θ is appropriately related to the blowing pressure P, so that the behavior shown is that of a pipe as the blowing pressure is increased, without other adjustments. The broken line indicates an overblown condition. In Fig. 5(i), the coupling is doubled and the pipe overblows to the octave for small θ. Note also that the phase lag θ, which is related to the jet travel time, is most appropriately thought of as specifying the pipe cut-up, or distance from the jet orifice to the pipe lip.

From these calculations several important conclusions arise.

(1) For a note which is reasonably well above its sounding threshold, the amplitude of the fundamental is most largely controlled by the thickness W of the jet. The amplitude of this fundamental increases steadily with W until, at a critical value (in this case $W \simeq 1.5$ mm), the pipe ceases to speak.

(2) The amplitude of the second partial (which is, in fact, the second harmonic of the pipe tone), in contrast decreases as the thickness W of the jet increases. The amplitude of this second harmonic increases very markedly, for a given pipe, as the blowing pressure P is increased and has its maximum value at a pressure just below that at which the pipe overblows. The amplitude of the second harmonic is greatest when the frequency of the second resonance is equal to twice the sounding frequency.

(3) The relative amplitude of the second harmonic also depends upon the asymmetry of the jet, as measured by the parameter x_0. In our simplified model the second harmonic component vanishes for a symmetric jet, $x_0 = 0$. For a real jet, it is likely that the difference in pressure environment for a jet blowing into or out of the pipe mouth is such that considerable asymmetry is always present.

(4) For a given blowing pressure, the sounding frequency of a pipe depends strongly upon the cut-up of the lip (as reflected in the phase parameter θ) and decreases as the cut-up distance is increased.

(5) For a given pipe, the sounding frequency depends strongly upon the blowing pressure and increases as this is increased. When overblowing occurs, the frequency jumps by a little less than a factor of 2, despite the fact that the upper resonance is at more than twice the frequency of the lower.

VIII. CONCLUSIONS

The approach to the nonlinear pipe excitation developed in this paper is a quite general one but its predictions in particular cases are quite explicit. The example displayed in the previous section is an arbitrary one, but its qualitative agreement with the accumulated experience of organ pipe voicers[14] and the subjective analysis of flute players is encouraging. A detailed experimental program is, however, clearly necessary to study the predictions of the theory and to clarify some of the physical parameters involved. Such a study is at present in progress in this laboratory.

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