Sound production by organ flue pipes*

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A nearly exact, nonparametric treatment is devised to allow calculation of the transient and steady-state acoustic spectra of a flue organ pipe, given the geometry of its construction and the pressure variation in its foot. Explicit formulation is given for the fundamental and the first and second upper partials. A scaling law is set out, relating the behavior of pipes of different fundamental frequencies, and several illustrative calculations are performed. Comparison between theory and experiment for a limited number of cases shows satisfactory agreement in semiquantitative terms but indicates that a more careful treatment of air jet behavior is required at low jet velocities.

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INTRODUCTION

The theory of sound generation in organ pipes has been studied previously by many authors, most notably by Cremer and Ising[1] and by Coltman,[2] but these treatments have dealt almost exclusively with the determination of sounding frequency and fundamental amplitude, rather than with harmonic development. The basis of an approach to calculation of harmonic development was set out by Benade and Gans[3] and applied to a reed-driven pipe by Worman.[4]

Application of these ideas to a theoretical treatment of the acoustic spectrum of flue pipes in both the steady state and during the onset transient has been made in two papers by the present author[5,6] which treat some parts of the problem with reasonable exactitude but rely upon a parametric description of other aspects. The presence of these adjustable parameters in the theory makes quantitative comparison with experiment impossible but the theory does predict behavior in good qualitative agreement with observation. For this reason it is of interest to develop it further into a fully quantitative treatment, and this is the objective of the present paper.

Those parts of the original treatment which were treated parametrically concerned the interaction of the acoustic disturbances in the pipe with the air jet, propagation of the effects of these disturbances along the jet, and interaction of the jet with the air column at the pipe lip. None of these matters is yet fully understood but the work which has been done since the time of Rayleigh[7–11] now allows at least a good semiquantitative treatment of the disturbance of the jet that is free from arbitrary parameters. A recent discussion by Elder[12] of the interaction between a modulated jet and a resonant column, when modified in some minor but important ways,[13] similarly allows this part of the problem to be treated realistically. Finally a more detailed consideration of the mathematical part of the development allows us to remove a limitation to small transverse jet displacements which was implicit in the earlier treatments.

I. GENERAL APPROACH

Following Cremer and Ising[1] and our own original approach[5,6] we consider the problem in three parts: The pipe as a passive linear oscillatory system having normal modes at angular frequencies n_i that are not generally in precise harmonic relation, the air jet along which disturbances propagate and are amplified in an approximately linear manner, and the highly nonlinear coupling between these two systems.

We suppose that the acoustic displacement in the pipe body has the form

$$x(t) = \sum_i x_i(t) = \sum_i a_i \sin(\omega_i t + \beta_i) , \tag{1}$$

where the frequencies ω_i are close to the normal-mode frequencies n_i, but not in general either equal to them, or themselves in exact harmonic relationship. The amplitudes a_i and phases β_i are slowly varying functions of time. The acoustic velocity derived from (1), after modification by the pipe mouth, interacts with the air jet and imposes on it a sinuous motion which increases in amplitude as it travels towards the pipe lip, leading there to a deflection of the jet core through a distance $y(t)$. Clearly $y(t)$ is related to $x(t)$ at some earlier time, but the relation is complicated by the dispersive nature of wave propagation along the jet. This problem has been treated, in essence, by Rayleigh.[7]

The displaced jet in its interaction with the pipe lip now gives rise to a fluctuating force $F(t)$ which is coupled to the oscillations in the pipe and tends to drive them with a frequency-dependent phase shift. The analysis of this problem is a modified version of that of Elder.[12,13]

Finally we shall express the problem as a set of coupled nonlinear differential equations which we recognize as related to the Van der Pol equation[6,14] and which we solve by the method of slowly varying parameters.[6,15] It turns out that the numerical integration of these equations, as well as giving information about the transient behavior of the pipe, is also rather more economical of computing time than is the direct solution of the steady-state problem. For ready reference an outline of this solution method is given in the Appendix.

II. THE DISTURBED JET

In our earlier discussion of this topic, the jet was given little detailed consideration but was treated parametrically. The problem of the acoustically disturbed jet does not appear to have yet been completely solved but we can make some progress.

Rayleigh[7] showed that, in the absence of viscosity, a plane jet with a square velocity profile V_0 and thickness W_0 emerging into still fluid of the same density supports the propagation of a sinuous disturbance y with wave number $k = 2\pi/\lambda$ according to the relation

$$y = A \exp(\pm(\tfrac{1}{2}kW_0)^{1/2}kV_0t)\cos[k(\tfrac{1}{2}kW_0V_0t - x)] \tag{2}$$

provided $kW_0 \ll 1$. The phase velocity of this disturbance, $u = \omega/k$, is then given by

$$u = \tfrac{1}{2}kW_0V_0 = (\tfrac{1}{2}\omega W_0V_0)^{1/2} \tag{3}$$

and the amplitude grows exponentially with time. This is in qualitative agreement with the observations of Brown[8] but the quantitative agreement is not good.

The main physical shortcoming of Rayleigh's treatment seems to arise from its neglect of viscosity, which will both modify the square velocity profile and cause the jet to spread and slow down. It is reasonable to approximate this behavior by assuming a Gaussian velocity profile[16]

$$V(y) = V \exp[-\pi(y - y_0)^2/4W^2], \tag{4}$$

where $y - y_0$ measures the distance from the jet center line y_0. The width of the jet at a distance x from the exit slit, or pipe flue, can also be approximated as

$$W = W_0 + 2x\tan\phi, \tag{5}$$

where φ is the semiangle of spread and, by imposing a condition of constant momentum transfer along the jet, we find for its central velocity

$$V = V_0(W_0/W)^{1/2}. \tag{6}$$

Measurements to be described later verify the appropriateness of these assumptions.

A jet with velocity profile similar, though not identical to this has been studied by Savic,[17] who has derived an expression for the phase velocity, for a neutrally stable disturbance, of the form

$$u = 0.55\,\omega^{1/3}W^{1/3}V^{2/3} \tag{7}$$

which agrees quite well with Brown's experimental data for growing disturbances. It is worth noting that, in view of (6), u is constant along the length of the spreading jet. In our subsequent discussion we adopt (7) as appropriate to an organ-pipe jet.

Rayleigh's equation (2), which assumes a sinuous disturbance all along the jet at $t = 0$, shows that, as well as propagating, this disturbance grows with time as $\exp[(\frac{1}{2}kW)^{1/2}kVt]$. This can be written in terms of a growth with propagation distance x as $\exp[(\frac{1}{2}kW)^{1/2}kVx/u]$. Savic did not investigate this aspect of wave behavior for his jet but considered only neutrally stable disturbances so that an equivalent expression is not available. There are also ambiguities involved in attempting to adapt Rayleigh's expression because of the relationship (3) which applies to his case but which must be replaced by (7) for Savic's jet. The most straightforward substitution of Savic's value for u in the form above gives growth as $\exp(\mu x)$ where

$$\mu = \alpha\omega^{2/3}W^{-1/3}V^{-2/3} \tag{8}$$

and the value of α is 3.15. Because of the uncertainty surrounding this approximation, however, we will retain α as a parameter in our theory, specifying only that it is probably of order of magnitude unity. We note again that, in view of (6), μ is constant along the jet.

One further modification is required to this exponential growth law for disturbances on the jet, and this arises from the fact that both Rayleigh and Savic assume that the amplitude of the displacement is small compared both with the thickness of the jet and the disturbance wavelength. When the first condition is violated the growth law probably tends to linear rather than exponential behavior, while violation of the second condition may lead to the formation of a vortex street. It is quite straightforward to include a transition to linear growth in the numerical solution of particular cases, and we shall assume that the production of vortices has little effect on the pipe behavior if they are entirely inside or outside the pipe lip.

To evaluate the way in which the acoustic field in the pipe mouth interacts with the jet, consider a jet emerging from a flue into a transverse acoustic velocity field $v\cos(\omega t + \beta)$. The jet is displaced bodily back and forth by an amount $(v/\omega)\sin(\omega t + \beta)$, except at the flue exit $x = 0$ where the displacement is held zero. This is effectively the same as applying a localized displacement $y(0) = -(v/\omega)\sin(\omega t + \beta)$ to the jet and, by Rayleigh's treatment, this displacement propagates along the jet, growing exponentially with time, or equivalently with x coordinate. Because the behavior is symmetrical about $x = 0$ if we neglect the divergence of the jet, the spatial growth factor is $\cosh(\mu x)$, rather than $\exp(\mu x)$.

Combining these terms we find for the total jet displacement at position x and time t

$$y(x,t) = (v/\omega)\{\sin(\omega t + \beta) - \cosh(\mu x)\sin[\omega(t - x/u) + \beta]\}\,. \tag{9}$$

The first term is the bodily displacement of the jet in the acoustic flow field and the second is the result of the displacement produced at the origin a time x/u before which propagates with velocity u and grows as $\cosh(\mu x)$.

At the pipe lip $x = l$ the factor $\cosh(\mu x)$ is generally much greater than unity so that, neglecting small terms, we can write the jet deflection at the pipe lip contributed by the ith mode as

$$y_i = G_i v_i \cos[\omega_i(t - \delta_i) + \beta_i - 3\pi/2], \tag{10}$$

where δ_i is the transit time

$$\delta_i = l/u_i \tag{11}$$

for waves of frequency ω_i and G_i is the amplification along the jet

$$G_i = \cosh(\mu_i l) \tag{12}$$

with appropriate modification if $\sum y_i > W$. The phase constant in (10) is $3\pi/2$ because we have chosen to write y_i in terms of a cosine function.

Now the acoustic velocity amplitude v_i in the pipe mouth is related to the acoustic displacement amplitude a_i of the ith pipe mode by

$$v_i = a_i \omega_i S_p / S_m, \tag{13}$$

where S_p is the cross-sectional area of the pipe and S_m that of the mouth. If we assume linearity in jet behavior, which is probably a reasonable approximation compared with other nonlinearities to be considered later, we can sum the contributions to the jet-tip deflection from all the modes to obtain, in the notation of (1),

$$y_0(t) = \sum_i y_i(t) = \sum_i (S_p/S_m) G_i \dot{x}_i (t - \delta_i - 3\pi/2\omega_i). \tag{14}$$

III. JET-PIPE INTERACTION

The next part of our problem is to examine how the jet, deflected as described by (14), can interact with the pipe at its lip to produce a driving force for the pipe modes. This was considered by both Cremer and Ising[1] and by Coltman,[2] but neither of these treatments is adequate for our purpose and we must use instead a modification of the more nearly rigorous treatment by Elder.[12]

The application of Elder's discussion to our present problem requires some modification of his basic model so that it applies to a jet whose flow into the pipe is modified by deflection rather than by a change in jet velocity. The necessary modifications to the theory have been dealt with elsewhere[13] and it has been shown that, as far as first-order terms are concerned, the results are not very different. They can therefore be applied with reasonable confidence to the real situation which is somewhere intermediate between these two extreme models.

As shown in our modification of Elder's treatment,[13] for the case of a high-velocity jet and a narrow flue, we can write the acoustic volume flow in the pipe Q_p in the form

$$Q_p \simeq Q_{\rm I} + Q_{\rm II} , \tag{15}$$

where $Q_{\rm I}$ and $Q_{\rm II}$ are given by

$$Q_{\rm I} = [Z_m/(Z_p + Z_m)] Q_j \tag{16}$$

$$Q_{\rm II} = [\rho V / S_p (Z_p + Z_m)] Q_j . \tag{17}$$

Here ρ is the air density, V the jet velocity, and S_p the pipe cross section. Z_p is the pipe impedance measured looking inwards from the lip and Z_m the mouth impedance looking outwards from the same point. As discussed by Elder[12] and the present author,[13] $Q_{\rm I}$ is a flow of the type described by Cremer and his co-workers, while $Q_{\rm II}$ is that described by Coltman. There is also a nonlinear term $Q_{\rm III}$ in both theories but this has been shown[13] to be negligibly small in comparison with other nonlinearities for a pipe with a narrow flue.

Now if Δl is the end correction to the pipe at its mouth, then

$$Z_m \simeq j\omega\rho\Delta l / S_p \tag{18}$$

so that (15) can be written

$$Z_s Q_p \simeq [\rho(V + j\omega\Delta l)/S_p] Q_j , \tag{19}$$

where

$$Z_s = Z_p + Z_m \tag{20}$$

is the impedance of pipe and mouth in series.

The right-hand side of (20) represents a forcing term which is not generally in phase with the jet flow Q_j but tends to lead it. For a typical pipe system $\omega\Delta l \gg V$ so that the lead is very nearly $\pi/2$. From (14), the phase lag of the displacement $y_0(t)$ of the jet at the pipe lip relative to the acoustic velocity in the pipe mouth is $\omega\delta - 3\pi/2$ so that the total phase lag around the loop is close to $\omega\delta - \pi$ and the simple feedback loop conditions are satisfied when there is about one-half wavelength along the jet so that $\omega\delta \simeq \pi$. This result was deduced from experiment by Coltman[2] but he did not consider the origins of the various phase shifts involved.

For our subsequent development let us rewrite (19) using $Q_p = S_p \dot{x}$ and (4) for the jet velocity so that Q_j is the total flow intercepted below the pipe lip, $y = 0$, across the full jet width D. This gives

$$Z_s S_p \dot{x} = H Q_j = F(t), \tag{21}$$

where F is the effective driving pressure generated by the jet. H has magnitude

$$H = (\rho/S_p)(V^2 + \omega^2 \Delta l^2)^{1/2} \tag{22}$$

and introduces a phase lead of

$$\theta = \tan^{-1}(\omega\Delta l / V) \tag{23}$$

and the jet flow $Q_j(t)$ is given by

$$Q_j = DV \int_0^\infty \exp[-\pi(y - y_0)^2 / 4W^2] \, dy \tag{24}$$

with $y_0(t)$ given by (14). We define H_i and θ_i to be H and θ with ω_i inserted for ω.

To evaluate H and Q we clearly need to know the end correction Δl at the mouth of the pipe. Various empirical or simplified theoretical expressions for Δl have been given, but these are known to hold over only limited ranges of mouth geometry.[18,19] The formulas given by Ingerslev and Frobenius[19] are, however, of adequate range and accuracy for our purpose. The end correction at the mouth is seen to be generally several times as large as that at the open end and to increase as the mouth area is decreased. A formula of practical use for many pipes gives

$$\Delta l \simeq 1.3 (S_p/S_m) r, \tag{25}$$

where S_p and S_m are mouth and pipe areas, respectively, and r is the radius of a circle with the same area as the mouth. Knowledge of Δl in a more precise way, including its variation with frequency, together with the pipe length and the end correction at the open end, allows us to calculate precisely the passive resonant frequencies n_i for the pipe. All these quantities can, however, be obtained directly or indirectly from experiment.

IV. THE NONLINEAR JET CHARACTERISTIC

In the development followed in our earlier papers[5,6] it was assumed that the jet was deflected into or out of the pipe lip by less than about twice its own width so that the exponential in an expression rather similar to (24) could be represented to adequate accuracy by only a very few terms of its series expansion. This may be satisfactory for a marginally oscillating system, but the photographic studies of Cremer and Ising,[1] of Coltman,[2] and others make it clear that in normal pipe speech the jet is usually fully switched from well inside to well outside the pipe lip. To treat this properly it is necessary to retain essentially all terms in any expansion of the exponential function. It is perhaps worthy of note, in passing, that although the jet is usually fully switched in normal pipe speech in the steady state, this is probably not true in the initial transient stage. It is also analytically more difficult to treat the driving force as a switch function than it is to perform the necessary expansion for an analytic treatment.

If we substitute (14) in (24), then we have the options of either expanding the exponentials first (giving series of modified Bessel functions) or of performing the integration first giving an error function. The second approach leads to more rapidly convergent results. From (24) we find

$$Q_j = (DVW/2)\,[1 + \mathrm{erf}(y_0\pi^{1/2}/2W)], \tag{26}$$

where the error function erf z can be expressed as a power series[20]

$$\mathrm{erf}\ z = \frac{2}{\pi^{1/2}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!\,(2n+1)} \tag{27}$$

and y_0 is given by the series (14), which we can write in the form

$$y_0\pi^{1/2}/2W = \sum_i b_i \cos\psi_i, \tag{28}$$

where

$$b_0 = a_0\pi^{1/2}/2W, \quad b_i = (S_p/S_m)G_i\omega_i a_i \pi^{1/2}/2W \tag{29}$$

are normalized amplitudes for the jet-tip displacement and

$$\psi_i = \omega_i(l - \delta_i) + \beta_i - 3\pi/2 + \theta. \tag{30}$$

Substituting (28) into (26) and using (27), it is now easy to show that

$$Q_j = (DVW/2)\Big\{1 + B_0(b_0) + B_1(b_0)\sum_i b_i\cos\psi_i + B_2(b_0)\sum_i\sum_j b_i b_j \cos(\psi_i \pm \psi_j) + B_3(b_0)\sum_i\sum_j\sum_k b_i b_j b_k \cos(\psi_i \pm \psi_j \pm \psi_k) + \cdots\Big\}, \tag{31}$$

where

$$B_0(z) = \mathrm{erf}\,(z) \tag{32}$$

and

$$B_n(z) = \frac{1}{2^{n-1}n!}\frac{d^n}{dz^n}B_0(z) \tag{33}$$

$$= \frac{(-1)^{n-1}}{2^{n-2}\pi^{1/2}n!}H_{n-1}(z)e^{-z^2} \tag{34}$$

$H_n(z)$ is a Hermite polynomial.[20] The series in (31) converges quite rapidly and only a small number of terms is required unless some of the b_i are very large compared with unity. This result is clearly a more general statement of the series expansion for the jet coefficient used previously.[5,6]

With these results we can now express Q_j as a set of quasi-Fourier components for either an open or a closed pipe and for an arbitrary number of pipe modes. For the illustrative calculations to be carried out here we consider an open pipe, so that $\omega_n \simeq n\omega_1$, and restrict our calculation to the first three modes. Addition of further modes involves extra algebra but no new effects or other difficulties.

To evaluate the jet flow component $Q_j^{(1)}$ consisting of all terms with frequency close to ω_1, we simply select from (31) all those terms $\cos(\psi_i \pm \psi_j \pm \psi_k \pm \cdots)$ for which

$$i \pm j \pm k \pm \cdots = \pm 1 \tag{35}$$

and obtain the result

$$\begin{aligned} Q_j^{(1)} = (DVW/2)\{&[B_1b_1 + B_3b_1(3b_1^2 + 6b_2^2 + 6b_3^2) + B_5b_1(10b_1^4 + 30b_2^4 + 30b_3^4 + 60b_1^2b_2^2 + 60b_1^2b_3^2 + 120b_2^2b_3^2) + \cdots]\cos\psi_1 \\ &+ [2B_2b_1b_2 + B_4b_1b_2(12b_1^2 + 12b_2^2 + 24b_3^2) + \cdots]\cos(\psi_2 - \psi_1) \\ &+ [2B_2b_2b_3 + B_4b_2b_3(24b_1^2 + 12b_2^2 + 12b_3^2) + \cdots]\cos(\psi_3 - \psi_2) + [3B_3b_1^2b_3 + \cdots]\cos(\psi_3 - 2\psi_1) \\ &+ [3B_3b_2^2b_3 + \cdots]\cos(2\psi_2 - \psi_3) + \cdots\}, \end{aligned} \tag{36}$$

where B_n implies $B_n(b_0)$. The leading terms in each factor have been written out explicitly; terms omitted at the end of the expression involve combinations of four or more different frequencies and are relatively small unless the b_i are considerably greater than unity, corresponding to a jet deflection which is large compared with the jet thickness. We shall see, when we come to discuss the actual numerical calculations, that it is possible to modify this expression and those for $Q_j^{(2)}$ and $Q_j^{(3)}$ so that their behavior is also correct for $b_i \gg 1$.

Proceeding similarly for the jet flow component with frequency near ω_2 we find

$$\begin{aligned} Q_j^{(2)} = (DVW/2)\{&[B_1b_2 + B_3b_2(6b_1^2 + 3b_2^2 + 6b_3^2) + B_5b_2(10b_2^4 + 30b_1^4 + 30b_3^4 + 60b_1^2b_2^2 + 60b_2^2b_3^2 + 120b_1^2b_3^2) + \cdots]\cos\psi_2 \\ &+ [B_2b_1^2 + B_4b_1^2(4b_1^2 + 12b_2^2 + 12b_3^2) + \cdots]\cos 2\psi_1 + [2B_2b_3b_1 + B_4b_3b_1(12b_1^2 + 24b_2^2 + 12b_3^2) + \cdots]\cos(\psi_3 - \psi_1) \\ &+ [6B_3b_1b_2b_3 + \cdots]\cos(\psi_3 - \psi_2 + \psi_1) + \cdots\} \end{aligned} \tag{37}$$

while for the component near ω_3

$$Q_j^{(3)} = (DVW/2)\{[B_1b_3 + B_3b_3(6b_1^2 + 6b_2^2 + 3b_3^2) + B_5b_3(10b_3^4 + 30b_1^4 + 30b_2^4 + 60b_3^2b_1^2 + 60b_2^2b_3^2 + 120b_1^2b_2^2) + \cdots]\cos\psi_3 + [B_3b_1^3 + \cdots]\cos 3\psi_1 + [2B_2b_1b_2 + \cdots]\cos(\psi_1 + \psi_2) + [3B_3b_1b_2^2 + \cdots]\cos(2\psi_2 - \psi_1) + \cdots\}. \quad (38)$$

Consideration of (36), (37), and (38) shows that several components, which may be of slightly differing frequencies, contribute to each term. These produce, in general, beat-like effects and a raucous sound. This may occur in practice, but generally pipes are voiced to produce a smooth steady speech in which all the components are locked together into harmonic relationship

$$\omega_n = n\omega_1 \quad (39)$$

the fundamental frequency ω_1 being determined in a complex way by the coupled pipe-mode equations.

V. PIPE-MODE EQUATIONS

The basic oscillation equation (21), which expresses the coupling between the nonlinear jet system and the pipe modes can be rewritten as

$$\dot{x} = Y_s HQ_j/S_p = Y_s F(t)/S_p, \quad (40)$$

where $Y_s = Z_s^{-1}$ represents the admittance function for the pipe as viewed from outside its mouth. Since this is a resonant system, Y_s has a number of poles in the frequency plane and, since the pipe itself is passive and necessarily slightly dissipative, these poles do not lie on the real axis but at points $\omega = n_i - \frac{1}{2}j\kappa_i$ in a notation in which time variation is represented by $\exp(j\omega t)$. If the quantities κ_i are small in comparison with their associated n_i, as is in fact the case for the lower resonances of an organ pipe for which $\kappa_i/n_i < 0.03$, the system exhibits sharp resonances with 3 dB width κ_i at frequencies n_i which are maxima for the admittance $Y_s(\omega)$ on the real axis. To a good approximation we can then write (40) as

$$\dot{x} = \sum_i (Y_s HQ_j/S_p)_{\omega\simeq n_i} = \sum_i (Y_s F/S_p)_{\omega\simeq n_i} \quad (41)$$

and, remembering (1), the individual quasi-Fourier components of this equation can be treated separately, except for the coupling implied by the nonlinear nature of the jet flow Q_j.

For the speech of an open flue pipe, which is our present concern, the resonances are nearly, but not quite, in harmonic relationship, $n_i \simeq in_1$, and (41) shows that the acoustic output should contain components at frequencies which are close to n_i, the admittance maxima. To examine this more closely and to elucidate the time variation we must write down the differential equations for the individual modes of (41). For the ith mode we can write

$$\ddot{x}_i + \kappa_i\dot{x}_i + n_i^2 x_i = \eta_i F_i(t), \quad (42)$$

where $F_i(t)$ is the pressure component with frequency near n_i derived from Eqs. (36), (37), and (38) for $i = 1, 2, 3$, and η_i is a constant such that κ_i/η_i is the specific acoustic impedance at the pipe mouth at the resonance $\omega = n_i$.

These quantities κ_i and η_i are in principle calculable from the geometry of the pipe and the mechanical and thermal properties of air.[21,22,24] Thus for the case of a pipe with radius r much less than the wavelength involved, when viscous and thermal losses are included[24] we find the approximate results

$$\kappa_i \simeq r^2\omega_i^2/Lc + 5\times 10^{-5} c\omega^{1/2}/r \quad (43)$$

$$\eta_i = \eta \simeq 2S_m/\rho L S_p, \quad (44)$$

where L is the pipe length, ρ the density of air, and c the velocity of sound. The approximation of dissipation mechanisms other than radiation, however, and the restriction to large wavelengths, limits the usefulness of these results. The more complex expressions given by Benade[22] are much more generally applicable.

An alternative approach is to measure the resonance curve of the pipe, which yields κ_i directly as the full width in radians per second between 3 dB points for the ith resonance. A measurement of the ratio between mouth pressure and acoustic velocity in the pipe similarly gives κ_i/η_i and hence, η_i. Such measurements are quite straightforward to perform, the acoustic pressure at the mouth at frequency ω being maintained constant and the acoustic velocity in the pipe determined from measurement of the radiation from its open end.

VI. SOLUTION OF EQUATIONS

The set of Eqs. (42), knowing that $F(t)$ is a nonlinear function of all the $\dot{x}_j$, as given by (21) and (31), can be recognized as a set of coupled equations of Van der Pol type,[14] and their approximate solution can be obtained by the method of slowly varying parameters.[14,15] Because we have considered this explicitly before,[6] we quote only the final results (see also Appendix).

Recalling from (1) the form

$$x_i = a_i \sin(\omega_i t + \beta_i) \quad (45)$$

we find the average values

$$\langle \dot{a}_i \rangle = (\eta_i/\omega_i)\langle F_i \cos(\omega_i t + \beta_i)\rangle - \tfrac{1}{2}\kappa_i a_i \quad (46)$$

$$\langle \dot{\beta}_i \rangle = -(\eta_i/a_i\omega_i)\langle F_i \sin(\omega_i t + \beta_i)\rangle + (n_i^2 - \omega_i^2)/2\omega_i, \quad (47)$$

where, on the right-hand side, F_i means the component of $F(t)$ with frequency near ω_i and $\langle\,\rangle$ implies an average in the sense that only the components with frequencies much less than ω_1 are retained.

The Eqs. (46) and (47) allow ready numerical evaluation of the velocity amplitudes $a_i\omega_i$ and frequencies $\omega_i + \dot{\beta}_i$ of the pipe oscillations as a function of time, once the initial conditions and the variation with time of the blowing pressure have been specified.

We may take, to a sufficient approximation, the initial conditions to be those generated by a sudden rise of the pressure in the pipe foot to a value P_1. The jet velocity V_0 at the flue is related to the pressure P in the pipe foot by

$$V_0=(2P/\rho)^{1/2}$$

and this decreases along the length of the jet according to (6) and (4). By Eqs. (21)–(24) then, the blowing pressure step P_1 gives rise to a driving pressure step of magnitude

$$F_1=HDVW\ \mathrm{erfc}(-\pi^{1/2}b_0/2W)\ , \qquad (49)$$

where b_0 is the static offset of the jet center inside the pipe lip and V is derived from P_1 by Eqs. like (48) and (6). We interpret erfc $(-x)$ as $1+\mathrm{erf}(x)$ if $x>0$.

This pressure step can be inserted into the mode equations (42), neglecting now any coupling between modes. Solving these equations by the method of Laplace transforms, on the assumption that $\kappa_i \ll n_i$, which is justified for practical organ pipes, we find

$$a_i(0)=\eta_i F_1^{(i)}/n_i^2 \qquad (50)$$

$$\beta_i(0)=\theta_i-\pi/2 \qquad (51)$$

$$\omega_i=n_i. \qquad (52)$$

These initial conditions, together with the Eqs. (46) and (47) would now seem to allow a complete solution of the problem. There are, however, several further considerations which we must first take up.

VII. FURTHER CONSIDERATIONS

In our formulation of the jet flow Eqs. (26)–(34) we treated the static offset b_0 of the jet center line as though it were an adjustable physical parameter, determined by the geometry of pipe flue and lip. A little thought shows that, while this may be true for the initial impact of the jet on the pipe lip, it is not true at later times.

Suppose the jet is directed with a particular offset b_0 so that the steady component of the flow into the pipe is γQ_j where $\gamma \le 1$. From the work of Coltman[2] and the analysis of Elder,[12] this flow builds up a static excess pressure $\gamma Q_j V/S_p$ in the pipe, where V is the jet velocity at the pipe lip and S_p the cross-sectional area of the pipe. This static-pressure component is not greatly dependent on the oscillatory motion of the jet once this becomes established, for the jet rapidly becomes nearly symmetrically switched so that $\gamma \simeq 0.5$.

The fraction of this static jet flow which returns to the mouth is probably about $S_m/(S_m+S_p)$ and this immediately leads to an estimate of the fractional deflection of the jet from its initial path by use of Eqs. (6) and (7) as

$$\Delta b_0 \simeq -\gamma[S_m/(S_p+S_m)](W_0/W)^{1/2} \qquad (53)$$

which is typically of order -0.1. The pipe is normally adjusted so that the jet blows somewhat outside the lip in any case ($b_0<0$), so that this argument tends to set an upper limit to normal values of b_0. We shall therefore expect b_0 initially to be negative by a moderate fraction of W and use this b_0 value for the initial conditions (50), but then immediately make b_0 more negative by an amount -0.1 for calculating the subsequent behavior. This is important since, if $b_0=0$, (34) shows that $B_2=0$ so that there is no second-order coupling between modes. In practice this would reduce acoustic stability.

One further difficulty arises from the fact that, in normal pipe speech, the jet may be fully switched across the pipe lip so that the b_i are large compared with unity. The expressions (36)–(38) for the components of the jet flow then require an inconveniently large number of terms to give a good approximation. A simple artifice allows us to overcome his difficulty.

Consider, first, the expansion given explicitly in (36). The convergence difficulty referred to above occurs for each of the expressions in square brackets [ ] since they all diverge as the b_i increase, instead of approaching some limiting value. This difficulty arises from the truncation of the error-function expansion (27) after a finite number of terms $n=N$ and can be avoided if we replace the rigorous expansion by an approximant

$$\mathrm{erf}\ z=\sum_{n=0}^{\infty} a_n z^n \simeq \sum_{n=0}^{N} a_n z^n(1+|a_N z^N|)^{-1} \qquad (54)$$

provided that, because of the monotomic nature of the error function, the sign of a_N is positive. This is the case for $N=1, 5, 9, \cdots$ so the obvious point at which to truncate our series is after the fifth term.

Applying this technique to the expansion Q_j given in (31), we see that we should replace Q_j by the approximant

$$Q_j[1+\langle|B_5(b_0)\sum_{ijklm} b_i b_j b_k b_l b_m \cos(\psi_i \pm \psi_j \pm \psi_k \pm \psi_l \pm \psi_m)|\rangle]^{-1}\ , \qquad (55)$$

where Q_j is now truncated after fifth-order terms and, as in (46) and (47), $\langle\ \rangle$ implies an average which retains only quantities varying slowly compared with n_1. The justification for performing the average in this way is that the neglected high-frequency terms would, if transferred into the numerator, lead only to terms of order equal to or greater than N, so that they can reasonably be neglected.

The major contribution to the denominator of (55) now comes, not from those few terms for which $i\pm j \pm k\pm l\pm m=0$, but rather from the sum of the constant terms produced from each oscillatory term by the action of taking its absolute value. It is not possible to find an exact value for this result because the oscillations are generally highly correlated, but we shall not make a very large error if we replace (55) by

$$Q_j\left[1+\left|(32/\pi)B_5(b_0)\left(\sum_i b_i\right)^5\right|\right]^{-1} \qquad (56)$$

the factor $32/\pi$ arising from consideration of the number of permutations occurring in (55) and the average value $2/\pi$ contributed by each. It is unlikely that (56) will be in error by more than about a factor 2, which is not very significant, for large values of the b_i and of course it becomes accurate for small b_i. We shall therefore use this form of approximant for Q_j and its components $Q_j^{(i)}$ in our numerical development.

From (46), and one of (36), (37), or (38) we see that the only possible steady-state solution (in which the amplitudes a_i are constant) is one achieved by a locking of the frequencies $\Omega_i \equiv \omega_i + \dot{\beta}_i$ into strict harmonic relationship so that $\Omega_i = i\Omega_1$. That such frequency locking occurs in normal organ pipes is readily verified from the steady shape of their output waveform. It is, however, possible to envisage a condition in which this locking does not occur but several modes are excited separately because of appropriate phase relations along the jet. The individual amplitudes a_i will then vary at relatively low frequencies $\Omega_i \pm n\Omega_j \pm m\Omega_k$ such that $i \pm nj \pm mk = 0$. The raucous noise so produced (known as "burbling") is familiar to pipe voicers and recorder makers and has been reported before in the literature.[23] It occurs most commonly in a marginally overblown situation when either modes 1 and 2 or else modes 2 and 3 are simultaneously strongly excited by the jet.

Finally we note that all modes are closely coupled through the nonlinearity of the jet interaction, and the fundamental frequency of any locked oscillation regime (in the sense discussed above) is determined by the positions of all the resonances and by the phase shifts along the jet at all the harmonics of this fundamental. This is the same conclusion as reached by Benade and Gans.[3]

VIII. NUMERICAL CALCULATIONS

Examination of the equations of our development shows that, with the exception of the viscous loss term in (43), they all scale precisely provided that all pipe dimensions are changed by a factor σ, the blowing pressure is maintained constant and time is also scaled by a factor σ. This means that frequencies are scaled by σ^{-1}, displacement amplitudes by σ, and velocity amplitudes are unchanged. The intensity (energy flux density) at a given distance from the pipe scales as σ^2 and we can overcome the difficulty of Eq. (43) by scaling κ_i by σ^{-1} so that the quality factors of the resonances remain fixed. These convenient rules mean that we need only carry out our calculations for one pipe length (or one fundamental resonance frequency). In practice we might expect of course that the scaling will not be exact, since the rigorous equations for jet motion may not scale as simply as the approximate treatments suggest.

Specific numerical calculations are now readily carried out, and integration of the equations is a rapid process on a computer of medium speed. For comparison with experiment we have chosen to compute not the acoustic velocity components v_i in the pipe, but rather the sound pressure levels associated with these components at an axial distance R from the open end of the pipe. The rms acoustic pressure at this point produced by a pipe mode with displacement amplitude a_i and frequency ω_i is

$$p_i(R) \simeq \rho\omega_i^2 a_i S_p / 2^{1/2} 4\pi R, \tag{57}$$

where ρ is the density of air and S_p the cross-sectional area of the pipe. The pressure amplitude (57) is just half that which would be produced by a flanged pipe, an approximation adequate for our present purpose, and applies only for R much greater than the pipe radius. We neglect the approximately equal radiation from the pipe mouth since in our experimental arrangement this is shielded from the measurement microphone. The sound pressure level L_i of this ith component, relative to $p_0 = 20\ \mu\text{Pa}$, at axial distance R is then given by

$$L_i = 20 \log_{10}[p_i(R)/p_0] \tag{58}$$

The assumed pipe parameters, chosen to agree with those of an adjustable experimental pipe which we describe in the next section, are shown in Table I. Several of these parameters can be varied independently and only a selection of the possible calculated results is shown in Figs. 1(a)–4(a). In each case the steady-state sound pressure level associated with each pipe mode at a point on the axis distant 1 m from the open pipe end is calculated, the blowing pressure being varied and the other parameters held fixed at the values shown in Table I, unless otherwise stated.

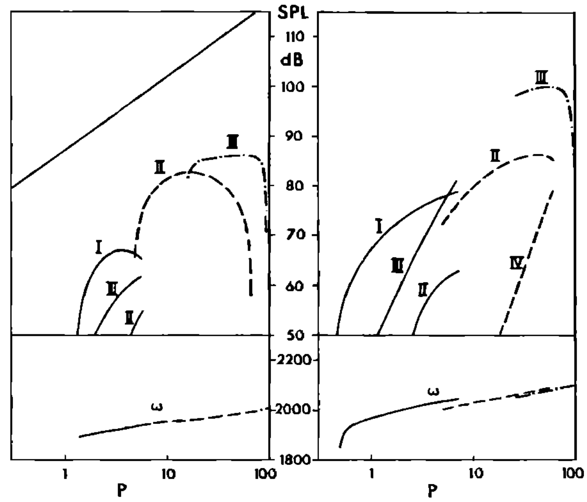

FIG. 1. (a) Calculated and (b) measured sound pressure levels, at a distance of 1 m from the open end, of the spectral components radiated from an organ pipe, as a function of the blowing pressure P in centimeters of water gauge (1 cm H_2O $\simeq$ 1 mbar = 100 Pa). The numbers I, II, III, IV are the normal mode numbers of the pipe and designate the approximate integral multiple of the lowest possible pipe frequency. Mode IV has been measured but not calculated. The normal (N) regime is shown with solid curves; underblown (U) regimes lie to the left and overblown (O) regimes to the right, each possible regime being given a distinctive line form. The angular frequency ω (in radians per second) has been divided by the mode number in each case. Hatched portions of curves indicate instability or inharmonicity. The oblique line at the top of (a) shows the maximum possible sound pressure level for 100% efficiency. In this figure the cut-up l is 5 mm, the flue width W_0 is 0.25 mm, and mouth width D is 4 cm.

TABLE I. Physical parameters adopted for calculation.

Pipe mode frequencies	$n_1 = 2020$	$n_2 = 4120$	$n_3 = 6230$ rad sec^{-1}
Resonance widths	$\kappa_1 = 40$	$\kappa_2 = 60$	κ_3 150 rad sec^{-1}
Pipe cross section	S_p 16 cm^2		
Pipe mouth width	$D = 4$ cm		
Flue width	$W_0 = 0.25$, 1.0 mm		
Mouth cut-up	l 5, 10, 15 mm		
Jet divergence	ϕ 5°		
Jet offset	$b_0 = -0.4$		
Jet sensitivity	$\alpha = 0.2$		

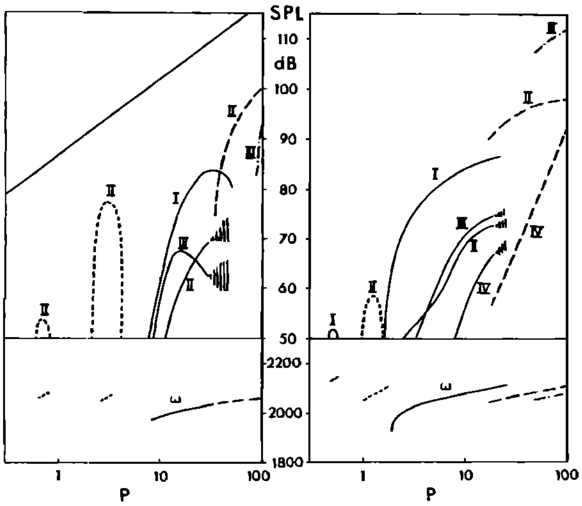

FIG. 2. As for Fig. 1 but with cut-up $l = 10$ mm, flue width $W_0 = 0.25$ mm.

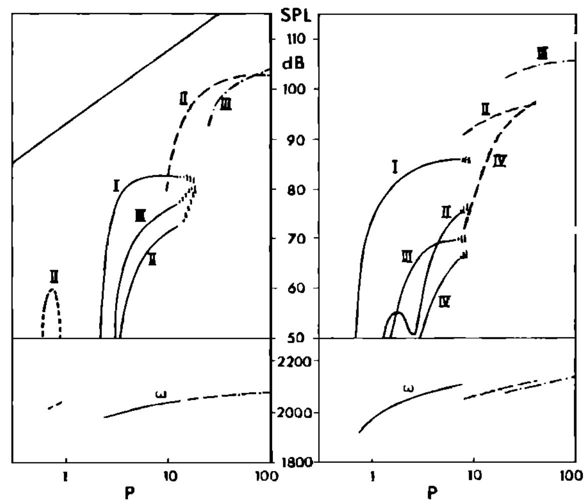

FIG. 4. As for Fig. 1 but with cut-up $l = 10$ mm, flue width $W_0 = 1.0$ mm.

Also shown in these figures is a line derived from the power input to the jet, and hence representing the maximum possible power output, as a function of the blowing pressure P. This input power Π is simply the blowing pressure multiplied by the volume flow through the flue, which has mouth width D and width W_0, so that

$$\Pi = PDW_0V_0. \tag{59}$$

The flow velocity V_0 in the flue is given by Bernoulii's equation (48) so that

$$\Pi = (2/\rho)^{1/2} D W_0 P^{3/2}, \tag{60}$$

where ρ is the density of air. This steady input power, after conversion to acoustic power in the pipe, is radiated nearly equally from the open end and the mouth. Our measurement microphone, however, was shielded from mouth radiation so that the measured radiated power has a maximum possible value of $\Pi/2$.

Sound radiation from the pipe end, which is much less than one wavelength in dimension, is nearly isotropic, so that the maximum possible acoustic intensity (or energy-flux density) at a distance R from the open end is

$$I_{max} = \Pi/8\pi R^2. \tag{61}$$

Because of the anechoic conditions of the measurement, the sound field is nearly a spherical wave so that sound pressure at a distance R is simply related to intensity at that point by

$$I = p^2/\rho c. \tag{62}$$

If the air density and speed of sound are assumed to be those for an ordinary room temperature and barometric pressure, with distances in meters and pressure P in pascals, a convenient relation results for the maximum sound pressure level, in decibels,

$$L_{max} \simeq 107 + 10 \log_{10}(DW_0/R^2) + 15 \log_{10} P \tag{63}$$

in dB relative to 20 μPa. It is this relation that is shown in the figures, except there the unit of P is approximately 100 Pa.

The theory contains only three unknown parameters, all related to the jet behavior: the gain coefficient β of (8), the limiting deflection up to which the exponential growth law applies, and the rate of growth for the subsequent linear growth. Variation of the last two of these over reasonable ranges has very little effect on the calculated results. Variation of α has little effect on the calculated behavior in the normal or overblown regimes but does markedly affect the underblown regime. The value α 0.2 adopted for calculation shows moderate development of the underblown regime while for $\alpha < 0.1$ it is suppressed entirely. The limit for exponential growth was taken equal to the spread width W of the jet at the pipe lip, and the subsequent linear growth to give an envelope with a semiangle of about 15°

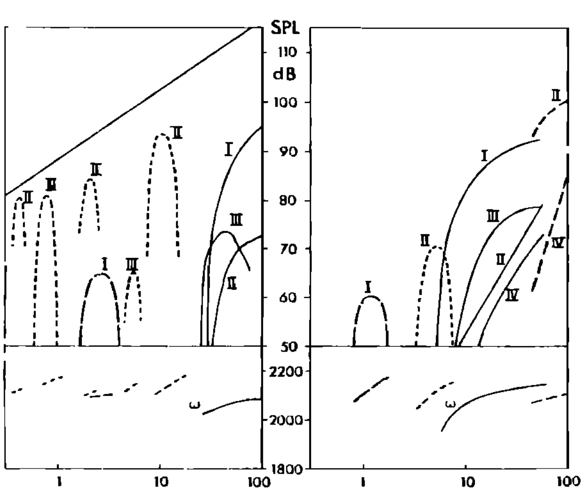

FIG. 3. As for Fig. 1 but with cut-up l 15 mm, flue width W_0 -0.25 nm.

IX. EXPERIMENTS

For the purposes of this study a flue pipe with adjustable geometry was constructed. This is shown in section in Fig. 5 with the sense of the adjustments indicated. The pipe was of about normal flute pipe scale, having inside dimensions 4×4 cm and length about 46

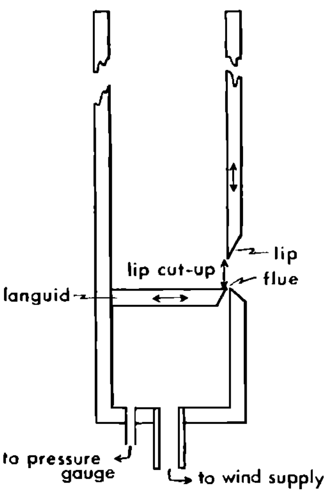

FIG. 5. Experimental organ pipe with the main components and the sense of their adjustments shown.

cm so that it sounded a fundamental near 330 Hz (about E_4). The languid was of the inverted type, as shown, with a bevel angle of about 60°, and entire construction was in perspex 6 mm thick.

The pipe was adjusted for normal speech with a cut-up near 1 cm and the form of the air jet was then examined by probing it with a fine capillary tube connected to a pressure sensor. Plots of this pitot-tube pressure (proportional to V^2) for traverses taken at various distances from the flue and for two different flue widths are shown in Fig. 6. It is clear that the transverse pressure distribution (and hence also the velocity distribution) is bellshaped, in good accord with our assumption (4). It is also evident that the jet spreads and slows down approximately in accord with (5) and (6), and the derived spreading angle is about $\phi \simeq 5°$. Checks show this angle to be almost independent of flue width. Finally we note that the jet center line is directed well outside the pipe lip ($a_0 < 0$) as we have assumed. For this particular adjustment of the pipe we find, from Fig. 6 and the definition (29), that $b_0 \simeq -0.4$ in both cases shown.

The resonance frequencies n_i and resonance widths κ_i for the pipe were determined for a lip cut of 1 cm by driving the mouth with a swept-frequency wave of constant sound pressure level and measuring the sound pressure radiated from the open end. These measurements were performed in an anechoic chamber, the mouth end of the pipe being further enclosed in an isolating box with a loudspeaker driver and control microphone. Table I shows the measured resonance frequencies n_i and 3-dB widths κ_i for the pipe. The inharmonicity is appreciable, as expected, and the κ_i values agree moderately well with those suggested by (43).

Finally the acoustic spectrum radiated by the blown pipe under various conditions was measured by sounding the pipe in an anechoic chamber, sampling the radiated sound with a calibrated pressure microphone placed on the pipe axis at a distance of 1 m from the open end and shielded from the mouth radiation by a baffle, and analyzing the resultant signal with a narrow-bandwidth wave analyzer.

The result of these measurements are shown in Fig. 1(b)–4(b) alongside the relevant calculated curves. The measurements and calculations are all given on an absolute basis and no adjustable parameters are involved.

X. DISCUSSION

A comparison of the theoretical and experimental curves in Figs. 1–4 immediately shows good semiquantitative agreement for the normal (N) and overblown (O) regimes, but there is considerable disagreement for the underblown (U) regime. Disregarding this latter for the moment, we note the following features which are common to both theoretical and experimental results.

(i) Increased blowing pressure leads from the normal N_I regime to the second mode O_{II} and the third mode O_{III} regimes. Metastable overlap of these regimes occurs.

(ii) The transition pressures between N_I, O_{II}, and O_{III} regimes increase as the cut-up of the pipe lip is increased.

(iii) The maximum SPL attainable in a given regime increases as the cut-up is increased.

(iv) The sounding frequency in each regime increases as blowing pressure is increased.

(v) A wide flue gives higher SPL and lower overblowing transition pressures than does a narrow flue.

(vi) The relative strength of upper partials in the N regime increases with increasing blowing pressure.

(vii) The maximum efficiency with which the blowing power of the jet is converted to acoustic radiation in the N and O regimes is of order 1%.

The existence of two or more stable oscillation modes in particular pressure ranges is expected from experiment and predicted by the theory. The particular mode

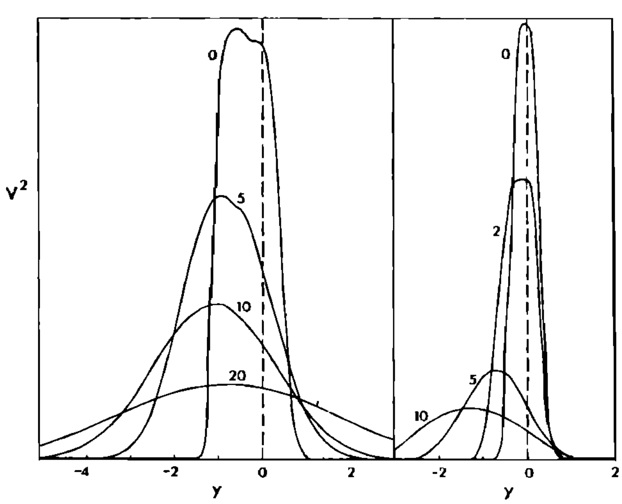

FIG. 6. Pitot-tube pressure, proportional to squared jet velocity V^2, as a function of transverse displacement y (mm) relative to the pipe lip for a jet emerging from a flue of width W_0 equal to (a) 1.00 and (b) 0.25 mm. The parameter on each curve gives the downstream displacement x (mm) away from the flue exit. Units for V^2 are not specified.

excited depends on the initial conditions and on the time variation of pressure. Thus in the overblown region O_{II} we can find either a pure second-mode oscillation, or a similar oscillation accompanied by low-level uncoupled oscillations of both modes I and II or, in the overlap region of N_I and O_{II}, a coupled mode in which the fundamental predominates. In practice, small adjustments of lip and flue influence these transitions and there is similar sensitivity to the precise assumptions made about jet behavior in the theory.

There are, however, several points of discrepancy between theory and experiment. The most serious are as follows:

(i) The observed low-pressure limit of the N regime is about a factor of 3 lower than calculated, and the cut off more gradual.

(ii) The measured jump in sounding frequency on transitions between regimes is significantly less than the required harmonic ratio, while in the calculations no such effect is shown.

(iii) The calculated sound pressure produced in the U regime is much greater than that observed, particularly for the case of a large lip cut up, and some U modes are calculated which do not occur in practice.

The origin of these defects in the treatment is easy to find and lies primarily with the Eqs. (7) and (8) for the jet behavior. Apparently (7), while satisfactory at higher frequencies and pressures, predicts too low a disturbance velocity for the fundamental at very low blowing pressures. Our two earlier discussions,[5,6] both based on a nondispersive jet behavior, were at least free from defect (iii), though the pressure range for the N regime was probably still too restricted.[5]

We have already mentioned that the behavior in the U regime is strongly influenced by the jet amplification parameter α. A more or less satisfactory suppression of this regime can be achieved by decreasing α to 0.1 or less, but this probably hides the difficulty rather than solving it. The U regime corresponds to a situation in which the jet length is approximately $3\lambda/2$, $5\lambda/2$, ... for the particular mode involved, while the N and O regimes have jet length near $\lambda/2$, where λ is the disturbance wavelength on the jet. The discrepancy probably arises because the simple theory of the jet supposes its displacement to remain sinusoidal whatever its amplitude, whereas in a real situation it breaks up into a double set of vortices once the displacement amplitude becomes comparable with the disturbance wavelength.[8] Clearly the interaction of such a vortex street with the pipe lip will be very different from that of a simple displaced jet.

Despite these shortcomings it is clear that the theory does go a long way towards describing the details of the sound production mechanism in flue pipes and the effect of pipe scale and of the various possible voicing adjustments[25] on the acoustic output.

XI. TRANSIENT BEHAVIOR

The nature of the initial acoustic transient when air pressure is applied to the pipe foot was discussed in an earlier publication[6] and our present development adds little to that treatment. Furthermore the particular pipe constructed for the experiments showed no very interesting behavior and, in particular, did not respond with a 'chiff' when set to a low lip cut up as in baroque voicing. This particular form of chiff, developed largely with slider chests which allow a rather slow build up of wind pressure, is distinct from the overblown transient produced by plosive application of wind pressure which was discussed before.[6] The baroque chiff seems to be closely related to pipe behavior in the U regime and cannot therefore be well calculated until this region of jet behavior is better understood.

The transition from normal to overblown regimes is not very well reproduced by the present theory—not so well, in fact, as in our simpler treatment.[6] The lower modes tend to continue sounding in nonharmonic relation below the dominant upper mode unless some steps are taken to remove them by starting the pipe sound as a nearly pure upper mode. It seems likely that there is an additional nonlinearity in jet behavior which aids the transition once a dominant mode has become established.

XII. CONCLUSIONS

This attempt to formulate a complete description of flue pipe behavior has proved only partly successful but has served to show up the areas of most critical inadequacy in our knowledge. It is not surprising to find that these are associated primarily with the behavior of the acoustically disturbed jet, which is the most complex component of the system.

More specifically we find that the descriptions of wave motion on the jet given by Rayleigh and by Savic appear to be inadequate, particularly for low stream velocites and low frequencies, for the particular magnitudes of other physical parameters encountered in organ pipes. We further note that the jet propagation is almost certainly very nonlinear once the disturbance amplitude exceeds the jet thickness and even more so when it approaches the disturbance wavelength. We have had to neglect or at least approximate both of these effects, but it seems that they may be important for a full understanding of mode stabilization in the overblown region and of the whole behavior in the underblown region.

APPENDIX

Consider the equation

$$\ddot{x}_i + \kappa_i \dot{x}_i + \eta_i^2 x_i = \eta_i F(t), \tag{A1}$$

where $F(t)$ is a nonlinear function of all the $\dot{x}_j$. We expect for x_i a solution of the form

$$x_i = a_i \sin(\omega_i t + \beta_i), \tag{A2}$$

where a_i and β_i are slowly varying functions of time. To restrict the freedom allowed by this specification, we also require that

$$\dot{x}_i = a_i \omega_i \cos(\omega_i t + \beta_i) \tag{A3}$$

which imposes the condition

$$\dot{a}_i \sin(\omega_i t + \beta_i) + a_i \dot{\beta}_i \cos(\omega_i t + \beta_i) = 0 \quad . \tag{A4}$$

Substitution of (A2) and (A3) into (A1), with the use of (A4), then gives

$$\dot{a}_i = \frac{\eta_i F(t) - \kappa_i \dot{x}_i}{\omega_i}\cos(\omega_i t + \beta_i) - \frac{a_i(\eta_i^2 - \omega_i^2)}{\omega_i}\sin(\omega_i t + \beta_i)\cos(\omega_i t + \beta_i) \qquad (A5)$$

$$\dot{\beta}_i = -\frac{\eta_i F(t) - \kappa_i \dot{x}_i}{a_i \omega_i}\sin(\omega_i t + \beta_i) + \frac{\eta_i^2 - \omega_i^2}{\omega_i}\sin^2(\omega_i t + \beta_i). \qquad (A6)$$

If we neglect the small variation of a_i and β_i over a single period $2\pi/\omega_i$ and average (A5) and (A6) over such a period, again neglecting all but the slowly varying terms, then we find the average values

$$\langle \dot{a}_i \rangle = (\eta_i/\omega_i)\langle F_i \cos(\omega_i t + \beta_i)\rangle - \tfrac{1}{2}\kappa_i a_i \qquad (A7)$$

$$\langle \dot{\beta}_i \rangle = -(\eta_i/a_i\omega_i)\langle F_i \sin(\omega_i t + \beta_i)\rangle + (\eta_i^2 - \omega_i^2)/2\omega_i, \qquad (A8)$$

where F_i implies the component of $F(t)$ varying with frequency near ω_i.

*This work is part of a program of study of the acoustics of traditional musical instruments that is supported by the Australian Research Grants Committee.

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