Mode locking in nonlinearly excited inharmonic musical oscillators

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Many musical instruments consist essentially of resonant systems having only approximately harmonic normal modes and excited by a force that depends nonlinearly on the velocities associated with those modes. An approximate condition is derived for the resulting sound spectrum to consist of components rigorously locked into harmonic relationship. Such mode locking is favored by nearly harmonic normal mode frequencies, by large mode amplitudes, and by large nonlinearity in the driving force.

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INTRODUCTION

Musical instruments capable of producing a sustained tone (e.g., wind instruments and bowed-string instruments) consist essentially of one or more resonant systems (air columns, cavities, strings) with very nearly linear acoustic behavior, excited by a nonlinear source (lips, reed, air jet, bow) with which they are coupled to produce a regenerative feedback loop. This nonlinearity is generally essential in determining the amplitude of steady-state oscillations of the resonant system, and it also has a major influence on the nature of the radiated acoustic spectrum.[1,2]

The general problem to be investigated in the present paper arises from the fact that the natural modes of any real acoustical resonant system are never in exact harmonic relationship, because of second-order effects like end corrections and string stiffness. It is common experience that nonharmonically related sounds ("multiphonics" or "burbles") can be produced on most wind instruments, though in normal tone production the overtones are accurately harmonic and locked in both phase and frequency to the fundamental. It is therefore our purpose to inquire into the features of the system that are responsible for this mode locking. The analysis will be kept as general as possible, though some specific examples will be discussed.

I. SYSTEM EQUATIONS

Suppose the resonant system possesses a set of normal modes with frequencies n_i, and let x_i be a generalized coordinate associated with the ith mode—we shall explain in a moment just how this coordinate should be interpreted in particular cases. Then, in the absence of an exciting force, these modes will obey equations of the form

$$\ddot{x}_i + \kappa_i \dot{x}_i + n_i^2 x_i = 0\,, \qquad (1)$$

where $\kappa_i > 0$ are the mode damping coefficients. We assume that all the mode equations of the resonator are linear, as shown, with the nonlinearities of the system concentrated in the driving mechanism. This is true to a very good approximation in most systems of interest.

Now suppose that the resonant system is acted on by an external force F which has no explicit time dependence of its own, but which is generated by the action of the resonator modes on a valvelike mechanism—again we give examples in a moment. Then we can write F as $F(\dot{x}_1, \dot{x}_2, \ldots)$, or as $F(\dot{x}_j)$ for convenience. The reason for choosing $\dot{x}_j$ rather than x_j will again be apparent in a moment, but we note that in any case $\dot{x}_j$ and x_j are simply related through (1). Under the action of this driving force the ith mode now obeys an equation of the form

$$\ddot{x}_i + \kappa_i \dot{x}_i + n_i^2 x_i = \lambda_i F(\dot{x}_j)\,, \qquad (2)$$

where the λ_i are coupling coefficients which we shall take as real, any complex phase shifts being included in the form of F. Thus, for example, the $\dot{x}_i$ on the left side of (2) refer to time t, while the $\dot{x}_j$ in F may be evaluated at some earlier time $t - \delta_i$.

To put some flesh on this skeleton we note that in ordinary musical instruments we commonly find one of two possible generators. The first is the velocity-controlled generator used in the bowed-string instruments or, in different physical realization, in air-jet driven instruments of the flute family. The second is the pressure-controlled reed or lip valve found in woodwind and brass instruments.

In the case of bowed strings, the frictional force F exerted by the bow on the string is a simple but highly nonlinear function of the relative velocity between the bow and the string at the bowing point. In this case x_i is the displacement coordinate for the ith string mode, and λ_i is large if the driving point is near to an antinode of string velocity $\dot{x}_i$.

In the flute family an air jet is deflected into and out of the pipe by the action of the acoustic volume flow out of the pipe mouth. Because the interaction between the jet and this flow takes place at the aperture from which the jet emerges, while the jet interacts with the pipe modes at the pipe lip, there is a phase delay δ_i, corresponding to wave propagation time along the jet together with any interaction phase delays, built into the $\dot{x}_j$ of F. In order that $\dot{x}_i$ imply an acoustic flow velocity, x_i must represent acoustic volume displacement and F should be expressed as an effective driving pressure. The coupling coefficient λ_i will be large if the driving point is close to an antinode of the acoustic velocity $\dot{x}_i$.

In woodwind and brass instruments the action of the generator, which controls acoustic flow into the system, is governed by the difference between the blowing pressure and the acoustic pressure inside the instrument mouthpiece. To use our Eq. (2) we must therefore choose $\dot{x}_i$ to represent acoustic pressure, so that x_i is the integral of pressure with time, and F is expressed as an acoustic volume flow. Again there may be a phase shift involved in F because of the resonance properties of the reed system. λ_i is again large if the driving point is close to an antinode of $\dot{x}_i$, but here this implies a pressure antinode.

By making these identifications of the generalized coordinates x_i, we see that the pressure-controlled system is essentially the dual of the velocity-controlled system. The former sounds at frequencies which are close to impedance maxima at the driving point, and the latter close to admittance maxima, as is well known.[1] We must be careful, however, not to carry the idea of duality too far, since the form of the nonlinear function $F(\dot{x}_j)$ may be very different in different cases. With this reservation we can now return to our formal development.

To solve this set of equations we use a method discussed by Bogoliubov and Mitropolsky.[3] Assume a set of quasi-steady-state solutions of the form

$$x_i = a_i \sin(\omega_i t + \phi_i) , \tag{3}$$

where both a_i and ϕ_i are slowly varying functions of time. To restrict the freedom allowed by this specification, we also require that

$$\dot{x}_i = a_i\omega_i \cos(\omega_i t + \phi_i) , \tag{4}$$

which implies the relation

$$\dot{a}_i \sin(\omega_i t + \phi_i) + a_i\dot{\phi}_i \cos(\omega_i t + \phi_i) = 0 . \tag{5}$$

If we substitute (3) and (4) into (2), make use of (5), and then neglect terms not varying slowly compared with ω_i, we find

$$\langle \dot{a}_i \rangle = (\lambda_i/\omega_i)\langle F(\dot{x}_j) \cos(\omega_i t + \phi_i)\rangle - \tfrac{1}{2}\kappa_i a_i , \tag{6}$$

$$\langle \dot{\phi}_i \rangle = -(\lambda_i/a_i\omega_i)\langle F(\dot{x}_j) \sin(\omega_i t + \phi_i)\rangle + (n_i^2 - \omega_i^2)/2\omega_i . \tag{7}$$

The brackets $\langle\ \rangle$ imply that only slowly varying terms are to be retained.

This approach has been applied elsewhere[4,5] to treat the transient and steady state speech of organ flue pipes. It is our purpose here to see what statements of a more general nature can be made. To this end we suppose that $F(\dot{x}_j)$ can be expanded as a generalized power series

$$F(\dot{x}_j) = \sum_n \sum_{j_1 \ldots j_n} c^{(n)}_{j_1 j_2} \dot{x}_{j_1} \dot{x}_{j_2} \ldots \dot{x}_{j_n} , \tag{8}$$

where the coefficients c have the operator property of introducing time shifts δ_j or corresponding phase shifts into the x_j with which they are associated. From the expression (4) for $\dot{x}_i$ it is clear that the general term in the summation (8) can be written as

$$\dot{x}_{j_1} \ldots \dot{x}_{j_n} = \sum 2^{1-n} a_{j_1} \ldots a_{j_n} \omega_{j_1} \ldots \omega_{j_n} \times \cos[(\omega_{j_1} \pm \cdots \pm \omega_{j_n})t + \phi_{j_1} \pm \cdots \pm \phi_{j_n}] , \tag{9}$$

where the summation is over all possible choices of $j_1 \ldots j_n$. The operator action of the coefficient $c^{(n)}$ may then introduce a phase shift into the argument of the cosine in (9) that depends on the particular set of modes involved in the term. Finally the averaging implied by the brackets in (6) and (7) selects only terms for which

$$\omega_i \pm \omega_{j_1} \pm \omega_{j_2} \pm \cdots \pm \omega_{j_n} \simeq 0 , \tag{10}$$

in the sense that this sum is very much smaller than ω_i.

II. MODE LOCKING

The case of primary interest in musical acoustics is that in which the normal modes form an approximate harmonic series. For such a case we can write

$$n_i = i\omega + \Delta_i , \tag{11}$$

where ω is the fundamental frequency of the related exactly harmonic series and is not generally equal to n_1, the frequency of the lowest mode. If only the odd members of the series are present, the argument can easily be modified. In more general cases the relation (11) with $\Delta_i < \omega$ can always be preserved by introducing null modes of infinite impedance and appropriately re-numbering the real modes.

For a system that is only weakly nonlinear and that is excited by only a small force F, the solution of (6) and (7) is easily appreciated. The only important term in (9), bearing in mind the restriction (10), is $a_i\omega_i \cos(\omega_i t + \phi_i)$ and the coefficient $c^{(1)}$ may introduce a phase shift δ_i into this. From (6) the oscillation at mode i will grow provided

$$\tfrac{1}{2}\lambda_i c_i^{(1)} \cos\delta_i - \tfrac{1}{2}\kappa_i > 0 , \tag{12}$$

and from (7) the oscillation frequency of this mode is

$$\omega_i + \dot{\phi}_i = n_i + \tfrac{1}{2}\lambda_i c_i^{(1)} \sin\delta_i . \tag{13}$$

Clearly, if there is no phase shift in the excitation mechanism, so that $\delta_i = 0$, the excitation conditions are optimal (assuming $c_i^{(1)} > 0$) and the mode displays its natural frequency. If phase shifts are involved, the mode oscillation shifts from its natural frequency.

None of this, however, is sufficient either to limit the vibration amplitude of the modes or to induce harmonic locking of their frequencies. Suppose that (12) is satisfied for only one mode, then the amplitude a_i will increase until the cubic term, which is the first that can satisfy (10), becomes appreciable. Instead of (12) we then have as condition for growth

$$\tfrac{1}{2}\lambda_i \cos\delta_i (c_i^{(1)} + \tfrac{1}{4} c_{iii}^{(3)} n_i^2 a_i^2) - \tfrac{1}{2}\kappa_i > 0 . \tag{14}$$

This condition will provide a limit to the value of a_i provided $c^{(3)}$ is of opposite sign to $c^{(1)}$. If this is not the case, then limiting will be accomplished by the first of the $c^{(2n+1)}_{i\ldots i}$ that is of opposite sign to $c^{(1)}$.

To discuss mode locking, first consider a system with just two modes i and j such that

$$qj = pi , \quad qn_j \simeq pn_i , \tag{15}$$

where p and q are integers. This could be the case, for example, in a multimode system for which the growth condition (12) is satisfied only for these two modes. Considering only leading terms, as given by (9) and (10), we have from (7)

$$\langle\dot{\phi}_j\rangle = (\lambda_j/n_j a_j)\{(c_j^{(1)} a_j n_j + \tfrac{1}{4} c_{jjj}^{(3)} a_j^3 n_j^3 + \cdots)\sin\delta_j + C_{i\ldots j\ldots}^{(p,q-1)} n_i^p n_j^{q-1} a_i^p a_j^q \sin[(pn_i - qn_j)t + p(\phi_i + \delta_i) - q(\phi_j + \delta_j) + \delta_j]\}, \tag{16}$$

$$\langle\dot{\phi}_i\rangle = (\lambda_i/n_i a_i)\{(c_i^{(1)} a_i n_i + \tfrac{1}{4} c_{iii}^{(3)} a_i^3 n_i^3 + \cdots)\sin\delta_j - C_{i\ldots j\ldots}^{(p-1,q)} n_i^{p-1} n_j^q a_i^p a_j^q \sin[(pn_i - qn_j)t + p(\phi_i + \delta_i) - q(\phi_j + \delta_j) - \delta_i]\}, \tag{17}$$

where

$$C_{i\ldots j\ldots}^{(r,s)} = [(r+s)!/r!s!2^{r+s}] c_{i\ldots j\ldots}^{(r+s)} . \tag{18}$$

In (16) and (17) δ_i and δ_j are the phase shifts produced in the excitation mechanism at frequencies n_i and n_j, respectively. We have also set $\omega_i = n_i$ and $\omega_j = n_j$ to eliminate the last term in (7), which effectively produces this shift in $\langle\dot{\phi}\rangle$ provided $\omega_i \simeq n_i$ and $\omega_j \simeq n_j$.

The condition for mode locking is now that there exist some set of phases ϕ_i and ϕ_j, and some common frequency base ω such that, dropping the $\langle\ \rangle$ from $\langle\dot{\phi}\rangle$ for convenience,

$$p(n_i + \dot{\phi}_i) = q(n_j + \dot{\phi}_j) = pi\omega , \tag{19}$$

which can be written

$$p\dot{\phi}_i - q\dot{\phi}_j = qn_j - pn_i . \tag{20}$$

Examination of (16) and (17) shows that each $\dot{\phi}$ has two parts. The first set of terms describes modification of the uncoupled sounding frequency of the mode, as in (13), because of the phase shift δ in the driving force. The second set of terms describes the locking interaction between the modes. Let us include the first part of $\dot{\phi}$ with the natural mode frequency n to give a sounded mode frequency n'. We note further that the locking interaction terms have a sinusoidal factor whose value swings between +1 and −1 until mode locking occurs. From (20), therefore, a necessary condition for mode locking is that for some θ,

$$|p\lambda_i C_{i\ldots j\ldots}^{(p,q-1)} a_i^p a_j^{q-1} n_i^p n_j^{q-2} \sin(\theta + \delta_j) + q\lambda_j C_{i\ldots j\ldots}^{(p-1,q)} a_i^{p-1} a_j^q n_i^{p-2} n_j^q \sin(\theta - \delta_i)| > |pn_i' - qn_j'| . \tag{21}$$

Provided this condition is satisfied for the mode amplitudes allowed by (14), mode locking can occur, with the phases ϕ_i and ϕ_j adjusting to give exactly the right conditions in (16) and (17). Once mode frequencies are momentarily harmonic, the subsequent amplitude and phase adjustments for the steady state occur gently and do not disturb the frequency locking. The result (21), which could of course be refined by addition of higher terms, is the basis of our further discussion.

III. DISCUSSION

To see the conclusions arising from (21) we should first note that for most ordinary nonlinear systems, the expansion coefficients $c^{(n)}$ in (8) will decrease rapidly in magnitude as the value of n increases. This is physically necessary if the force F is to have a reasonable behavior. In many cases of physical interest, therefore, it will be only n values of 1, 2, or 3 that are significant. This in turn means that only values of 1, 2, and 3 will generally enter for p and q in (21).

From (21) we can immediately specify, therefore, the conditions required for mode locking:

(a) The integers p and q specifying the relation between the modes must be small ($p+q<4$);

(b) The sounding frequencies n_j' and n_j' (and thus also the natural frequencies n_i and n_j) of the modes involved must be nearly harmonically related;

(c) The coupling coefficients λ_i and λ_j between the two modes and the driving force must be large;

(d) The driving force must be highly nonlinear, as expressed by the magnitude of $c^{(2)}$ and $c^{(3)}$ relative to $c^{(1)}$;

(e) The mode amplitudes a_i and a_j must be large.

Conversely, if the instrument is to produce a multiphonic effect with nonharmonically related tone components, it is desirable for the resonator to possess at least two resonances capable of oscillation whose frequencies differ by less than a factor of 2. This situation can be achieved by a random opening of finger holes on many woodwind instruments. The precise mechanism by which nonlinearity is reduced while maintaining adequately large coupling coefficients λ_i and λ_j to cause the two modes to sound depends in detail on the instrument involved. For flutelike instruments the player generally uses a wide air jet to reduce nonlinearity, and a carefully controlled blowing pressure and jet length to adjust δ_i and δ_j so as to satisfy (12) for both modes. The physical adjustments in reed and lip-driven instruments are somewhat less clear.

Another situation that sometimes arises with strong blowing in organ pipes is that in which modes 2 and 3 satisfy the sounding condition (12) but mode 1 does not. The nonlinear coefficient linking modes 2 and 3 has $p+q=5$, and so is relatively small even in a normal pipe. The modes may therefore remain uncoupled if they are appreciably inharmonic, and produce an unpleasant beating sound[4,5] near the frequency of mode 1.

The behavior of bowed strings is somewhat different from that of pipes because the bowing mechanism always provides a highly nonlinear force, even for very small oscillation amplitudes. In addition, the modes of a thin string are very nearly harmonic, so that condition (21) is almost always satisfied and the modes are always locked in normal playing.

An exception to these statements occurs in the case of a narrow bow applied exactly at a node of some mode i. It then follows that the corresponding λ_i and any of the $C_{i\ldots j\ldots}^{(r,s)}$ containing the subscript i are zero. Mode i is therefore not coupled in the same way as the other modes. There are, however, driving terms of frequency $i\omega \simeq n_i$ generated by coupling between the other modes and these can further couple to string mode i through their effect on string tension, giving a locked mode i of appreciable amplitude.

Many other cases could be discussed in a rather similar manner. The object of the present note has been to provide a framework for doing so.

[1]For an extensive discussion see A. H. Benade *Fundamentals of Musical Acoustics* (Oxford U.P., New York, 1976).

[2]An extensive collection of reprints relating to string instruments can be found in *Musical Acoustics*, Parts 1 and 2, edited by C. M. Hutchins *Benchmark Papers in Acoustics*, Vols. 5 and 6 (Dowden, Hutchinson & Ross, Stroudsburg, PA, 1975–1976). A similar collection relating to wind instruments is found in *Musical Acoustics–Piano and Wind Instruments*, edited by E. L. Kent, Vol. 9 in the Benchmark series (1977).

[3]N. N. Bogoliubov and Y. A. Mitropolsky, "Asymptotic Methods in the Theory of Non-Linear Oscillations" (Hindustan, New Delhi, and Gordon and Breach, New York, 1961).

[4]N. H. Fletcher, "Transients in the Speech of Organ Flue Pipes—a Theoretical Study," Acustica **34**, 224–233 (1976).

[5]N. H. Fletcher, "Sound Production by Organ Flue Pipes," J. Acoust. Soc. Am. **60**, 926–936 (1976).