Axisymmetric wave propagation on a conical shell

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Equations are derived to describe the propagation of axially symmetric waves generated at the apex of a conical shell, taking into account the coupling between longitudinal and transverse waves. A form of approximate solution is proposed, based upon the Green's function for wave propagation on a flat plate but containing several variational parameters, the number of which can be reduced to one if additional approximations are made. It is shown that the region of the cone near the vertex moves amost like a rigid body, but the motion becomes wavelike at a distance from the vertex which decreases rapidly with increasing frequency, and at large distances the propagation is essentially the same as on a flat plate. The one-parameter approximation is investigated numerically for two realistic situations, and appropriate solutions are presented and discussed.

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INTRODUCTION

The vibrations of conical shells have been studied by many people,[1–7] partly because of the intrinsic interest of the problem and partly because of the obvious practical importance of the subject in the design of loudspeakers. For this reason attention has been concentrated on evaluating eigenfrequencies and mode shapes for conical frustra either clamped or free at their truncation planes.

The present study has initially been developed towards an understanding of the propagation of axisymmetric waves generated at the apex of a thin conical shell extending to infinity. This approach, which does not appear to have been followed before, gives considerable insight into the vibrational behavior of conical shells, from both physical and mathematical viewpoints, without the complications inherent in considering reflection from boundaries.

In the interests of simplicity, we consider only the case of waves generated by an axial simple-harmonic displacement of the vertex of the cone. Torsional wave propagation could be treated rather similarly but the mathematical framework is quite different, as also is the physical context. Waves lacking circular symmetry have been excluded from consideration for the same reasons.

I. WAVE EQUATIONS

To derive the equations of motion in the simplest and most physically illuminating manner possible, consider a thin conical shell of semiangle α and define coordinates as shown in Fig. 1. Here r is the distance measured from the apex, and the motion of the conical surface has been resolved into a normal component ξ and a tangential component η.

The equations of motion for both ξ and η involve several terms which can be usefully considered separately. For a given displacement (ξ, η) we can indeed identify separate contributions from plate stiffness, from hoop stress, from material compression, and from an additional Poisson stress.

The displacement ξ, for example, gives rise to a local plate-stiffness force $-A\nabla^4\xi$, where from the usual theory

$$A = Eh^3/[12(1-\sigma^2)], \tag{1}$$

where E is the Young's modulus, σ the Poisson's ratio, and h the thickness of the cone material. The displacement (ξ, η), however, causes an expansion in the hoop of material distant r from the apex of amount $\xi \cos\alpha + \eta \sin\alpha$, and this gives rise to a hoop stress in a direction normal to the axis of magnitude

$$H = (B/r^2 \sin^2\alpha)(\xi \cos\alpha + \eta \sin\alpha), \tag{2}$$

where

$$B = Eh/(1-\sigma^2). \tag{3}$$

The component of this stress in the ξ direction is $-H\cos\alpha$.

Finally, though the effect of Poisson contraction associated with the displacement ξ is already included in the stiffness modulus A in (1), this is not true of the Poisson stress in the ξ direction arising from η, or rather from $\partial\eta/\partial r$. This effect also contributes an inward hoop stress of magnitude

$$P = \frac{\sigma B}{r \sin\alpha}\frac{\partial\eta}{\partial r}, \tag{4}$$

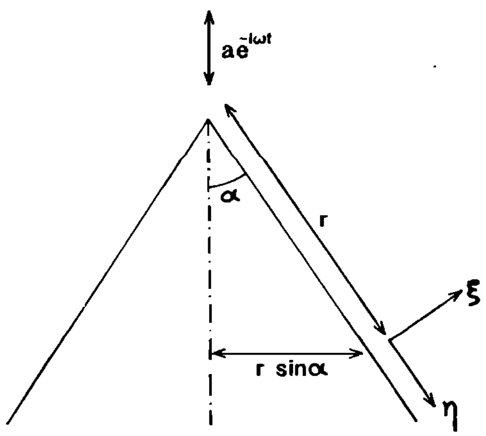

FIG. 1. Coordinate definitions for the conical shell.

and its component in the ξ direction is $-P\cos\alpha$.

Summing all these contributions, we find for the equation of motion in this approximation, when the force F is concentrated at the apex as shown in Fig. 1,

$$\rho h \frac{\partial^2 \xi}{\partial t^2} = -A\nabla_r^4 \xi - \frac{B}{r^2}\left(\xi \cot^2\alpha + \eta \cot\alpha + \sigma r \cot\alpha \frac{\partial \eta}{\partial r}\right) + \frac{F}{2\pi r}\sin\alpha\, \delta(r-0), \tag{5}$$

where ρ is the density of the material of the shell and ∇_r^4 is the radial part of ∇^4 in two dimensions, namely

$$\nabla_r^4 = [\nabla_r^2]^2 = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]^2. \tag{6}$$

The form chosen for the applied force, which acts axially at the apex and will be assumed to be sinusoidal, leads to a total force amplitude F. In the real physical situation the conical shell becomes solid within a distance of about $h\cot\alpha$ of the apex so that the precise form of the force distribution within this region is not significant.

The equation of motion in the η direction is found by an analogous argument. The ordinary compressive strain contributes a stress of $B\nabla^2\eta$ and the hoop stress a component $-H\sin\alpha$ in the η direction. The η part of the Poisson stress is included in the modulus B in (3) but the ξ part, or rather its gradient $\partial\xi/\partial r$, contributes a tangential force in the η direction of

$$T = \frac{\sigma B}{r\sin\alpha}\frac{\partial \xi}{\partial r}\cos\alpha. \tag{7}$$

The resulting equation of motion for η is

$$\rho h \frac{\partial^2 \eta}{\partial t^2} = B\nabla_r^2 \eta - \frac{B}{r^2}\left(\xi\cot\alpha + \eta - \sigma r\cot\alpha\frac{\partial\xi}{\partial r}\right) - \frac{F}{2\pi r}\cos\alpha\,\delta(r-0), \tag{8}$$

where ∇_r^2 is the radial part of ∇^2 in two dimensions, as given by (6).

The two coupled equations (5) and (8) now specify the complete vibration behavior of the system in the linearized thin-shell approximation. The arguments by which they were derived may appear to lack rigor even in this approximation, but it is comforting to note that the resulting equations are identical with those derived more carefully by Goldberg and others,[1,2] once some notational modifications and other minor algebraic corrections have been performed.[8]

It is also worth noting that in the limit $\alpha = 90°$, corresponding to a flat plate, the two equations are no longer coupled and reduce to the expected simple forms. At the other extreme, $\alpha = 0°$, the cone becomes a cylinder of radius $R = r\tan\alpha$ and, if R is taken as finite, the equations once again have their expected simple uncoupled forms.

II. APPROXIMATE SOLUTION OF THE EQUATIONS

Solution of the coupled equations (5) and (8) over the whole domain $0<r<\infty$ and with boundary conditions corresponding to outgoing waves generated at $r=0$ would appear to present difficulty because of the singularity in the coefficients of both equations at $r=0$. This problem was noted by Goldberg[1] in developing a series solution for a conical frustrum. Paradoxically it turns out that actual inclusions of the point $r=0$ in the physical problem removes the difficulty provided the cone angle α is finite.

Before examining this point, however, we observe that the asymptotic solution to the equations in the limit of large r is simple, for they then reduce to the forms

$$\rho h \frac{\partial^2 \xi}{\partial t^2} = -A\nabla_r^4 \xi \tag{9}$$

and

$$\rho h \frac{\partial^2 \eta}{\partial t^2} = B\nabla_r^2 \eta. \tag{10}$$

The solution appropriate to outgoing waves of angular frequency ω can be expressed in terms of Hankel functions $H_n = J_n + iN_n$, where J_n and N_n are, respectively, Bessel and Neumann or Weber functions, and we have assumed a time dependence of the form $\exp(-i\omega t)$. The asymptotic forms are then[9]

$$\xi \sim C_1 H_0(k_t r - \delta_1) \sim C_1 (2/\pi k_t r)^{1/2} \exp[i(k_t r - \delta_1 - \tfrac{1}{4}\pi)] \tag{11}$$

and

$$\eta \sim C_2 H_0(k_l r - \delta_2) \sim C_2 (2/\pi k_l r)^{1/2} \exp[i(k_l r - \delta_2 - \tfrac{1}{4}\pi)], \tag{12}$$

where C_1 and C_2 are amplitude coefficients, δ_1 and δ_2 are phase shifts which depend on behavior near the origin, and the transverse and longitudinal wavenumbers k_t and k_l are given by

$$k_t = (\rho h/A)^{1/4}\omega^{1/2} \tag{13}$$

$$k_l = (\rho h/B)^{1/2}\omega. \tag{14}$$

Transverse wave progagation is dispersive while longitudinal is not.

Turning now to consider behavior of the equation very close to the origin, we note that there are obvious physical difficulties with the model because of the finite thickness of the shell. These can, however, be ignored by simply assuming a shell of infinitesimal thickness and mechanical properties defined by the superficial density ρh and the elastic moduli A and B. The neglect of nonlinear terms would be serious if, as in treatment of a simple membrane, the linear approximation predicted infinite displacement at the origin. However, the coupling between ξ and η in (5) and (8), and particularly the hoop-stress terms derived from (2), save the situation by constraining the apex of the cone to move as a rigid body, while the known solution of (9) for the case of an infinite flat plate[10] has a finite amplitude and no singularity at the origin. We shall return to discuss this later, and for the present we simply note that continuity at the apex of the cone requires

$$\lim_{r\to 0}(\xi\cos\alpha + \eta\sin\alpha) = 0. \tag{15}$$

Assuming now a time variation $\exp(-i\omega t)$ and considering the open region $r>0$ in which the force is zero, it is convenient to seek solutions to (5) and (8) of the form

$$\xi = \sum_n a_n (k_t r)^n, \tag{16}$$

$$\eta = \sum_n b_n (k_t r)^n. \tag{17}$$

Substitution immediately gives the recurrence relations

$$(n+2)^2(n+4)^2 a_{n+4} + [(k_t/k_l)\cot\alpha]^2 a_{n+2} + \cot\alpha[\sigma(n+2)+1]b_{n+2} - a_n = 0 \tag{18}$$

$$(n+3)(n+1)b_{n+2} + \cot\alpha[\sigma(n+2)-1] \times (k_t/k_l)^2 a_{n+2} + b_n = 0. \tag{19}$$

Each of the functions ξ and η has two parts based, respectively, on even and odd terms. The leading terms a_0 and b_0 are determined by (15) which requires

$$a_0 = a\sin\alpha, \quad b_0 = -a\cos\alpha, \tag{20}$$

where a is the amplitude of the impressed axial motion at $r=0$. It is then necessary to choose a_2 so that the two even series asymptotically approach $C_1 J_0(k_t r - \delta_1)$ and $C_2 J_0(k_l r - \delta_2)$, respectively, convergence being the criterion rather than any knowledge of δ_1, δ_2, C_1, or C_2. The two odd series are then determined, without knowledge of the leading coefficients, by the requirement that they asymptotically approach $iC_1 N_0(k_t r - \delta_1)$ and $iC_2 N_0(k_l r - \delta_2)$, respectively.

Unfortunately, this process does not appear analytically or numerically feasible because of the instability of the series to small errors in the low-order coefficients. The reasons for this will emerge presently and lead us to an appropriate approximate solution.

Returning to the asymptotic form (9), we see that it is formally identical with the equation for the transverse vibration of a flat plate with the same physical properties as the shell material. The Greens function corresponding to outgoing waves when the plate is excited by a force $F\exp(-i\omega t)$ concentrated at the origin is[10]

$$\xi_0 = C_0[J_0(k_t r) + iN_0(k_t r) - J_0(ik_t r) - iN_0(ik_t r)] \tag{21}$$

or equivalently

$$\xi_0 = C_0\{J_0(k_t r) + i[N_0(k_t r) + (2/\pi)K_0(k_t r)]\}, \tag{21'}$$

where K_0 is a modified Bessel function and

$$C_0 = (iF/8\omega)(\rho/hA)^{1/2}\exp(-i\omega t). \tag{22}$$

In this expression, the singularity in $iN_0(k_t r)$ as $r\to 0$ is cancelled by an identical singularity in $iN_0(ik_t r)$ so that ξ_0 is finite at $r=0$. Similarly the exponential divergence of $iN_0(ik_t r)$ as $r\to\infty$ is canceled by a similar divergence in $-J_0(ik_t r)$ leaving an overall behavior of these two terms as $r^{-1/2}\exp(-k_t r)$ for large r. It is from difficulty in achieving exactly the correct values of the coefficients to effect these cancellations that the problem with the series solution (16)–(19) arises.

Considerations of continuity between the plate and the conical shell as $\alpha\to 90°$ suggest that the form (21) might serve as the basis for an approximate solution of (5) and (8) by generalizing the argument of the functions involved. An appropriate generalization is to write

$$\xi = a\sin\alpha\{J_0[f(r)] + iN_0[f(r)] - J_0[if(r)] - iN_0[if(r)]\}, \tag{23}$$

where

$$f(r)\to k_t r \quad \text{as } r\to\infty, \qquad f(r)\to k_1 r \quad \text{as } r\to 0. \tag{24}$$

The demonstrable division of $\xi(r)$ into a series of odd and even terms as in (18) and (19) suggests that $f(r)$ should contain only odd powers of r. An appropriate smooth function satisfying these requirements is

$$f(r) = k_t r - [\pi^{1/2}(k_t - k_1)/2\lambda]\,\mathrm{erf}(\lambda r), \tag{25}$$

where erf(z) is the error function

$$\mathrm{erf}(z) = \frac{2}{\pi^{1/2}}\int_0^\infty \exp(-z^2)dz. \tag{26}$$

The parameter λ will depend on frequency and on the semiangle α of the cone, and we expect that $k_1 < k_t$ because the hoop stresses make the cone stiffer than the flat plate near its vertex. The phase shift δ_1 in the asymptotic form (11) is then

$$\delta_1 = \pi^{1/2}(k_t - k_1)/2\lambda. \tag{27}$$

The asymptotic form (10) for the longitudinal vibrations implies simply a solution involving $J_0(k_l r)$ and $N_0(k_l r)$, both of which must be present in order to give a solution appropriate to outgoing waves. However (15) requires that the singularity in N_0 as $r\to 0$ be eliminated, and this can be achieved only by adopting a form for η based upon (21). The extra terms are appreciable only in the region near the origin where the coupling between η and ξ is important. We therefore adopt the trial function

$$\eta = -a\cos\alpha\{J_0[g(r)] + iN_0[g(r)] - J_0[ig(r)] - iN_0[ig(r)]\}, \tag{28}$$

where

$$g(r) = k_l r - [\pi^{1/2}(k_l - k_2)/2\mu]\,\mathrm{erf}(\mu r). \tag{29}$$

The phase shift in the longitudinal wave at large distances is given by

$$\delta_2 = \pi^{1/2}(k_l - k_2)/2\mu \tag{30}$$

by analogy with (27).

If we substitute the trial functions (23) and (28) into the original equations of motion (5) and (8), we find immediately that the singular behavior at the origin can be removed only if

$$k_1 = k_2 = 0. \tag{31}$$

We therefore adopt this restriction.

Since we have abandoned the search for an exact solution in favor of a reasonably compact analytical approximation to such a solution, we must now devise an appropriate procedure to determine the two remaining parameters λ and μ. Unfortunately, since the domain of the problem is open rather than closed and the boundary conditions are inhomogeneous rather than homogeneous, we cannot use the standard variational methods of eigenfunction theory. We must therefore, instead, devise a heuristic approach appropriate to our particular problem.

To do this we note that, if we had by chance achieved in (24) and (28) a set of functions including among them the exact solutions of (5) and (8) with the boundary conditions appropriate to our situation, then for this set of values of the parameters the two functions,

$$Z_1(r) = \rho h\omega^2\xi - A\nabla_r^4\xi - \frac{B}{r^2} \times\left(\xi \cot^2\alpha + \eta \cot \alpha + \sigma r \cot \alpha \frac{\partial \eta}{\partial r}\right) \quad (32)$$

and

$$Z_2(r) = \rho h\omega^2\eta + B\nabla_r^2\eta - \frac{B}{r^2} \times\left(\xi \cot \alpha + \eta - \sigma r \cot \alpha \frac{\partial \xi}{\partial r}\right), \quad (33)$$

would vanish identically for all r. In the real situation in which (24) and (28) may approximate the exact solutions but can never be identical with them, the values of the residual functions Z_1 and Z_2 measure the extent of the deviations from the exact solutions. It is, therefore, reasonable to define some weighted combination Z of Z_1 and Z_2, integrated over the whole domain of r, and to vary the parameters contained in ξ and η so as to minimize Z.

Definition of Z poses some problems since the variations we are considering are not arbitrary but are restricted to a very limited class. If this were not so, then the obvious course would be to construct a sort of Lagrangean density $L = Z_1\xi + Z_2\eta$ and integrate this over the whole surface of the cone. However, the restricted forms of ξ and η would allow cancellation of regions of positive and negative L, with misleading results, while the particular forms adopted for ξ and η make it desirable to concentrate upon fit in the region near the apex of the cone rather than on more distant regions as in a simple integration over area.

With these considerations in mind, we adopt for Z the linear mean-square error as defined by

$$Z = (\rho h\omega^2)^{-1}\int_0^\infty (|Z_1|^2 + |Z_2|^2)\, dr. \quad (34)$$

Standard relations[9] are used to separate the real and imaginary parts of Z_1 and Z_2 and the initial factor is included to simplify comparison of numerical results. Clearly any functions $\xi(r)$ and $\eta(r)$ which make $Z = 0$ are exact solutions to our problem, while the minimization of Z with respect to the available parameters in trial functions ξ and η produces, in some sense, a best fit to the real solution.

III. NUMERICAL EXAMPLE

While it is clear from our discussion what the form of the general solution must look like, it is highly desirable to compute a specific case. Before attempting this we simplify the calculation still further by noting that the primary reason

TABLE I. Parameters for brass cones.

Semiangle	α	60°, 80°
Wall thickness	h	0.5 mm
Density	ρ	8.5×10^3 kg m^{-3}
Young's modulus	E	1×10^{11} N m^{-2}
Poisson's ratio	σ	0.3

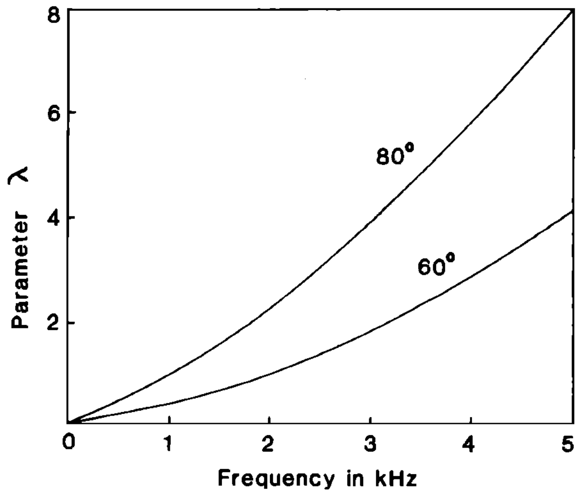

FIG. 2. Values of the parameter λ minimizing the mean-square solution error Z for brass conical shells of semiangle 60° and 80°, respectively.

for deviation of the solutions from their simple flat-plate forms is the coupling between transverse and longitudinal vibration modes. We might therefore expect the two parameters λ and μ of (25) and (29) to be rather similar in magnitude. We can therefore greatly simplify the computational problem by making the assumption

$$\lambda = \mu \quad (35)$$

thus reducing the number of variational parameters to one.

To relate our problem to reality, we now consider the excitation of a cone made from sheet brass 0.5 mm in thickness, with a semiangle of either 60° or 80°. Other relevant physical parameters are given in Table I. Approximate minimization of Z over an integration range $R = 3$ m (effectively infinite except for the lowest frequencies) gave the values

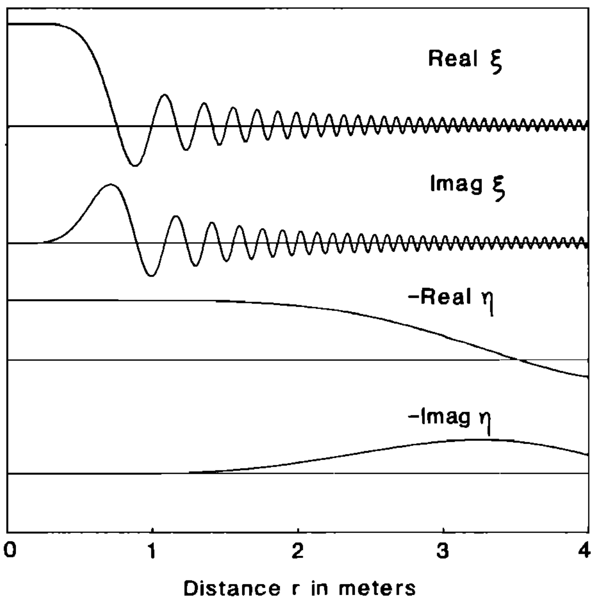

FIG. 3. Real and imaginary parts of the transverse and longitudinal wave displacements ξ and η on a conical shell of semiangle 60° at frequency 1000 Hz at the instant when displacement is a maximum at the apex $r = 0$. Both ξ and η are always real at the apex.

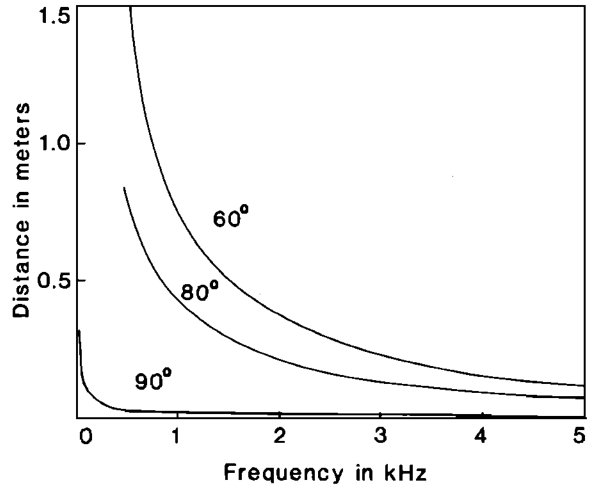

FIG. 4. Distance to the first zero of $\xi(r)$ when $\xi(0)$ is a maximum for cones of semiangle 60°, 80°, and 90° (a flat plate).

shown in Fig. 2 for the parameter λ for the two cone angles considered. For the integration range R, the function Z has the value R if we take $\lambda = 0$, and increases without limit for very large λ. The value of Z at its computed minimum ranged from 0.1–0.9 depending on frequency.

To illustrate the form of the solutions, these are plotted in Fig. 3 for the particular case of the 60° cone excited at a frequency of 1000 Hz. It is clear that, near the apex, the cone vibrates almost like a rigid body, with breakup into wavelike behavior at larger distances. The distance to the first sign change in the real part of ξ, and thus to the first reversal in the phase of the wave relative to its rigid-body behavior near the apex, is plotted in Fig. 4, for several cone angles. Clearly the breakup into wavelike behavior approaches the apex of the cone very rapidly as the frequency is raised—a phenomenon well known to the designers of loudspeakers.

IV. DISCUSSION

This treatment of wave propagation on a conical shell is admittedly crude and preliminary in nature, but it has the advantages of producing both a qualitative picture of the behavior based upon simple physical concepts and a set of numerical predictions that can, at least in principle, be simply checked. Work is now being begun in our laboratory to do this.

By producing relatively simple analytical approximations to the propagating-wave solutions, this approach also allows simple calculation of the shapes and eigenfrequencies of the axisymmetric normal modes of a conical shell of finite length with given boundary conditions at its free edge, the apex of the cone however being necessarily intact.

ACKNOWLEDGMENTS

This problem came to my attention during a study of the nonlinearity of the normal modes of certain Chinese gongs, initiated with T.D. Rossing of Northern Illinois University under the US–Australia Cooperative Scientific Exchange Program. The work reported here was supported by the Australian Research Grants Committee.

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