Nonlinear Dynamics and Chaos in Musical Instruments

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Abstract.

Nonlinear phenomena are essential for the production of harmonic sounds from musical instruments with sustained tone through the phenomenon of mode locking. Under conditions where mode locking is circumvented, the sound of the instrument is 'multiphonic', an effect sometimes used in modern compositions. Simple impulsively excited instruments such as guitars and bells have nearly linear behaviour, with all modes simply decaying exponentially with time. Gongs and cymbals with shallow curvature, however, exhibit a range of striking auditory effects such as pitch glide and energy cascade towards high frequencies, which are commonly used in Eastern music. These effects all depend upon dynamic nonlinearity, and in some cases upon chaotic vibration. More detailed investigation of the forced vibration of an orchestral cymbal shows period multiplication and a transition to chaos, both of which are clearly and characteristically audible.

1. Introduction

The simple first-order linear theory of musical instruments, with which we are all familiar, is remarkably successful in explaining their acoustic behaviour. Briefly, the vibrating elements of musical instruments are chosen to have normal modes in harmonic frequency relationship, the mode frequencies all being integral multiples of the lowest or fundamental mode frequency. Stretched strings and cylindrical or conical pipes fulfil these requirements. Some sort of generator — a moving bow with a strong frictional force, a vibrating reed, or the vibrating lips of the player — is coupled to the system and provides some sort of negative resistance because of feedback effects, and the whole instrument then goes into sustained oscillation. The relative levels of the harmonics in the radiated sound are determined by the radiating properties and incidental resonances of the instrument body, in the case of violins or other stringed instruments, or of the instrument horn in the case of a wind instrument. Instruments that are excited in a transient manner, such as guitars or pianos, can be treated similarly. Only in the case of bells are the mode frequencies recognised as being inharmonic, and this contributes their characteristic non-blending sound.

So apparently successful is this linear theory, that it comes as rather a surprise to discover its fundamental defects. Indeed, as we shall see presently, nonlinearity in several forms is essential to the operation of all sustained-tone musical instruments, and contributes greatly to the characteristic sounds of the more interesting percussion instruments. We shall deal with these two classes of instruments in turn. The reader who would like more information on the basic acoustics, both linear and nonlinear, of musical instruments is referred to one of the several major texts on this subject [1, 2].

2. Sustained-tone instruments

There is one rather trivial aspect of nonlinearity that is obviously required in a discussion of sustained-tone instruments, and that is some more-than-linear increase in the dissipation, or some decrease in the negative resistance of the driving mechanism, with increasing amplitude. Without such a nonlinearity the oscillation amplitude would either die away or else increase without limit. Such a nonlinearity is easily accommodated in a quasi-linear theory, so we accord it no further attention. Two very much more significant aspects of nonlinearity concern the origin of the negative-resistance characteristic of the driving mechanism and the nature of its coupling to the resonant acoustic system.

2.1. Nonlinear driving mechanisms

While positive resistances occur naturally in all systems because of frictional or viscous damping, the origin of negative resistances, whether mechanical or acoustic, is more obscure. Some external energy source is clearly essential, since the negative resistance supplies energy to the system, and some sort of strongly nonlinear behaviour is often involved. We consider just two examples — the bowing mechanism of a violin and the reed driving a clarinet.

2.1.1. The violin

Of all the instruments of music, the violin has perhaps attracted most attention, including that of such well-known physicists as Helmholtz, Raman and Saunders. An excellent modern discussion is given by Cremer [3]. Much modern interest centres on the resonances of the violin body and the distinction between excellent and poor instruments, but here we are concerned rather with fundamentals of the playing mechanism.

It is a commonplace of elementary physics that static friction is generally greater than dynamic friction, but the way in which the frictional force varies with relative velocity is usually not discussed. Figure 1 shows a plausible relationship, with only a simple discontinuity when the relative velocity changes sign, but it is possible that there is an additional antisymmetric-spike discontinuity at this point. At any rate, the relationship is extremely nonlinear in this region.

If we take this curve to represent the frictional force between the bow hairs and the string of a violin, then we see that the frictional force is larger when the string is moving in the direction of the bow motion, so that the relative velocity is lower, than it is when the string is moving backwards against the bow. Since the average displacement of the string under the action of the bow is small and constant, the amplitude of the motion in the two directions must be equal, which means that energy is supplied to the string from the bow at a rate:

$$P = \int F(v)v\,dt = \int F(v)\,dx \tag{1}$$

which is positive. This energy can offset viscous and other losses in the string and maintain it in vibration.

If the velocity of the bow is large and the bowing pressure small, and particularly for bowing an object such as a wine glass rather than a string, then the oscillation may

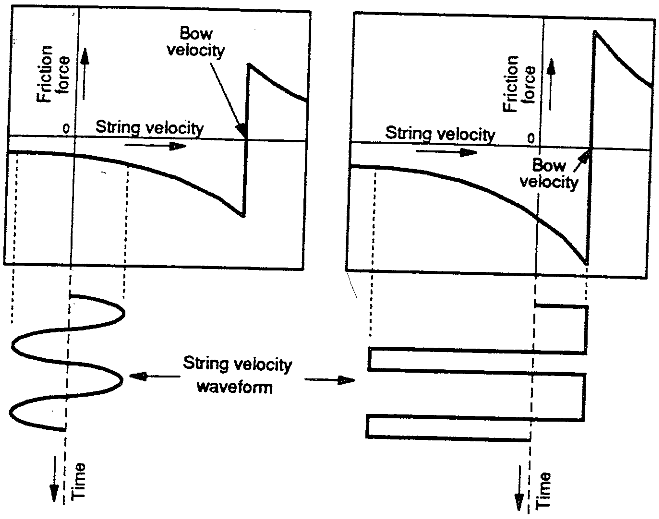

Figure 1: Nonlinear frictional characteristic between a violin bow and a string, and the response of the string (a) for very light bow pressure and high bow velocity, and (b) for normal heavier bow pressure and moderate bow velocity.

remain stable at moderate amplitude. If, however, these conditions are not fulfilled, which is normally the case in bowed-string musical instruments, then the vibration amplitude of the string grows with time until it catches up with the bow in its forward excursions. The string is then trapped during these excursions at the discontinuity in the friction curve, and its motion consists of repeating periods of stick-slip motion in which it is either carried along by the bow or slips backward against it.

It is not particularly easy to analyse this motion, and its elucidation is due to Helmholtz and Raman. The sticking motion at the bowing point is clearly characterised by a constant string velocity equal to the bow velocity, but it turns out that the slipping phase of the motion also occurs with a constant velocity that is related to the distance of the bowing point from the end of the string. This rectangular-wave velocity at the bowing point leads to a triangular-wave displacement at this point and a saw-tooth force wave at the bridge terminating the string. This saw-tooth force drives the body of the violin, and its response, as determined by its internal resonances, influences the radiated sound.

The fundamental frequency of the string vibration is determined by its length, mass and tension and, because the motion is exactly repetitive, the radiated sound spectrum is necessarily exactly harmonic. For the moment this is not a surprise, but we shall see presently that it is a remarkable result.

2.1.2. The clarinet

The clarinet consists essentially of a cylindrical tube at one end of which is a mouthpiece with a flexible reed valve which is blown by the player with a constant pressure p_0. The volume flow U through the reed depends upon the difference between the blowing pressure p_0 and the pressure p inside the instrument mouthpiece. At low blowing pressures the flow increases as $(p_0 - p)^{1/2}$ according to Bernoulli's equation but, as

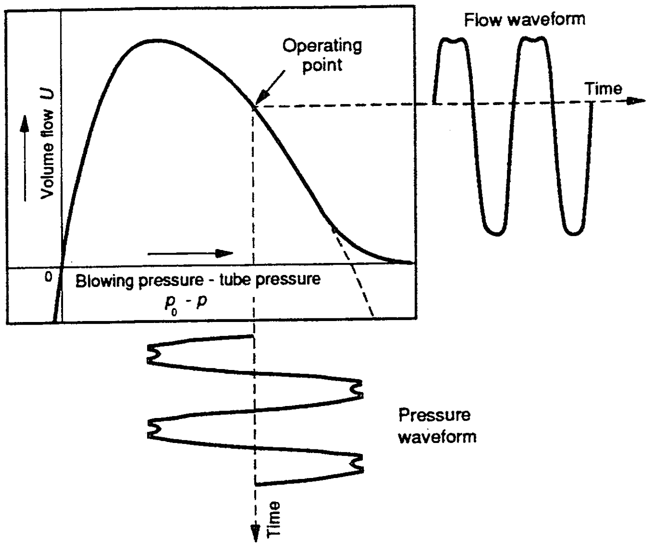

Figure 2: Nonlinear flow characteristic for a clarinet reed. The flow waveform is asymmetrical, but the pressure waveform is nearly symmetrical because the cylindrical instrument tube selectively reinforces odd harmonics.

the blowing pressure is increased, the reed is forced closed and its aperture varies as $[x_0 - \beta(p_0 - p)]$ where x_0 is the original reed opening and β measures its elastic compliance. The total behaviour is thus given by:

$$U = A(p_0 - p)^{1/2}[x_0 - \beta(p_0 - p)] \tag{2}$$

where A is a constant. This relation is plotted in Figure 2, and is clearly very nonlinear.

We are interested in the way in which the flow U varies with p, the acoustic pressure inside the mouthpiece of the instrument, taken for the moment to be an infinitesimal quantity. At low blowing pressure p_0 the volume flow U into the pipe decreases when p increases, which is the simple resistive behaviour we expect, for example, for a rigid aperture with superimposed flow. At high blowing pressures, however, an increase in p actually increases the volume flow U, because it has more influence on the reed opening than on the flow speed. This reversal of behaviour, as indicated by the slope of the curve in Figure 2, corresponds to a negative acoustic resistance, so that the reed acts as a generator and supplies energy to acoustic modes in the instrument tube. This mechanism will work at any frequency below the resonance frequency of the reed itself, which is generally several times the actual sounding frequency. Details need not concern us here, but are available in the literature [4]. In the simple theory, the modes of a cylindrical pipe are harmonically related, though missing the even modes since we require a pressure inside the pipe to drive the reed. The acoustic output is thus a regularly repeating harmonic wave.

2.2. Mode locking

It is at this stage that we must look a little more closely at the model. In each case the nonlinearity provides a negative resistance which transfers energy from the steady source to the oscillating modes of the string or pipe. But if we look more carefully at these string or pipe modes, we find that they are, in fact, not exactly harmonically related. For a pipe with finger holes, as in a real clarinet, the mode frequencies may be very far from integer multiples of a common fundamental, while even for a simple string the string stiffness and incomplete rigidity of the end supports introduce appreciable inharmonicity. How, then, can we get an exactly harmonic acoustic output? For it is exactly harmonic and phase locked, as we readily verify by noting that the waveform remains precisely unchanging so long as we maintain the bow speed or the blowing pressure constant. The answer comes from the nonlinearity.

Suppose that a nonlinear negative-resistance generator drives a resonator (string or pipe) having modes with natural frequencies ω_n. The forcing term driving each mode is a nonlinear combination of the velocity or pressure associated not just with that mode, but with all the others as well, so that we can write an equation for each mode of the form:

$$\frac{d^2x_n}{dt^2} + 2\alpha_n\frac{dx_n}{dt} + \omega_n^2 x_n = F_n(x_1, x_2, \ldots x_n, \ldots) \tag{3}$$

where the x_n are pressures, velocities or displacements associated with the individual modes, the α_n are damping coefficients, and the F_n are nonlinear forcing functions.

We can solve these equations to good approximation by assuming for each mode a behaviour of the form:

$$x_n = a_n \sin(\omega_n t + \phi_n) \tag{4}$$

where a_n and ϕ_n are slowly varying functions of the time t. Inserting these expressions in the mode equations, averaging over time, and omitting all except slowly varying terms, we find that:

$$\frac{da_n}{dt} \approx \frac{1}{2\pi\omega_n}\int_0^{2\pi} F(x_i)\cos\theta_n d\theta_n - \alpha_n a_n \tag{5}$$

$$\frac{d\phi_n}{dt} \approx -\frac{1}{2\pi a_n\omega_n}\int_0^{2\pi} F(x_i)\sin\theta_n d\theta_n \tag{6}$$

where we have written, for convenience, $\theta_n = \omega_n t + \phi_n$.

These equations can be solved numerically, starting from an assumed initial state such as one in which all modes are excited by sudden application of a step function, corresponding to the beginning of bow motion or the application of breath pressure. We find [5] quite generally that, provided several of the normal mode frequencies ω_n are moderately closely in the ratio of small integers, the whole oscillation will settle down to a mode-locked steady state with constant amplitudes a_n and mode response frequencies $\omega_n + d\phi_n/dt$ that are in precise integral relation. This explains the strictly harmonic regime that characterises the normal acoustic spectrum of musical instruments.

When the frequencies of two or more important low-lying modes are very far from integrally related, however, the mode locking fails and the two modes are excited nearly independently, depending to some extent upon the initial conditions. The generator nonlinearity then generates sum and difference frequencies of all orders, and the result is a peculiar inharmonic and sometimes roughly beating sound. While such 'multiphonic' sounds are not a part of classical performance technique, they are being increasingly used by modern composers [6].

Armed with these analysis techniques, we can proceed to calculate the amplitudes of the phase-locked modes inside the instrument and then the properties of the radiated sound. We can also calculate the transient behaviour at the beginning of each note, which is a vital part of the timbre of a musical instrument. This is just one of the analytical techniques leading to these results, other possibilities being to carry out the calculation completely in the time domain [7] or to use mixed representations for the steady state such as that of harmonic balance [8].

3. Impulsively excited instruments

Impulsively excited instruments include those with plucked strings, such as the guitar, harp or harpsichord, and those with hammered strings, such as the piano and dulcimer, as well as those instruments that are normally considered as belonging to the percussion section of the orchestra. In this group we find a widely varied set of acoustic phenomena, depending upon the degree of nonlinearity encountered.

3.1. Stringed instruments

Instruments such as the guitar and the piano are well described in a nearly linear approximation. The string modes are not precisely harmonic, a circumstance that contributes to the slightly bell-like sound of the piano and to its stretched tuning at the extremes of the keyboard, and the interactions between the multiple strings used for each note give peculiar effects which are, however, entirely linear [9]. The timbre variation obtainable using different plucking points in the guitar is also a completely linear phenomenon. In both instruments, once the string has been excited, the normal modes simply decay exponentially without interaction.

Nonlinearity in the piano is essentially confined to the nonlinear elastic properties of the hammer felt, which becomes stiffer for heavy blows and influences the tone quality accordingly. In a guitar or similar instrument, particularly if one uses metal strings tuned to rather low tension, it is possible to induce nonlinear behaviour through the stretching of the string by large-amplitude vibrations. If a string of length L carries modes with amplitudes a_n, then the tension stress increases from its initial value σ_0 to a new value:

$$\sigma = \sigma_0 + \frac{E}{L}\int_0^L \left\{ \left[1 + \left(\frac{dy}{dx}\right)^2 \right]^{1/2} - 1 \right\} dx \approx \sigma_0 + E\sum_n \left(\frac{n\pi a_n}{2L}\right)^2 \qquad (7)$$

together with rapidly fluctuating terms. Here E is the Young's modulus of the string material. Since the mode frequencies are proportional to the square root of the tension stress σ, this nonlinearity causes an unpleasant falling 'twang' as the string is released and its amplitude decays, and is minimised by using strings with a low Young's modulus, such as gut or nylon, and increasing the tension stress to nearly the breaking point.

3.2. Bells and gongs

Much more interesting nonlinear phenomena are encountered in the family of percussion instruments, and indeed there is a nice progression in behaviour as we reduce both the wall thickness and shell curvature of the vibrating shell. The situation is illustrated in Figure 3.

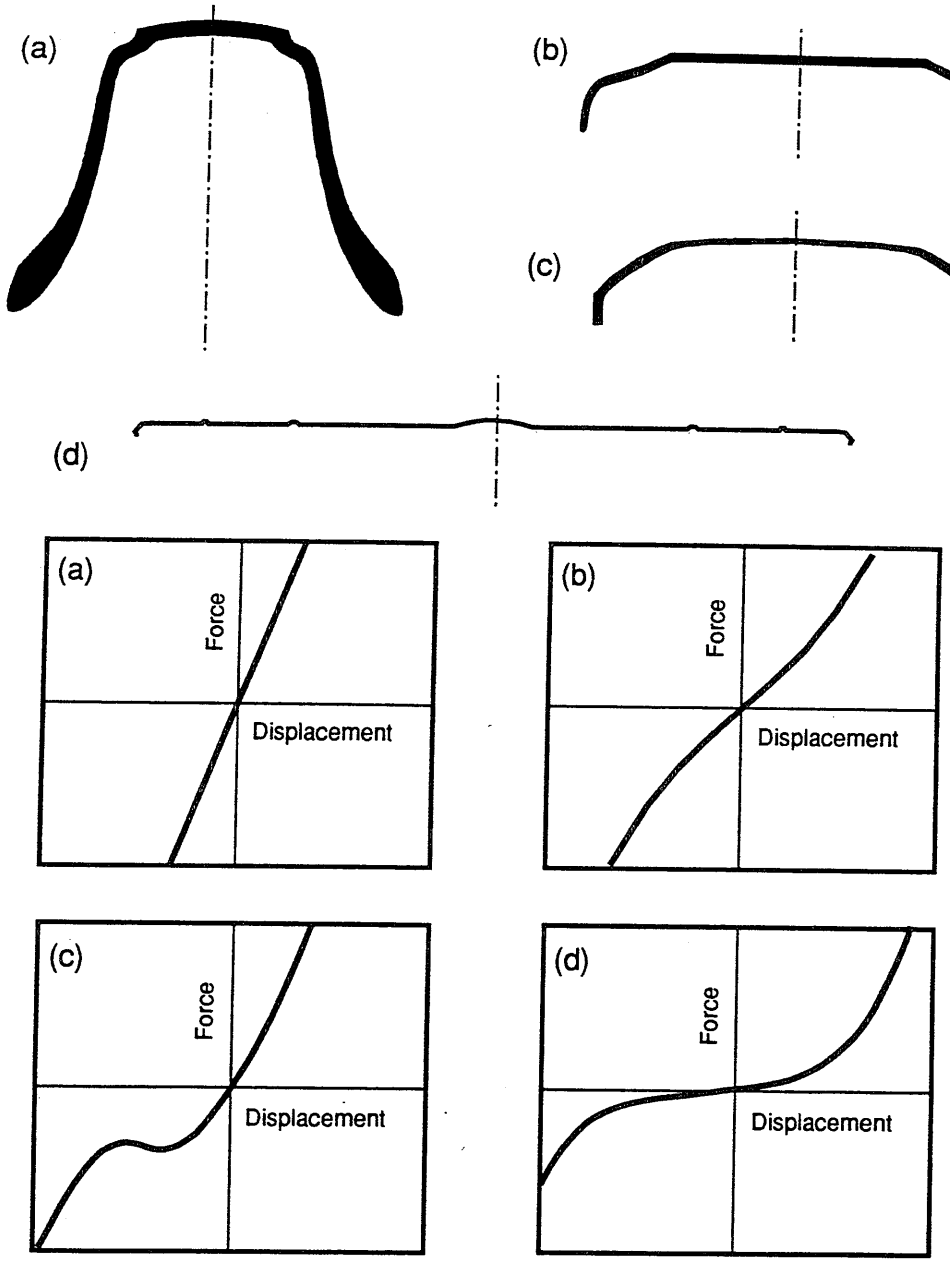

Figure 3: Cross sections of a range of typical bells and gongs, with the associated force/displacement curve for each. (a) A European church bell; (b) a Chinese opera gong with flat top; (c) a smaller Chinese opera gong with domed top; (d) a Chinese tamtam.

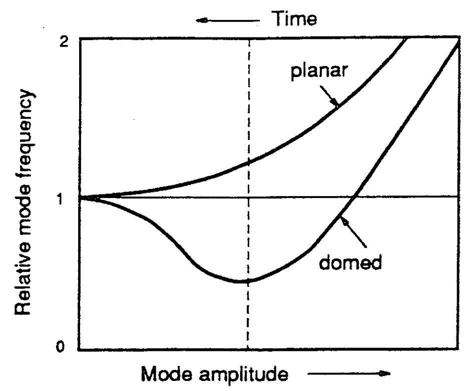

Figure 4: Calculated dependence of the mode frequencies of two Chinese opera gongs upon excitation amplitude. When the gong is struck, the initial amplitude is large and then decays, giving a pitch glide.

The normal European church or carillon bell is cast from heavy bronze to the thick-walled traditional shape shown in Figure 3(a). This shape is chosen so as to bring the prominent lower mode frequencies into nearly harmonic relationship, including a minor-third interval 6 : 5, though small tuning adjustments are necessary after the casting is complete. Elastic restoring forces are provided overwhelmingly by the stiffness of the thick walls, and the characteristic is essentially linear, as shown in (a). The bell is excited by a metal clapper and the mode amplitudes simply decay away exponentially. Audibly there may be a slow 'warble' in the sound because pairs of nominally degenerate modes differ slightly in frequency because of casting irregularities.

A set of interesting nonlinear phenomena has been built into the characteristic gongs widely used in the Peking Opera [10]. The larger gong, shown in Figure 3(b), has a plane central section surrounded by a conical and then a cylindrical flange. Only the central section vibrates appreciably. In this case, the central metal is moderately thin, and restoring forces are contributed both by material stiffness and by tension forces generated by stretching, as discussed above for the string. The restoring force thus has a cubic nonlinearity, as shown in (b). The gong is normally struck vigorously in its centre with a padded stick, exciting the centrosymmetric modes to high amplitude and raising the average tension and with it the mode frequencies. As the mode amplitudes decay, so the frequencies decrease towards their small-amplitude values. This behaviour is illustrated in Figure 4. In a typical gong, the achievable pitch glide may be as much as 20 percent, or a minor third, giving an unusual and characteristic effect.

In the smaller opera gong of Figure 3(c), the central section is slightly domed, which introduces a quadratic as well as a cubic nonlinearity, as shown in the curve (c). In this case, as the amplitude is increased the mode frequencies first fall and then rise, as shown in Figure 4, though in fact the attainable amplitudes are limited to the falling part of the behaviour. When the gong is struck, therefore, the pitch rises as the mode amplitudes decay. The extent of attainable pitch glide is again typically about a minor third. Used in turn, these two gongs provide musical commentary upon the action of the opera.

3.3. *The cymbal*

As the gongs become progressively shallower in design, and their wall thickness progressively thinner, their inherent wall stiffness becomes less and the importance of tension effects greater. This leads to even more complex nonlinear phenomena [11, 12].

The ordinary orchestral cymbal, shown in section at the top of Figure 5, is a shallow spherical-cap shell with a central boss. It is ordinarily struck near its edge, and produces a broad-band shimmering sound with no obvious modal basis, even though small-signal modes can be readily studied. Because of the domed shape, we expect the force curve to resemble the quadratic plus cubic curve of Figure 3(c), though the behaviour may be very different from that of the small Chinese opera gong because the edge of the cymbal is free, rather than being clamped by a stiff flange.

For the simplest investigation of nonlinearity, the cymbal can be mounted horizontally and excited with a sinusoidal shaker attached to its centre. Typical results with this setup are shown in Figure 5. At low amplitudes, as in (a), the radiated sound is concentrated at the fundamental of the exciting frequency, though there is some radiation generated at harmonics of this frequency by the vibrational nonlinearity. With increasing excitation level, as in (b), the relative levels of all the harmonics increase as expected. Then suddenly, at a critical excitation amplitude, the spectrum develops a complete set of subharmonics of order five as shown in (c). The sound change is remarkable! The harmonically distorted sound of (b) is a simple rich tone based upon the musical note E_5 with frequency about 600 Hz, while the period-multiplied spectrum of (c) is a full C major chord based on C_3 with E_5 appearing as the fifth harmonic (or, musically, as the major 17th). With further increase in excitation level, the vibration once again makes a sudden transition, this time to chaotic behaviour as shown in (d). In this regime the vibration is no longer centrosymmetric, and the sound is closely similar to that of an edge-struck cymbal.

This behaviour is sensitively dependent upon the excitation frequency. At slightly different frequencies we have observed apparently direct transition to chaos, as well as the development of subharmonics of orders two and three, or even of orders three and five simultaneously.

The mathematical problem of analysing cymbal behaviour is difficult, because the large number of active or potentially active modes makes the dimensionality of the problem large. It is possible, however, to gain some insight into the behaviour by studying a simple problem with quadratic and cubic nonlinearity but only a single degree of freedom, as expressed by the equation:

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + x + \alpha x^2 + \beta x^3 = F \sin \omega t \tag{8}$$

where α and β measure the strengths of the quadratic and cubic nonlinearities respectively, and F is the magnitude and ω the normalised frequency of the external force. Numerical investigation of this system [12] shows that it can exhibit simple bifurcation, splitting into five, and a transition to chaos, thus mimicking the behaviour of the physical system.

3.4. *The tamtam*

The large Chinese gong, or tamtam, shown in Figure 3(d) is almost completely flat, with a shallow central dome, a turned over edge, and several rings of hammered bumps. It is typically nearly a metre in diameter and is made from metal less than 1 mm in

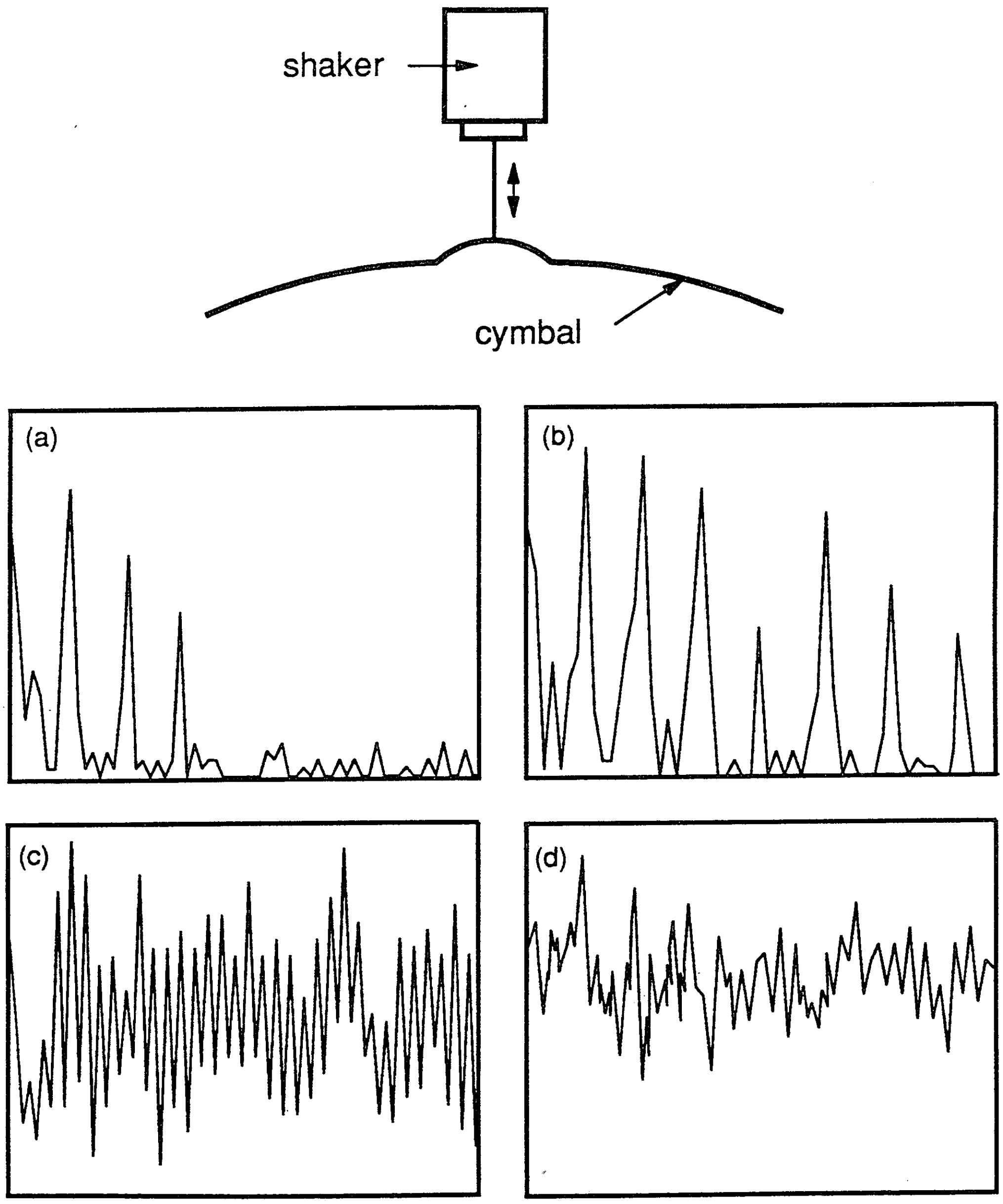

Figure 5: Cross section of an orchestral cymbal (top), set up for the vibration experiment. Below is shown the response of the cymbal when excited sinusoidally at its centre at a frequency close to its first resonance. (a) Low-level excitation shows mild harmonic distortion of the fundamental signal; (b) at a higher level the harmonic distortion becomes very large; (c) at a critical level, depending on the exact frequency, the period of the vibration increases by a factor 5, giving multiple subharmonics; (d) for an even higher excitation level, the vibration becomes chaotic and exhibits a characteristic cymbal sound. In each case the vertical scale range is 80 dB and the horizontal scale is 0 to 5 kHz. The excitation frequency is about 600 Hz.

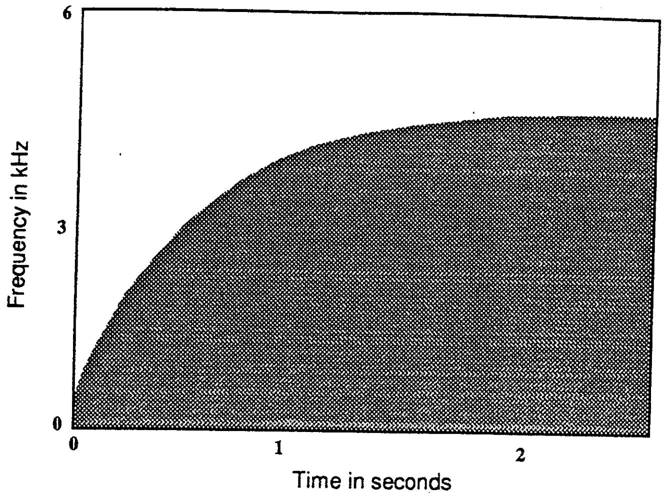

Figure 6: Idealisation of a Sonagraph analysis of the evolution of the sound of a large tamtam. The grey-scale indicates the energy at various frequencies, with a filter weighting emphasizing high frequencies.

thickness. The hammered bumps help to stiffen the structure, but also appear to have a role in increasing nonlinearity and in breaking down the circular symmetry of the gong. The force/displacement curve is very flat, as shown in (d), but with extreme cubic nonlinearity.

The tamtam is played by striking it in the centre with a large and softly padded beater, thus exciting only low-frequency centrosymmetric modes with frequencies below about 100 Hz. The development of the sound is then most impressive, as shown by the idealised Sonagraph record in Figure 6. The low-frequency modes initially excited simply decay steadily, but some set of nonlinear processes pumps energy from these modes into modes of progressively higher frequency over a time of about 3 seconds. At the end of this time the vibrations of the tamtam are concentrated around its edge, presumably as non-centrosymmetric vibrations, and the sound has progressed from a deep rumbling thump to a silvery sheen which takes many seconds to decay to inaudibility. FFT spectra taken at various times after the initial strike confirm this behaviour in detail and also indicate that there is no identifiable mode structure, suggesting that we have once again a chaotic vibration.

Detailed analysis of the behaviour of the tamtam and its mode cascade has not yet proved possible, but we have been able to show rather similar phenomena in much simpler situations, such as strings or bars with sharp kinks in their shape. The extent to which the hammered bumps of the tamtam are responsible for the mode cascade is unclear, but an important role does seem likely.

4. Conclusions

Musical instruments have been developed through the ages to make best use of simply available acoustic and vibrational phenomena to produce striking and novel auditory effects. Subsequent refinement of design and performance technique has then enhanced the individuality of the behaviour so that they often provide unmatched exemplars of interesting physical phenomena. The end is not yet in sight, but it is particularly striking to see the twin roles that nonlinearity plays in music. On one hand it welds together the misaligned modes of traditional wind and string instruments to provide the fully developed and precisely aligned harmonic series upon which Western music is based, while on the other it exploits the phenomena of chaotic vibration to produce superbly evocative percussive effects. We have still a lot to learn!

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