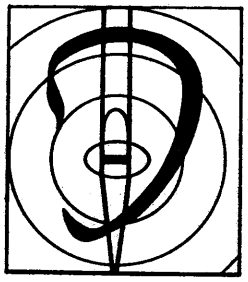

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Nonlinearity, transients and spectra

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Abstract

The role of nonlinearity in generating the steady spectrum and defining the transient behaviour of musical instruments is discussed. A simple theory for analysing, in the time domain, a multimode inharmonic system driven by a nonlinear self-excited generator is presented and it is shown that this can lead to a mode-locked harmonic regime, a multiphonic regime, or even chaotic oscillation. The theory is applied to discuss the steady state and transients in bowed-string, woodwind and brass instruments. Brief reference is made to impulsively excited instruments.

Introduction

Many of the design and performance aspects of musical instruments can be understood on the basis of linear theory, and indeed this theory is perfectly adequate to treat in detail the resonances and radiation properties of the air columns in wind instruments and the complex bodies of stringed instruments, although this is not generally true for percussion instruments such as gongs and cymbals. The excitation mechanism in sustained-tone instruments is, however, intrinsically nonlinear, and most of the mechanisms that determine radiated power, acoustic spectrum, and transient behaviour rely upon this nonlinearity.

A sustained-tone instrument consists essentially of a linearly-behaved resonator, coupled at one place to a nonlinear driving mechanism, and at another place to a linear, though possibly complex, radiating mechanism. The nonlinear exciter is itself supplied with steady power by the player, either in the form of mechanical motion or of air at more than atmospheric pressure. Each stage of this system is coupled back to earlier stages, the ultimate control coupling being between the radiated sound, which typically constitutes less than one percent of the input mechanical power, and the ears of the player.

In this discussion I want to concentrate on the way in which the nonlinear driving mechanism excites the linear primary resonator and the strong coupling between these two, for this determines the basic behaviour of the instrument. Once this is understood, the remainder of the system is nearly linear and has essentially a passive filtering action. There has been a great deal of progress made in this understanding in recent years, largely on the basis of time-domain computer simulation methods (McIntyre et al, 1983). Here I want to take a different approach and treat the problem in the frequency domain in terms of excitation of the normal modes of the resonant system. I make no claim that this method is superior, but it is complementary and gives different insights into the behaviour of the system.

General theory

The behaviour of any linear system, such as a vibrating string, an air column or a metal bell, can be described in terms of its normal modes or characteristic vibrations $\xi_n(\mathbf{r},t)$ where ξ might be a displacement, a velocity, a pressure or a flow, depending on the system considered. These modes have shapes and frequencies that can be measured, or calculated once the shape and elastic properties are defined. Because the system is linear, any possible free vibration that it may undergo can be expressed as a superposition of these normal modes, oscillating sinusoidally at their natural frequencies but with amplitudes and phases that depend upon the initial excitation. The vibration x of the system at a particular point $\mathbf{r}'$ can therefore be written

$$x = \sum_n x_n = \sum_n a_n \sin(\omega_n t + \phi_n) \qquad (1)$$

where $x_n = \xi_n(\mathbf{r}', t)$ is the contribution of mode n at this point.

When this simple system is coupled at the point r' to a generator producing an excitation $F(x)$, each mode responds as a simple oscillator, and can be described by an equation of the form

$$\ddot{x}_n + k_n \dot{x}_n + \omega_n^2 x_n = F(x) \qquad (2)$$

where k is a damping coefficient. I have written F as possibly a function of the variable x at the excitation point, because I want to allow for feedback. In a musical instrument we must have such feedback, because the excitation supplied by the player is simply a steady force and cannot excite vibrations. The function $F(x)$ can be quite complicated and involve phase shifts and time delays. Any part of F that is in-phase with $\dot{x}_n$ will influence the damping of the system and cause the amplitude of mode n to grow or decay, depending on its sign and upon the intrinsic damping k_n, while any component of F that is in-phase with x_n will shift the response frequency of the mode away from ω_n. We can formalise this by assuming that the amplitudes a_n and frequencies ω_n of the mode functions x_n in (1) are actually functions of time. Rather than changing ω_n however, if is better to assume that the phase ϕ_n varies with time, so that the response frequency of mode n becomes $\omega_n + (d\phi_n / dt)$. We can then do some algebra to sort out the parts of F that are in-phase with $\dot{x}_n$ and x_n respectively, to get

$$\frac{da_n}{dt} = \frac{1}{\omega_n}\langle F(x)\cos(\omega_n t + \phi_n)\rangle - k_n a_n \qquad (3)$$

$$\frac{d\phi_n}{dt} = -\frac{1}{a_n \omega_n}\langle F(x)\sin(\omega_n t + \phi_n)\rangle \qquad (4)$$

where the brackets $\langle ... \rangle$ imply that we take an average in such a way as to retain only terms that vary slowly relative to ω_n. This reflects the fact that each mode will respond only to excitation approximately matching it in frequency.

In any sustained-tone instrument $F(x)$ is a nonlinear function of x, as we see presently from examples, so that the situation represented by the large set of equations (3) and (4) is actually very complicated. For any nonlinearity that is not too extreme, F can be expanded as a series

$$F(x) = \sum_i c_i x^i$$

$$= \sum c_N a_n^p a_m^q \ldots \sin[(p\omega_n \pm q\omega_m \pm \ldots)t] \qquad (5)$$

The second form of writing shows that there is a complex mixing of frequencies of all orders caused by the nonlinearity. Since these terms all enter into (3) and (4), all equations of the set are coupled together in a complicated way by the nonlinearity.

Although we are concerned here primarily with sustained-tone instruments, it is worthwhile to note that equations (1)–(5) apply equally to impulsively-excited systems. The only difference is that F no longer involves an external energy supply but simply describes the nonlinear interactions between modes following their initial impulsive excitation to prescribed amplitudes.

Study of (3) and (4) shows that three possible types of steady-state behaviour, or regimes of oscillation as Art Benade called them, are possible.

- The harmonic regime, in which all mode frequencies are shifted by the nonlinearity to become exact integer multiples of the fundamental frequency, and locked together in phase (Fletcher, 1978). This is the normal sounding regime of musical instruments.
- A multiphonic regime, in which, generally because of use of peculiar fingerings and modified blowing, several strongly-excited modes are too far from harmonic relationship to be locked together by the limited nonlinearity. These modes then oscillate at nearly their natural frequencies, but the nonlinearity generates multiple sum and difference frequencies according to (5) giving a "multiphonic" sound, sometimes favoured by modern composers (Keefe & Laden, 1991).
- A genuinely chaotic regime, in which the sound has an unpleasant rasping quality. This is exemplified by the harsh "crow" of an isolated double reed, but can rarely be produced on an assembled instrument. Chaotic oscillation does, however, occur in nonlinear percussion instruments such as cymbals (Fletcher, 1993 b).

Musical instruments

Figure1 shows the nonlinear excitation characteristics of three common types of musical instruments—bowed strings, reed-driven woodwinds, and air-jet excited instruments. The physical basis of these curves is familiar and has been discussed in detail elsewhere (Fletcher & Rossing, 1991)—the bowed string characteristic depends essentially upon the difference between static and dynamic friction, the reed characteristic upon Bernoulli flow through an aperture which is itself controlled by the pressure difference across it, and the jet flow by the velocity profile of the deflected jet. All have a negative slope near the operating point O, and this corresponds to a negative resistance which subtracts from the natural damping factor k_n and, provided the magnitude of the first term of (3) is greater than $k_n a_n$ for some mode, generates a self-sustained oscillation. The amplitude of possible oscillation, however, is limited in all cases: in the case of the bowed string by the bow velocity, in the case of the reed by its closing, and in the case of the jet by its passing entirely into or out of the pipe lip.

In instruments such as the clarinet and the flute which can sustain small-amplitude vibrations, the harmonic structure of the excitation is described initially by the small-nonlinearity expression (5)—the fundamental is dominant and the amplitude of the nth harmonic is proportional to the nth power of the amplitude of the fundamental. This result, well known in nonlinear theory, was first remarked upon in relation to clarinet sound by Benade's student Worman (1971).

The bowed-string nonlinearity is more extreme than the other two—it exhibits a discontinuity in the force itself, while the reed shows a discontinuity in the derivative of the force and the jet a discontinuity in the second derivative. Furthermore, the slope of the frictional characteristic of the bowed string is such that the steady amplitude always reaches the discontinuity, so that there is no regime of "small nonlinearity" and the excitation function has a switching-type discontinuity, which implies an excitation spectrum which rolls off at 6dB/octave at high frequencies. This is modified, of course, by the resonant response of the instrument body and by its radiation efficiency, but the general limiting spectral shape is clear in the sound as well.

Double-reed conical woodwinds, such as the oboe and bassoon, also tend to traverse the whole

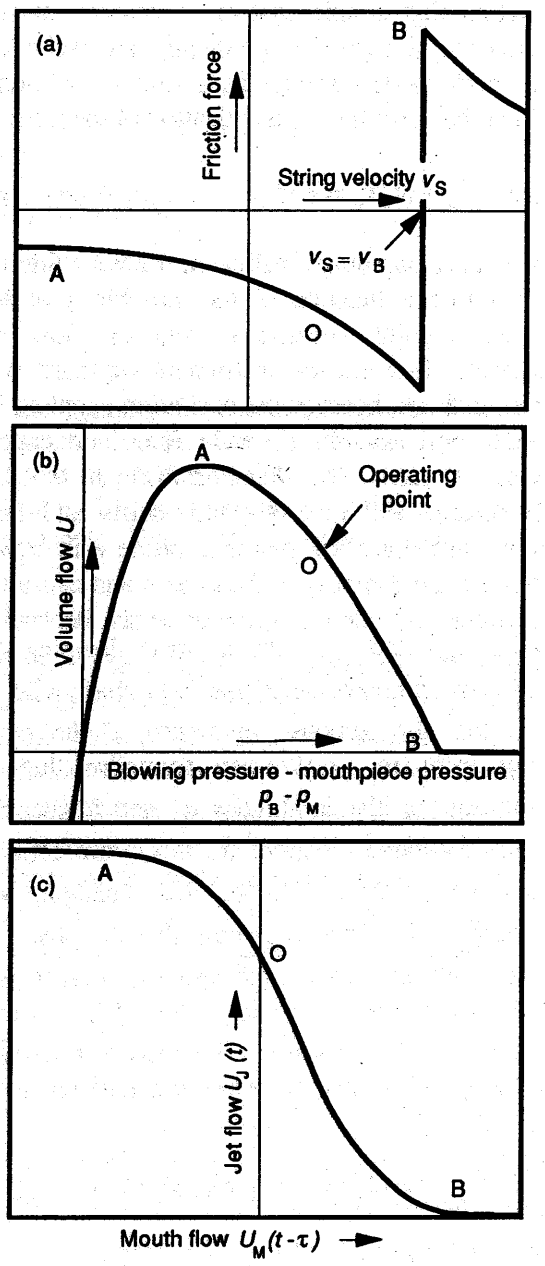

Figure 1: Nonlinear excitation characteristics of (a) a bowed string, (b) a reed generator, and (c) an air jet generator. The operating point is set near O in each case, and the large-signal switching behaviour is between points A and B. In (a) there is a discontinuity at B when the string velocity equals the bow velocity; in (b) there is a discontinuity of slope at B when the reed closes; in (c) there may be a discontinuity in curvature near the edges of the jet at A and B.

range of reed motion up to closing, since the slope of the characteristic at normal blowing pressures increases with increasing oscillation amplitude. Because there is a discontinuity only in the slope of the excitation function on closure, this leads to a limiting spectral roll-off of 12dB/octave, though a "soft" closure could increase this to 18dB/octave. Again this must be modified by the mode tuning of the instrument at low frequencies and by the radiation characteristics, including the tone-hole lattice cutoff at high frequencies. This radiation characteristic reduces the amplitude of the lower harmonics of each note in the radiated sound.

Single-reed cylindrical woodwinds, such as the clarinet, have rather different properties because the air column resonator responds only weakly to the even harmonics of the excitation. This causes the reed vibration to be nearly symmetrical about the operating point and makes it possible to sustain low-amplitude vibrations without closure of the reed. The instrument can therefore exhibit the small-nonlinearity behaviour noted above. As the dynamic level is increased the reed comes closer to closing, though the curve of the lay probably makes this always a soft closure with consequently greater rolloff at high frequencies.

Instruments such as the flute, excited by an air jet, have an even softer nonlinearity when they are blown vigorously enough to exhibit switching behaviour. Viscous eddy diffusion ensures that the velocity profile of the jet has at most a discontinuity in its second derivative, so that the simple theory of jet excitation, which depends upon jet volume flow into the pipe, predicts a rolloff of at least 18dB/octave at high frequencies. Again the total radiated spectrum is modified by the radiation characteristics of the finger holes.

In real situations various other effects enter to smooth off the discontinuities and further reduce the amplitudes of the extreme higher harmonics in the radiated spectrum.

Transients

Equations (3) and (4) can describe the transient as well as the steady-state behaviour of musical instruments, and indeed they are primarily adapted to do just this. There is always some impulsive excitation when a player begins a note, if only because the bow begins to move, or the air jet to flow, more or less abruptly. This always excites all the normal modes of the resonator to some low level, and these vibrations are close to their natural frequencies since the influence of other modes is initially small. One of the modes generally grows more rapidly than the others and begins to influence their frequencies until the whole oscillation settles down into one of the possible steady state regimes discussed above. The initial transient typically takes about 20 cycles of the fundamental frequency, though this depends upon the nature of the nonlinearity. Depending upon the form of the transient in the force supplied by the player, the dominant steady-state mode may differ from that which is dominant during the early part of the transient. Some of these effects are illustrated in Figure 2, which is calculated for abrupt application of pressure to a particular organ pipe (Fletcher, 1976). Similar effects can be observed and calculated for other musical instruments.

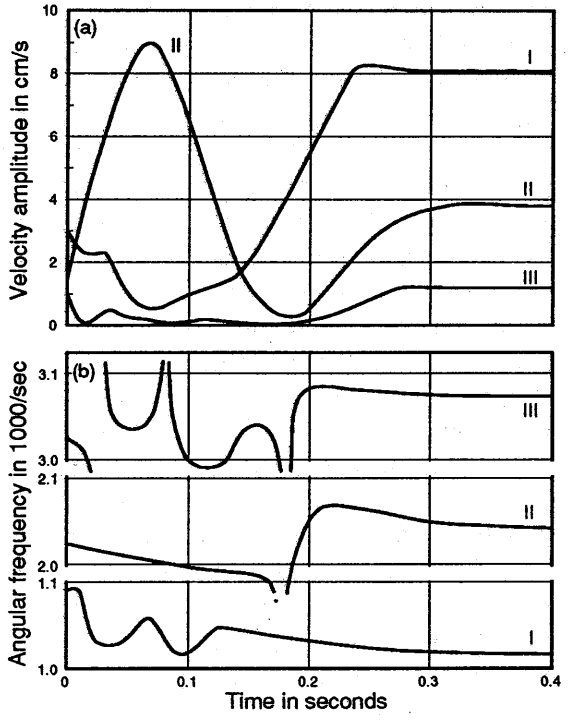

Figure 2: Calculated initial transient for a flue organ pipe excited by abrupt application of pressure with a slight overshoot. Note the initial dominance of the second partial mode II, and the final harmonic mode-locked regime.

It is of particular interest to examine the initial transient in lip-driven brass instruments, for this differs considerably from other cases. Such instruments are often played using horn modes as high as the sixteenth. This means that as many as sixteen cycles of the note being played elapse between its initiation and the time that the first reinforcing reflection is received from the instrument bell. Analysis of the autonomous vibration of valves shows that the lip valve, which in the simplest model is blown open by mouth pressure and closed by pressure in the instrument mouthpiece, can oscillate close to its resonance frequency with the aid only of the capacitive impedance of the player's mouth (Fletcher, 1993a). The lips can therefore launch a tone burst at the frequency of the desired note. Analysis is simple in the time domain, and the importance of a cleanly defined reflection is clear, but it is instructive to see the analogous treatment in the frequency domain.

The tone burst generated by the lips has a spread of frequencies centred on its tone frequency and excites a collection of horn modes also centred on that frequency. For a long horn, the separation of mode frequencies is small, and quite a number of modes are excited. These oscillate at their natural frequencies and return to their original phase relations after a time equal to the reciprocal of their frequency separation. This is just the reciprocal of the fundamental frequency of the horn, as in the time-domain treatment. In fact, an analysis of the horn vibrations at an intermediate time will show a tone burst propagating down the horn, though possibly smeared out by dispersion effects. There is an additional fact that is clear from the frequency-domain analysis, however, and this is that the cleanness of the reflected pulse is determined by the degree of harmonicity of the group of modes centred around the playing frequency that are excited by the tone burst. It is possible to have good mode alignment around one playing frequency but not around another at some distance from it, leading to good notes and bad notes on the instrument.

Impulsively excited instruments

There is not space here to discuss the role of nonlinearity in impulsively-excited instruments in any detail. Instruments such as the guitar and the harpsichord are very nearly linear in behaviour, and in the piano the nonlinearity is in the hammer blow rather than in the instrument itself. Similarly bells, which derive their elastic stiffness from thick metal walls with considerable curvature, are substantially linear. As the walls become thinner and the curvature less, however, a bell transforms into a gong or a cymbal, and the characteristic sounds of these instruments owe a great deal to nonlinearity.

If the nonlinearity is small, as in some Chinese gongs, the most noticeable effect is a glide of mode frequencies—either upwards or downwards according to the geometry of the gong—with changing amplitude (Fletcher, 1985). In gongs with greater nonlinearity, such as the large tamtam, we find a cascade of energy from the low-frequency modes, excited by the initial strike with a soft beater, into high-frequency modes, giving a most effective shimmering sound. Even the common cymbal, which shows simple mode behaviour at very small excitation amplitudes, behaves with great complexity at higher amplitudes and shows subharmonic generation of many orders followed by a transition to chaotic oscillation (Legge & Fletcher, 1989, Fletcher 1993b). Some of these effects are best studied in the time domain and some in the frequency domain, but understanding comes more easily when both approaches are used.

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