VIBRATO IN MUSIC

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ABSTRACT. Vibrato, which is an oscillation in the pitch, loudness or timbre of a musical tone, is a very important aspect of musical performance. This paper discusses the ways in which vibrato can be analysed, and also the ways in which it can be produced by performers on musical instruments and by singers.

1. INTRODUCTION

Transients have an important place in determining the subjective qualities of musical sounds. Most important are the attack and decay transients, without which most sounds lose their individuality – a recording of a piano played backwards sounds like some sort of organ – and these have received considerable attention in the literature. Within a nominally steady musical sound, however, the performer may insert a periodic modulation of some kind with a frequency typically of around 5Hz that is called vibrato. Not all instruments or performances use vibrato, but those that do not, such as pipe organs (mostly by necessity), Renaissance viols (again by necessity), and classical orchestral clarinets (by tradition), gain individuality by this very lack.

Vibrato is in many cases produced by a conscious physical manipulation, such as the regular oscillation of the left hand of a violinist where it stops the string against the fingerboard, but in some situations, such as elderly singers, the vibrato seems to arise naturally through oscillation of abdominal and laryngeal muscles and to be largely uncontrolled. More skilled musicians are able to vary the amplitude, and to some extent the frequency, of the vibrato and do this for musical purpose as the notes of the melody develop. In most cases, however, the frequency is in the range 5 to 8Hz, and it is perhaps significant that this is the typical frequency range of muscular tremors in neurological disorders such as Parkinson's disease and not too far from the resting alpha rhythm of the human brain. This suggests that both the generation and the perception of vibrato are closely related to innate human physiological and psychological characteristics. A classic discussion of psychological aspects has been given by Seashore [1].

It is not the purpose of this paper to investigate these subtle matters, but rather to examine the phenomenon of vibrato from a purely physical and mathematical viewpoint. In the course of this study a careful distinction (acoustical rather than musical) will be made between various types of vibrato, though it is not certain that these can be clearly related to rather vague musical distinctions such as that between 'vibrato' and 'tremolo'. The term 'vibrato' will be used here to encompass all varieties of the effect.

2. ANALYSIS OF VIBRATO

While the steady sound produced by a sustained-tone instrument such as a flute, a violin, or the human voice, is strictly harmonic, the same is not true of impulsive sounds produced by instruments such as harps or guitars, in which all vibrational modes have frequencies close to the nominal mode frequencies of the primary vibrating element (the string in both these cases), and these overtones are not ever in exact harmonic relationship to the fundamental [2]. In both types of instrument, however, the effect of vibrato is to impose a cyclic variation upon some important physical parameter such as string length or blowing pressure and this results in a cyclic variation of acoustic parameters such as the amplitudes and frequencies of the fundamental and overtones constituting the sound. The vibrato may well destroy the exact harmonicity of the overtones of sustained-tone instruments, and this is one of the possibilities to be investigated here.

Consider an infinitely prolonged note with some sort of vibrato. To the ear the sound may vary in three different ways, alone or in combination. The first is a cyclic variation in the loudness, which in music is generally called tremolo; the second is a cyclic variation in the pitch, generally called vibrato, and the third a cyclic variation in tone quality or timbre, to which a musical term has not been assigned. It is helpful to examine the ways in which each of these possibilities can be measured and specified.

TIME-DOMAIN ANALYSIS

This is the most straightforward but least informative way in which to describe the acoustic signal. At some specified location in the sound field the acoustic pressure $p(t)$ is measured at a sampling rate at least twice that of the highest frequency component of interest, ideally after passing the signal through a band-pass filter at that cut-off frequency in order to eliminate aliasing effects. This signal contains all the necessary information about the sound, but is of little use except for further analysis.

FOURIER ANALYSIS

In Fourier analysis the signal $p(t)$ is converted into the frequency domain by performing a Fourier transform, ideally upon an infinite length of signal but in practice on a length containing an integral number of vibrato cycles. This yields a complex frequency spectrum $p(\omega)$ that also contains all the signal information. Generally this complex spectrum is converted for display to a power spectrum $P(\omega) = |p(\omega)|^2/2$ which discards the phase information.

A simple sinusoidal amplitude modulation of a signal of frequency ω and amplitude a by a vibrato frequency Ω and amplitude Δa gives rise to two side-bands at frequencies

$\omega \pm \Omega$ along with the original signal at frequency ω, as shown in Fig. 1(a). The relative amplitudes of the three frequency components depend upon the modulation index $\Delta a/a$, and if this becomes much greater than unity then the component at frequency ω vanishes.

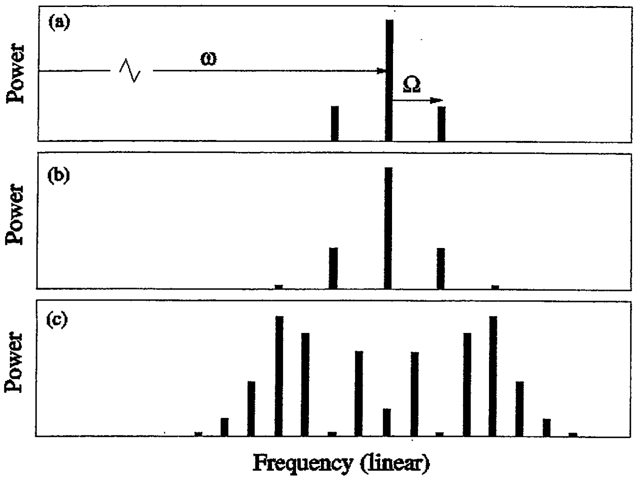

Figure 1. (a) Fourier power spectrum of an amplitude modulated signal with $\Delta a/a=1$. (b) Power spectrum of a frequency modulated signal with $\Delta\omega/\Omega=1$. (c) Power spectrum of a frequency modulated signal with $\Delta\omega/\Omega=5$.

A simple sinusoidal frequency modulation by an amount $\Delta\omega$ at a frequency Ω gives rise to multiple sidebands at frequencies $\omega \pm n\Delta\omega$ with amplitudes proportional to $J_n(\Delta\omega/\Omega$, where J_n is a Bessel function of order n. If $\Omega \ll \omega$ and $\Delta\omega \ll \omega$ as is the case in musical vibrato, then only the carrier frequency ω and the first two sidebands at $\omega \pm \Delta\omega$ are prominent, as shown in Fig. 1(b), so that it may be difficult to distinguish frequency modulation from amplitude modulation simply by examining the power spectrum. At the particular modulation index for which $\Delta\omega = 2.4\Omega$ the component at frequency ω vanishes. If the vibrato is very slow, so that $\Omega \ll \Delta\omega$, then the spectrum spreads over a band of width about $2\Delta\omega$, as shown in Fig. 1(c).

Fourier analysis, it should be noted, does away with the time element entirely, since it deals only with an infinitely long signal (or the same signal endlessly repeated) and yields a frequency spectrum that is time-independent. It is therefore of limited assistance in the study of musical vibrato.

GALERKIN ANALYSIS

Since it is known on general grounds that the sound signal from a musical instrument is based upon a superposition of overtones $a_n(\omega_n)$ at frequencies ω_n that may or may not be in harmonic relation to the fundamental frequency ω_1, it is often more useful to maintain this view and regard the vibrato tone as a superposition of these modes so that

$$p(t) = \Sigma_n\, a_n(t)\cos[\omega_n t + \phi_n(t)]$$

but the amplitudes a_n and phases (n are now relatively slowly varying functions of time. The apparent frequency of mode n is then

$$\omega_n' = \omega_n + d\phi_n/dt.$$

This modal decomposition of the signal, known as the Galerkin approximation, has the great advantage that it yields an 'instantaneous amplitude' and 'instantaneous frequency' that both correspond closely with psychophysical perception, though the terms themselves are not analytically respectable. It is possible to use this approximation to calculate the behaviour of many nonlinear systems of the kind found in musical instruments [3]. The approach gives a readily interpreted picture of the amplitude and frequency of all components of the sound without the complication of sidebands.

One possible problem with this approach is that, if the phase ϕ_n jumps suddenly, then this appears as an infinity in the frequency. An example of this is the case of amplitude modulation or beating with $\Delta a \gg a$. Here the signal has the form $a\sin\omega t\cos\Omega t$ and, if the amplitude a is taken as always positive, then there is a phase jump of π twice in each period, with consequent frequency infinities.

FAST FOURIER TRANSFORM ANALYSIS

While a fast Fourier transform (FFT) is simply a rapid and convenient numerical algorithm for performing a Fourier transform, it differs practically in that this transform is generally performed repetitively on successive small sections of signal and displayed as a time-resolved power spectrum. The frequency resolution $\Delta\omega$ is related to the length Δt of the transformed sample by the condition $\Delta\omega\Delta t \approx 2\pi$, while the Nyquist cut-off frequency is $\omega^* = \pi N/\Delta t$ where N is the number of data points in the transform. Since N is normally fixed by the software used for the computation, the result is a simple trade-off between frequency resolution and time resolution.

If time resolution is sacrificed in favour of frequency resolution so that the sample length is greater than twice the vibrato period, then the FFT approach behaves like the normal Fourier transform and shows a 'carrier frequency' and two sidebands for each mode. If, on the other hand, time resolution is made significantly less than the vibrato period, the FFT will display a set of modes that vary cyclically in frequency and amplitude, following the Galerkin approximation. Because the FFT approach effectively averages the Galerkin approximation over the sample time, if this is short, the possible infinities in frequency are reduced to simply large jumps, but these jumps need to be carefully interpreted.

SONAGRAPH ANALYSIS

The most useful analysis tool derives from the Sonagraph, which in its early forms rotated a sensitive paper on a drum bearing the recorded track to be analysed. The rotation slowly swept an analysing filter through a frequency range from zero to about 5 kHz, and the stylus imprinted the signal level on the paper, giving a time-resolved spectrum of selected bandwidth. Modern signal analysis programs perform the same operation digitally. The figures in the present paper are derived from one such program [4].

HUMAN AUDITORY PERCEPTION

Since the object of this analysis is to relate perceived vibrato effects to physical parameters of the performance, it is important than a method of analysis is chosen that adequately approximates human auditory perception. Numerous psychophysical studies [5] show that human auditory resolution is rather less fine than 50ms and that, while a frequency resolution of about 3 cents, or about 0.2%, is possible near the mid-range of the frequency spectrum though such resolution requires sounds that are steady for several seconds. (One semitone is a change in frequency by a factor $2^{1/12}$ or about 6% and is divided logarithmically into 100 cents.) When the tone duration is 1s or less, the frequency resolution declines rapidly. Similarly, changes in sound level of 1dB are perceptible when they occur at intervals of a second or more, but become progressively less obvious when they occur more rapidly.

These considerations suggest that an method of analysis with a time resolution of about 100ms and a corresponding frequency resolution of about 10Hz, which corresponds to about 2% or 30 cents near the middle of the treble stave (about 400Hz) is probably about optimal for analysing vibrato. An FFT analyser with 1024 data points adjusted to meet these criteria will have an upper cut-off frequency of about 5kHz, which is adequate for the analysis of most musical sounds, though of course the audible components of these sounds extend to much higher frequencies.

3. VARIETIES OF VIBRATO

The most musically and acoustically revealing method of analysis of musical vibrato is an appropriate form of FFT analysis, with the sample length of about 100ms, so that the frequency resolutions is about 10Hz, as discussed above. Applied to a typical musical vibrato, this analysis generally indicates a combination of frequency and amplitude modulation of the sound, which is indeed what the listener hears, though it is possible to concentrate perceptive attention on one or other characteristic. A musical note, however, is not generally a simple sinusoid with a single frequency, but rather consists of a fundamental accompanied by an array of overtones. The effect of vibrato may differ from one overtone to another, so that a third form of vibrato can be identified that might be termed 'timbre' vibrato, where the musical word 'timbre' refers to tone colour.

When considering vibrato, we can identify two basically different classes of musical systems. In the first class, exemplified by plucked or bowed string instruments and by the human voice, it is the frequency of the primary oscillator (the string or the vocal folds) that is varied; associated resonators (the instrument body or the vocal tract) serve simply as shaped filters that modify the spectral envelope of the sound. In the second class, exemplified by woodwind and brass instruments, the primary resonator (the air column) actually determines the frequency of the sound, and what is modified in vibrato is the behaviour of a subsidiary negative-resistance oscillator (the air jet, the reed, or the player's lips) that is slaved to the primary resonator. Frequency deviations are thus much easier to produce in the first class of instruments than in the second, as we see in the examples that follow.

Impulsive stringed instruments

A piano has an inherent amplitude-modulation, though not really a vibrato, for each overtone of the sound by virtue of the fact that most notes are sounded by several strings vibrating in unison. The interaction between the strings is complicated [6] and arises because the bridge is necessarily not completely rigid, since it must transmit the string vibrations to the soundboard. The player, however, has no control over this effect, so it will not be considered further here, despite the fact that it is important to the quality of piano sound.

Something similar happens in the harpsichord and the harp but has a different origin because these instruments have only one string per note (although large harpsichords may have additional strings at octave or sub-octave pitches). Since the string is not generally plucked exactly at right angles to the bridge, it has a tendency to oscillate in an elliptical path, and this ellipse precesses slowly, because of both nonlinear effects and also the direction-dependence of the bridge impedance [2,3]. This precession gives a quasi-periodic amplitude modulation to the normal force on the soundboard. Again the player has no control over this effect, so that it is not a real vibrato.

In a guitar, however, the player uses one finger to 'stop' the string being plucked, and this finger has a position between two of the frets on the neck of the instrument so that the vibrating length is determined by the lower fret position. If, however, the player rocks this finger backwards and forwards, then this has an effect on the tension in the string because of slight variation in the displaced length between the frets. This tension variation in turn varies the vibration frequency of all of the string modes by exactly the same fractional amount, giving a coordinated frequency modulation to the string vibration.

The matter is, however, not as simple as this. The string vibration must be communicated to the instrument body for sound radiation, since the string itself radiates almost no sound because its diameter is so small compared with the sound wavelength involved. The guitar body, however, has many resonances – indeed it is the distribution of these resonances that distinguishes a fine guitar from a poor one. As the frequency of any mode varies under the effect of changing tension, therefore, this alters a little the response of the instrument body as the frequency moves closer to or away from the nearest resonance. There is also an associated change of phase, which adds to the initial frequency modulation. The result is that the simple frequency modulation of the string acquires an amplitude modulation as it is transferred to the body and radiated. When this sound signal is analysed by the FFT method, those parts of the signal with higher amplitude are given higher weight, with the result that there may appear to be a slight shift in the median vibration frequencies of individual modes in addition to the vibrato.

Bowed strings

In a bowed string instrument such as the violin, the string vibration is maintained by a stick-slip frictional phenomenon between the moving bow and the string – hence the importance of rosin to enhance the friction. This stick-slip motion is highly nonlinear, with the result that the vibrational motion of the string repeats regularly, giving a precisely harmonic sound for sustained notes. Vibrato is again introduced by rocking the active finger tip against the fingerboard as in the guitar but, because there are no frets, the result is not a change in tension but rather a change in string vibrating length. Analysis of this situation is very difficult, because it constitutes a 'moving boundary problem' but, because the vibrato frequency is very much lower than the fundamental string vibration frequency (5Hz compared with 200–2000Hz for a violin), it is a reasonable approximation to perform a calculation using a quasi-static approximation. The string frequency is then seen to vary approximately sinusoidally at the finger-motion frequency. The fact that the violin body is intimately involved in sound radiation, and that it possesses pronounced resonances of its own, affects the vibrato in the same way as for the guitar, making the final effect one combining frequency, amplitude, and timbre variations. The maximum frequency variation in vibrato is typically about ±3% or about ±50 cents, as shown in Fig. 2. Note that the vibrato extent, when measured in frequency rather than pitch, increases in proportion to the frequency of the overtone involved, thus maintaining a harmonic relationship to the fundamental at all times.

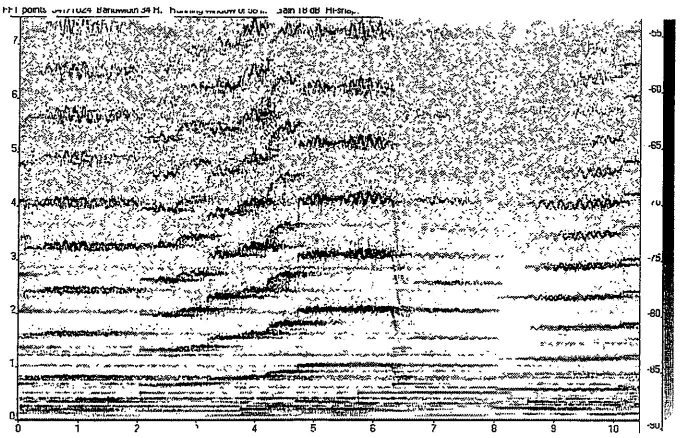

Figure 2. Soundswell analysis of Nigel Kennedy (violin) playing an excerpt from the Meditation from Massenet's Thais. The time span is about 10 seconds and the frequency range 0–7 kHz. Maximum vibrato amplitude is ±3% or about half a semitone in each direction.

Since the violin is a sustained-tone instrument, vibrato is an important feature of its sound quality, and is used almost always. This contrasts with Renaissance bowed-string instruments of the viol family, which have cords tied around the fingerboard to constitute very shallow frets, and are played without any vibrato at all. Adjustment of the frets allows notes to be played consistently in tune, a feat which is much more difficult on the violin.

Violins and other bowed-string instruments are often heard in groups, as in an orchestra, and here the vibrato takes on another role. The string players make no attempt to coordinate their individual vibratos, so that the result is a sound consisting of many superimposed signals with slightly differing frequencies and vibrato rates. When this is considered on the basis of Fourier analysis, the signal is seen to be rather like narrow-band noise. This is called a 'chorus effect' and is particularly pleasant to most listeners.

Flute-like instruments

In instruments of the flute family, a tube resonator with finger holes to adjust its acoustic length is excited by an air jet from the player's lips which blows alternatingly into and out of the instrument mouth-hole. The air jet itself is very complex, and its motion involves the propagation of displacement waves excited upon it by acoustic flow out of the mouth-hole. The interaction of the jet with the sound modes in the tube at the upper lip of the mouth-hole is similarly complex. To sound a given note, the player must control the air-jet length and blowing pressure within fairly narrow limits, or the instrument will either not sound or will sound a higher mode than the one intended.

Vibrato in flute instruments is generally produced by a cyclic variation of about 10% in the blowing pressure. The relative levels of the upper harmonics of the sound depend quite sensitively upon the blowing pressure, while the amplitude and frequency of the fundamental varies by only a very small amount. The result is a vibrato that has been characterised as being a 'timbre vibrato' since there was relatively little change found in either pitch or radiated sound power [7]. Timbre variations do, however, have an effect upon perceived loudness.

A more recent study using FFT techniques [8] has, however, shown periodic variations of about (30 cents in the frequency of the fundamental and rather large and erratic variations in the apparent frequencies of the higher modes, these variations increasing in extent with the mode number. As discussed above, it is possible that these frequency variations are produced by changes in phase, due perhaps to associated variations in the exact blowing angle of the jet in relation to the edge of the mouthpiece [9]. Such phase changes increase in magnitude in proportion to the mode number. The FFT analysis reported in this paper raises some questions about the reality of the frequency fluctuations, however, since the displayed time resolution is about 0.01s and the frequency resolution better than 10Hz rather than the expected 100Hz. The analysis shown in Fig. 3 shows a maximum vibrato shift of about ±25 cents, which confirms the figure given in the referenced publication, but no anomalies are evident in the higher harmonics of the sound.

In the flute, as in other wind instruments that use vibrato, the rate and extent of this vibrato is under the control of the player. Often a sustained note at the beginning of a phrase will start with almost no vibrato, but this will build up in frequency and amplitude during the course of the note and lead on to the next note in the phrase. Conversely, near the end of a phrase this sequence may be reversed. The normal frequency of vibrato, generally in the range 5 to 6Hz, is also often characteristic of the individual player.

Other wind instruments

Reed wind instruments, such as oboes or clarinets, can also produce vibrato, either by oscillation in blowing pressure or, less commonly, by lip pressure on the reed. The vibrato is under the control of the player to the extent that bassoons, for example, may use vibrato when playing duets with oboes but not when playing with clarinets, simply because it is traditional for orchestral clarinets to play without vibrato. There does not appear to have been any detailed acoustic study of this vibrato, but the analysis given in Fig. 4 suggests that the frequency variation is only about ±40 cents and that variations in loudness and timbre may also be important.

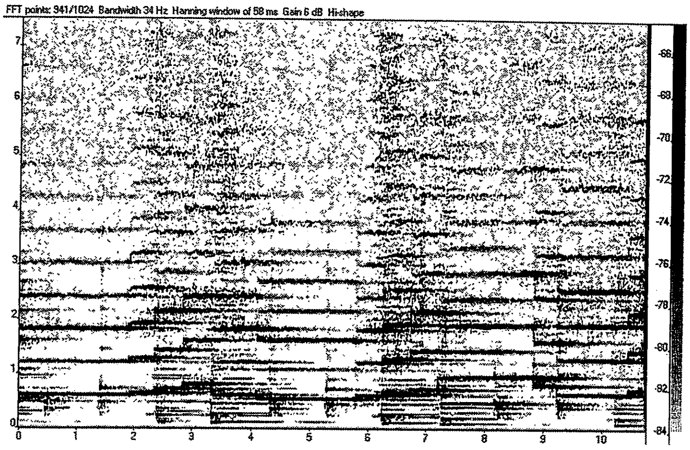

Figure 3. William Bennett (flute) playing part of the Largo from Bach's Concerto for Flute and Strings BWV1056. Maximum vibrato amplitude is ±1.5% or about one-quarter of a semitone. The apparent overlap of notes is due to reverberation.

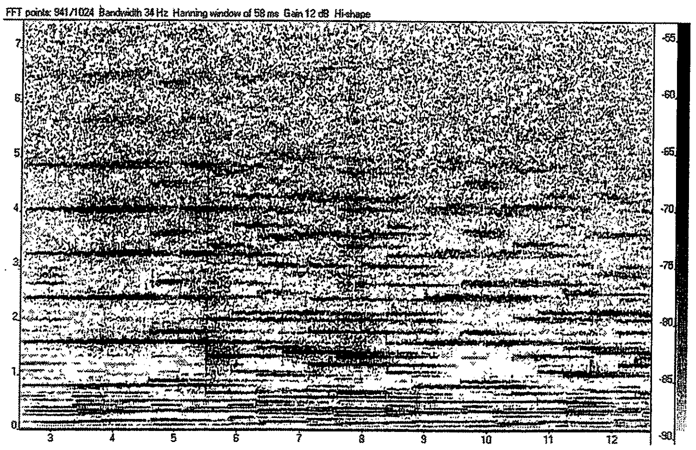

Figure 4. The oboist of the Stuttgart Chamber Orchestra playing the introduction to the Echo Duet of Bach's Christmas Oratorio. Maximum vibrato amplitude is ±2.5% or about 0.4 semitones.

Brass instruments do not use vibrato to any great extent, perhaps partly because of the physical requirements on blowing pressure and lip tension necessary to produce the desired sound and partly because of tradition, which has established that these instruments sound better when played 'straight'.

The singing voice

Vibrato in singing has received a good deal of attention from teachers but less from acousticians. Typically the singing voices of children make no use of vibrato, and this creates the 'pure' or 'simple' sound characteristic of English cathedral choirs. The voices of girls continue to develop smoothly as their age increases, and it is usual for a small amount of vibrato to develop. After the age of 20 or so, the extent of vibrato depends upon artistic choice and physical development. Some professional female singers maintain a voice with very little vibrato for many years, and this style goes very well with the music of composers such as Purcell and with much folk music. Other singers follow a more operatic tradition and use pronounced, and even exaggerated, vibrato in all their singing. After many years of singing in this style, it seems impossible for these singers to revert to simple sounds, and the vibrato intensity generally continues to increase as they grow older. While this is perhaps appropriate in some music with dramatic emotional content, it is felt by many to be an unfortunate defect in singing style. At the other end of the artistic spectrum, singers in some Eastern European traditions eschew vibrato altogether, giving a most striking effect to the music characteristic of that tradition.

Vibrato in male singers, particularly basses, sounds rather different from that of sopranos, probably because the basic sound frequency is typically lower by a factor of nearly four. Certainly, however, some well-known bass singers have developed with age a style with a wide and rapid vibrato, with the result that the pitch of the note being sung is largely obscured in rapidly moving music such as some of that by Handel.

The physiological mechanisms of vibrato generation in singing have been the subject of detailed study [10], but the results vary somewhat from one singer to another. The pitch of a vocal tone is determined almost entirely by tension in the muscles supporting the vocal folds, though this tension is itself influenced to some extent by sub-glottal pressure. The primary origin of vibrato thus lies with the muscles controlling the larynx, though there is evidence of coordinated oscillation in muscle tension in the chest and abdomen, leading to synchronised oscillations in sub-glottal pressure. Because the fundamental frequency of the human voice is not locked to any resonance of the vocal tract, the singer has a great deal of freedom in pitch variation during vibrato.

Quantitative studies of vocal vibrato have been made by several people, and are discussed by Sundberg [10], while a more recent analysis of prominent artists singing Schubert's *Ave Maria* has been reported by Prame [11]. For the quiet mood of the Schubert song, the vibrato rate was 6.0±0.4 Hz and the average vibrato extent 71±9 cents, though this varied from 34 to 123 cents for different notes and different singers. In the wider and more operatic repertoire [10] some well-known sopranos actually use vibrato as large in extent as ±2 semitones! (If the vibrato is larger in extent than this it is called 'trillo'.) For such a large vibrato, particularly if the vibrato rate is rather slow, the perception is of an actual fluctuating pitch, rather than a variation of tone quality on a particular note. For smaller vibratos, however, the pitch perceived by a listener is very close to the average frequency of the sound, so that a wide vibrato does not allow the singer to be significantly out of tune without this fact being evident.

Figure 5 shows a typical example of vocal vibrato for a distinguished soprano (Joan Sutherland) singing a quiet meditative piece of music. Even here the frequency variation is about ±170 cents, or 1.7 semitones in either direction, but the listener senses just the average pitch with quite high precision. Note again that the vibrato extent, when measured in frequency, increases in proportion to the frequency of the overtone involved, thus maintaining a harmonic relationship to the fundamental at all times.

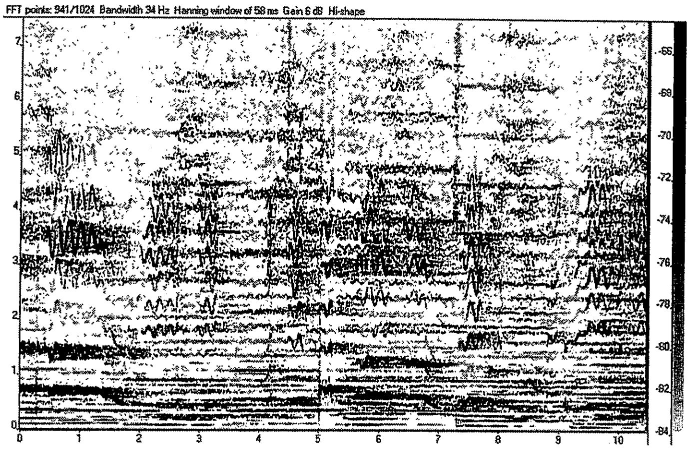

Figure 5. Sound spectrogram of Dame Joan Sutherland singing a tranquil section of Puccini's Suor Angelica. The time span is 10 seconds and the frequency range 0 to 7 kHz. Maximum vibrato amplitude is ±10% or about 1.7 semitones in each direction.

Despite what appears to be the almost autonomous nature of the muscle vibrations responsible for vibrato in singing, the performers do have some measure of control over its amplitude and frequency. The vibrato intensity generally increases with loudness and emotional content of the music, though whether this is conscious or subconscious is not clear. Another level of control is shown in a study of duet singing by pairs of distinguished sopranos, as recently reported by Duncan et al. [12]. They found that in some cases the singers adjusted their singing so that their vibratos were approximately synchronised, sometimes in-phase and sometimes anti-phase.

When, as often happens, mature singers combine to form a choir, their individual vibratos are not synchronised, so that, as for groups of violins, the result is analogous to a narrow-band noise signal. This 'chorus effect' is by no means unpleasant, and indeed adds characteristic beauty to such combined singing. The resulting auditory effect is in sharp contrast to the nearly 'pure-tone' effect produced by groups of boy sopranos in cathedral choirs, where vibrato is not generally used.

4. CONCLUSIONS

Vibrato is an important component of many musical sounds and allows the performer to impose subtle variations upon the quality of notes. It has become so nearly universal, however, that some performances, particularly of early music, gain distinction from the absence of vibrato! In the best performances, the nature and extent of the vibrato are under the close control of the musician and are varied to suit the demands of the item being performed, and indeed help to shape the style of individual phrases within that performance. Unfortunately, many singers appear to develop an uncontrolled and excessive vibrato with increasing age, which detracts from the beauty of their songs.

This brief survey has shown that only some aspects of musical vibrato are understood in detail – there is ample scope for a comprehensive and comparative study. As well as benefiting performers on traditional instruments, a proper understanding can perhaps add life to the otherwise often mechanical sounds of much electronic and computer-generated music.

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