Hyperhelices: A classical analog for strings and hidden dimensions

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(Received 24 June 2003; accepted 9 January 2004)

A hyperhelix is a structure consisting of a rod coiled into a helix, coiled into a helix,..., through a finite or even infinite number of orders. An examination of the transverse vibrations of such a structure shows that the macroscopic behavior is accounted for by waves on the rod that are confined to an extremely small range of wave numbers centered about a value equal to the reciprocal of the smallest helical radius involved. All other dynamical aspects of the behavior and their associated physical dimensions are completely hidden at the level of the final helix. It is suggested that the study of the dynamics of such a structure might provide a fruitful analogy for understanding the string theory.
[DOI: 10.1119/1.1652038]

I. INTRODUCTION

As is well known to those working on the string theory of elementary particles or cosmic structures, many of the possible physical dimensions of the string are hidden because they are coiled up into some sort of small structure concealed within the string. In this paper, we examine the classical behavior of string-like structures with many hidden classical dimensions to provide an analog of quantum string theory.

Classical structures do not have hidden dimensions, but an instructive analog can be found in the behavior of conceptual structures called ''hyperhelices''. A hyperhelix is a helical structure where the coils of the helix are found to also consist of a helix with a much smaller helical diameter though a similar helical pitch angle (see Fig. 1). In turn, the coils of this smaller helix consist of an even smaller helix, and so on. In the discussion to follow, we proceed in the opposite direction where we take a rod, coil it into a helix, coil this helix into a very much larger helix, etc. For convenience, we shall refer to a simple helix as a hyperhelix of order 1, a simple helix coiled into a larger helix as a hyperhelix of order 2, and so on. A simple rod is then a hyperhelix of order zero. If this super-structuring (or sub-structuring) is allowed to continue indefinitely, then the resulting object is fractal, but it is also possible for the structuring to end after a finite number of generations, as is the case with the hidden dimensions of string theory, and the resulting object might be termed quasi-fractal.

The vibrational behavior of hyperhelices has been investigated by the author and colleagues,[1] and the purpose of this note is to discuss the way in which the hidden dimensions influence the behavior of the object as a whole. The hope is that the hyperhelix might be a fruitful analogy by which physicists can convey to non-experts a picture of part of the nature of string theory.

II. WAVE PROPAGATION ON A HELIX

Wave propagation on a simple straight elastic rod is a classical problem in mechanics to which approximate solutions have been known for a long time. There are many treatments in literature of the much more complex problem of wave propagation on a simple helix, because of its practical importance in engineering. A helpful survey has been given by Wittrick.[2]

There are several forms of elastic waves that can propagate on a straight rod. The simplest are longitudinal waves, torsional waves, and flexural or bending waves for which there are two orthogonal polarizations. Each of these waves has a characteristic speed, with the speed of the flexural waves being much smaller than the speed of the other wave types.

If ω is taken to be the angular frequency and c is the phase velocity of a wave of the type considered along the elementary rod from which the helix is formed, then $k=\omega/c=2\pi/\lambda$, where λ is the wavelength measured along the rod. Consider the propagation of bending waves when the rod is bent into a helix of radius R. If $k \ll R^{-1}$ so that the wavelength is much longer than the helical radius, there are two forms of propagating waves, corresponding to the two polarizations of the wave displacement relative to the helical axis. Referring to Fig. 1, if the displacement is perpendicular to the helical axis as represented by T_0, then the result is a ''varicose'' wave, which we define as type A for convenience, in which the radius of the helix expands and contracts. This motion is necessarily closely coupled to a longitudinal motion of much larger displacement amplitude on the helical rod when it is viewed with respect to the helical coordinate system, but the propagation velocity is essentially that of the bending wave. In the second type of wave, defined as type B, the displacement is parallel to the helical axis, as shown by T_0', and the helical loops do not expand. The elastic strain in the elementary rod is then actually torsional, though this is not immediately obvious. The complicated couplings between transverse, longitudinal, and torsional modes have been examined by several authors,[1,3,4] and are clearly important in the engineering dynamics of helical springs.

An interesting aspect of the behavior of type A and B waves emerges when these waves are regarded on the scale of the helix, considered as a straight tube-like element with its helical structure ignored. Type A waves are then seen to be simple torsional waves on this super-structure, while type B waves are longitudinal. No transverse waves of types T_1 and T_1' on the helix as a whole appear to emerge, and this appears to represent a problem since a helical spring can

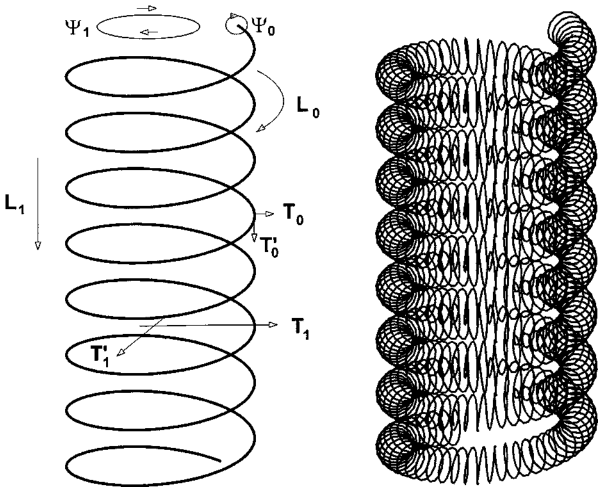

Fig. 1. (a) A simple helix, or hyperhelix of order 1. The torsional, Ψ_0, longitudinal, L_0, and transverse, T_0 and T_0', motions at the level of the elementary curved rod are identified, as are their descriptions as torsional, Ψ_1, longitudinal, L_1, and transverse, T_1 and T_1', modes of the first-order helix. (b) A hyperhelix of order 2. There is a similar group of macroscopic modes Ψ_2, L_2, T_2, and T_2' on this structure, as well as hidden microscopic modes of orders 1 and 0 on the elementary helix and elementary rod (from Ref. 1).

clearly be displaced sideways and can support macroscopic transverse waves. The origin of these macroscopic displacements must be sought in a more careful examination of the elementary waves.

For $k \ll R^{-1}$, the propagation of type A and B waves along the helix is nearly non-dispersive.[2] Dispersion sets in at larger wave numbers, and the phase velocity $c = \omega/k$ falls to zero when $k = R^{-1}$, as shown in Fig. 2(a), which gives dispersion curves plotted in the style of phonon dispersion curves in crystals, though with a normal rather than a reduced wave number. The surprising occurrence of frequency zeros in these curves, confirmed by a study of the progressive bending of a rod,[5] has a simple macroscopic interpretation when the helix is viewed as a simple thick tube or rod and its internal helical structure is ignored. When k is equal to R^{-1}, type A waves, in which there is a displacement normal to the helical axis, have a wavelength equal to the circumference of a single helical turn. This means that opposite halves of the turn are displaced in opposite directions relative to the helical axis, as shown in Fig. 3. This displacement is coupled, in the helical coordinate system, to a longitudinal displacement of the same wavelength which differs in phase by $\pi/2$ along the turn, so that there is no major elastic strain. The result is that the whole turn is simply displaced sideways. Because all turns of the helix are displaced in exactly the same way, the whole helix simply moves sideways. There is no elastic strain and the propagation velocity on the elementary rod is zero.

When k is not exactly equal to R^{-1}, the direction of the displacement precesses with distance along the helix, so that the resultant macroscopic displacement is helical. In this case, there is some elastic strain and the propagation velocity, although small, is no longer zero. The angular sense of the helical wave depends on the sign of $k - R^{-1}$; if $k > R^{-1}$, then the helical turns are all slightly expanded by the wave, while if $k < R^{-1}$ they are all slightly compressed.

Consider now the simultaneous propagation of two such

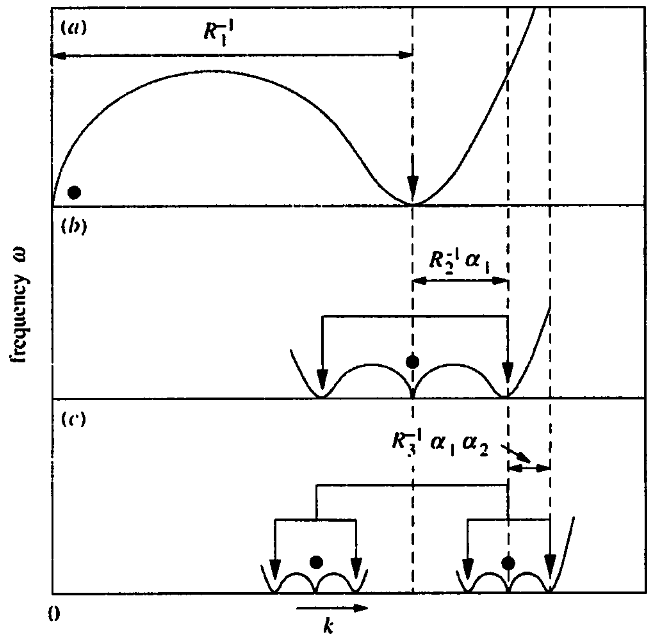

Fig. 2. (a) Dispersion curve for transverse wave propagation on a simple helix. The frequency is plotted as a function of the wave number on the elementary rod from which the helix is made. (b) The related curve for a hyperhelix of order 2, showing the splitting near $k = R_1^{-1}$, again plotted in terms of the wave number on the elementary rod. (c) A similar curve showing further splitting for a hyperhelix of order 3. In each case, the arrows show the minima that are coupled to produce simple transverse waves on the macroscopic hyperhelix, and bullets show the wave numbers for which varicose (torsional) behavior occurs (from Ref. 1). In (b) and (c) the spread along the k axis near the point R_1^{-1} is greatly exaggerated.

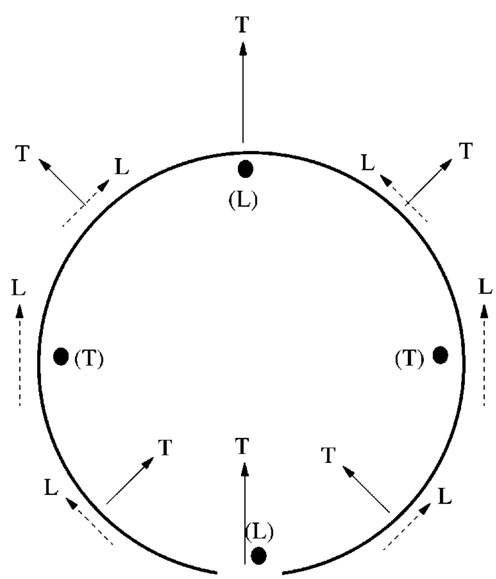

Fig. 3. A single helical turn, showing the displacements due to type A waves when the wavelength is exactly equal to the circumference of the turn ($k = 2\pi/R$ where R is the radius of the turn). Solid arrows show the transverse-wave displacements, T, and dotted arrows show the closely coupled longitudinal-wave displacements, L, which are equal in amplitude but displaced in phase by $\pi/2$. (Black dots show the zeros of the respective waves.) The result is a simple transverse (upward) displacement of the whole turn with no elastic strain being involved. All other turns behave in the same manner, and there is no elastic strain in the whole wave, which therefore has zero frequency.

waves on a helix with helical pitch angle α. When a helical type A wave of wave number $k = R^{-1} + \delta$ is associated with a wave of the same amplitude with wave number $k = R^{-1} - \delta$, the resultant wave is a macroscopic plane transverse wave with wave number δ/α on the helix as a whole. Once more, there are two polarizations of this wave, depending on the signs with which the component helical waves are coupled. There is, however, a small complication. If $\delta \ll R^{-1}$ so that the macroscopic helical waves have a wavelength extending over many helical turns, the propagation speeds of the two helical components of a planar transverse wave of low frequency ω will be very nearly, but not exactly, equal. This means that the polarization plane of the transverse waves on the helix will gradually precess as they propagate. The distance over which this precession becomes appreciable is, however, very large for a finely coiled helix, and so can generally be ignored.

The important conclusion from this discussion is that it is possible to disregard the internal structure of the helix and to regard it as a macroscopic straight rod with appropriately low density and elastic modulus. All of the normal macroscopic transverse wave distortions on this helical rod are then represented by waves of wave number very close to R^{-1} on the invisible elementary rod that makes up the helix. All of the geometrical dimensions that refer to this elementary rod have been hidden, and remain hidden until the frequency of the waves on the macroscopic helix is increased to the extent that the slow precession of the polarization plane of transverse waves as they propagate becomes detectable. A simple physical example is the behavior of the helical cord often used to connect the handset of a telephone to its base. This cord behaves in almost all ways like a simple rope or a rod with minimal stiffness, and its helical sub-structure can be ignored provided waves excited on it have a wavelength much greater than the spacing between the helical turns.

III. WAVE PROPAGATION ON A HYPERHELIX

If a straight rod is regarded as a hyperhelix of order zero and a simple helix as a hyperhelix of order 1, then a hyperhelix of order 2 can be formed by coiling an ordinary helix of radius R_1 into a much larger helix of radius R_2. Because, as noted, the ordinary helix actually behaves like a simple rod carrying waves with wave number $k_1 = k - R_1^{-1}$ for $k \approx R_1^{-1}$, the hyperhelix will behave like an ordinary rod when k_1 is close to R_2^{-1}. Because $R_2 \gg R_1$, the macroscopic behavior of the second-order hyperhelix is therefore contained within the two coupled regions very close to $R_1^{-1} + R_2^{-1}$ and $R_1^{-1} - R_2^{-1}$, respectively, as shown in Fig. 2(b). Within these two neighborhoods, all of the complexities of the wave motions on hyperhelices of orders 0 and 1 are hidden.

This progression, with the size of the helix increasing by several orders of magnitude at each order, can be carried through an arbitrary number of orders, each order effectively hiding one dimension of the behavior and reducing further the region of elementary wave number space involved on the initial rod. The situation is, however, more complex than it initially appears, because the relevant region of wave number space splits into two at each hyperhelix stage, so that the final dispersion curve is a quasi-fractal object, terminating after a finite number of stages if the order of the final hyperhelix is also finite. This is illustrated in the dispersion curves for hyperhelices of order 1, 2, and 3 shown in Fig. 2. For a hyperhelix of order N, the macroscopic wave behavior can be shown[1] to be confined to the wave number range

$$R_1^{-1} \pm \sum_{n=2}^{N} R_n^{-1} \prod_{m=1}^{N-1} \alpha_m , \tag{1}$$

on the elementary rod, where the subscripts refer to the hyperhelical orders. On the hyperhelix of order N, the physical dimensions are so large that the macroscopic wave numbers k_N are confined to a very small region near $k_n = 0$.

For the simple case in which the pitch angles α_n and size ratios R_n/R_{n+1} are taken to be constant and the sequence is continued so that $N \to \infty$, the Hausdorff (or capacity) dimension of the resulting fractal structure is[6]

$$D = \frac{\log(\alpha R_n/R_{n+1})}{\log(R_n/R_{n+1})}. \tag{2}$$

Because geometrical considerations require that $R_n/\pi R_{n+1} < \alpha < 1$, the dimension D lies in the range $1 < D < 2$.

This discussion has modeled the hyperhelix as being successively coiled into larger and larger helical structures, so that one has to take a successively more "global" view of its behavior. This is the reason that the wave number range on the elementary rod that describes this behavior is progressively more and more limited. It would, of course, be possible to proceed in the opposite direction: To regard the final hyperhelix of order N as an ordinary macroscopic object and then to take a progressively more highly magnified view of its structure. If this were done, then the macroscopic waves of wave number k_N would be seen to be made up from waves on the underlying hyperhelix of order $N-1$ and having wave numbers k_{N-1} lying very close to $(R_N/R_{N-1})k_N$. This progression would then continue as further underlying hyperhelical orders were examined. Although this interpretation is, perhaps, the more logical way in which to approach hidden dimensions, it has not been adopted here because of the greater difficulty in describing the progression of wave types. The dispersion curves would still have the form shown in Fig. 2 except that the scale would change immensely in going from Figs. 2(a) to 2(b) to 2(c).

IV. CONCLUSIONS

Although the relevance to string theory is not readily apparent, quasi-fractal hyperhelices clearly have the essential ability to hide dimensions and mode details within an apparently simple structure. Perhaps the richness of these possibilities may provide some analogies from which string theory, or other areas of theoretical physics, might benefit.

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