Nonlinearity and Chaos in Acoustics

Neville Fletcher*, Robert Perrin† and Katherine Legge‡
Acoustics and Vibration Centre
Australian Defence Force Academy
Campbell, ACT 2601

ABSTRACT: A brief survey is given of the nature of nonlinearity and the transition to chaotic behaviour in vibrating systems of interest in acoustics. Chaotic behaviour is illustrated by considering the response of a circular plate or thin axisymmetric shell excited sinusoidally at its centre. Chaos sets in at an unexpectedly small amplitude and leads to large excitation of non-driven modes. Some practical implications are considered.

1. INTRODUCTION

If there is one development in basic understanding that, over the past ten years, has had greater impact than any other on the way we look at physical phenomena, then it is the theory of nonlinear and chaotic phenomena. Along with its associates, catastrophe theory and fractal geometry, it is one of the most exciting areas of research today in a whole range of classical areas of study such as mathematics, mechanics, acoustics and fluid dynamics, and it is beginning to penetrate into the world of quantum phenomena. There was a special section on acoustic chaos at the 13th ICA in 1989, a major conference on more general aspects of chaos was held in Sydney early this year, and we even read about the subject in the weekend newspapers!

It is not possible, in an article as short as this, to give any extensive discussion of either chaos theory or its implications. What we have tried to do, therefore, is to give an outline of the basic background, and to illustrate it with some examples from our own experience of the chaotic behaviour of a vibrating panel. Not only is this potentially simple enough to understand in detail — though we are as yet a good way from such understanding — but it has important applications in the real world of acoustics and vibrations. For those who wish to delve deeper, we recommend the popular non-technical book by Gleick [1] and the extensive set of more technical papers edited by Cvitanovic [2]. There have also been numerous articles on chaos and fractal geometry in the pages of Scientific American.

2. NONLINEARITY

A physical system is linear if its response amplitude is proportional to the stimulus amplitude, all other things being kept constant. A simple linear spring (extension proportional to applied force) is a familiar example, but linearity is assumed in mechanics (force $\propto$ acceleration) and in electric phenomena (Ohm's law). Predictions of system behaviour based on such linear assumptions generally work well provided we do not depart too far from equilibrium, but for extreme cases a linear theory is inadequate — springs unwind, beams buckle, resistors get hot.

In acoustics and ordinary vibration applications we are generally in a domain where linear theory is adequate, though there are exceptions for such things as the sound production mechanism of musical instruments [3]. Nonlinearity is more noticeable when the pressure amplitude of a sound wave becomes appreciable in comparison with normal atmospheric pressure, say greater than 10 kPa or about 174 dB, as in an explosion or a lightning flash or the passage of an aircraft at supersonic speed. Then the temperature of the air in the pressure crests rises significantly relative to that in the troughs, which falls similarly. Because sound travels more quickly at higher temperatures, this leads to distortion of the pressure wave to an N-shaped shock wave.

The nonlinearity with which we shall be concerned here, however, is of a much less extreme variety, and concerns only the gradual stiffening of various types of springs as their deflection is increased. This is illustrated in Figure 1. This sort of behaviour is found in many ordinary springs, and also in the sideways deflection of plates, in which a tension force builds up to assist the stiffness arising from simple bending. If f is the distorting force and x the spring deflection, then this type of behaviour can be written

$$f = ax + bx^3 \tag{1}$$

where a is the normal spring stiffness and b/a measures the severity of the nonlinearity. There are, of course, many more complex forms of nonlinearity than that shown in equation (1); the deflection of a slightly dished plate, for example, requires the addition of a term in x^2.

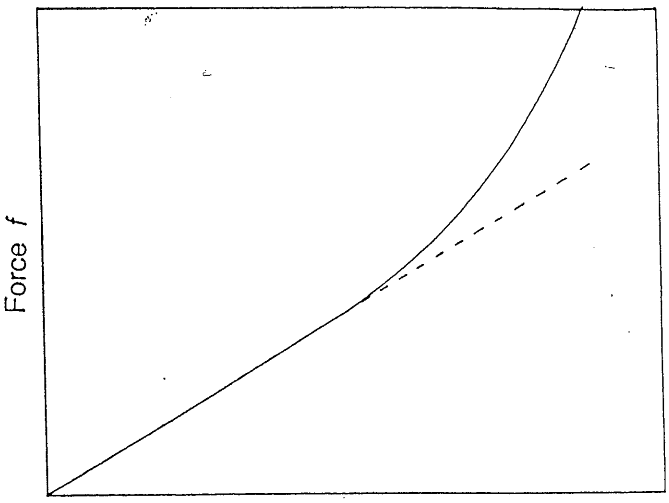

Figure 1: Behaviour of a stiffening spring, as in equation (1).

* *Permanent address:* CSIRO Australia, Research School of Physical Sciences, Australian National University, Canberra 2601
† *Permanent address:* Department of Physics, Loughborough University of Technology, Loughborough, Leicestershire, England
‡ *Permanent address:* Department of Physics, Bendigo College of Advanced Education, Bendigo, Victoria 3550

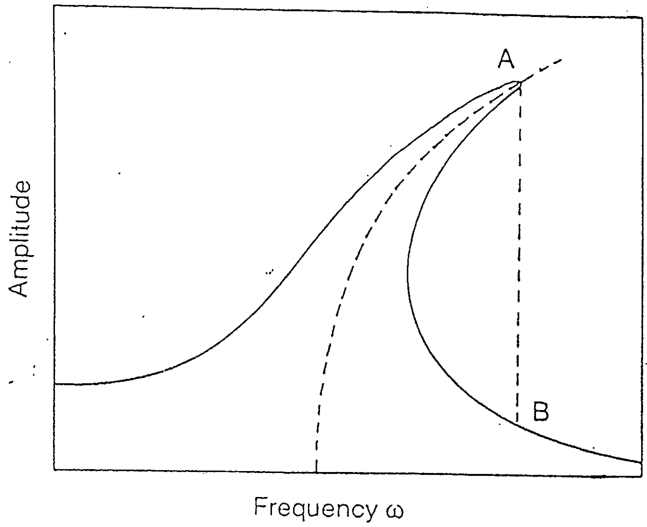

Figure 2: Nonlinear resonance curve, as described by equation (3) with $\alpha = 0$.

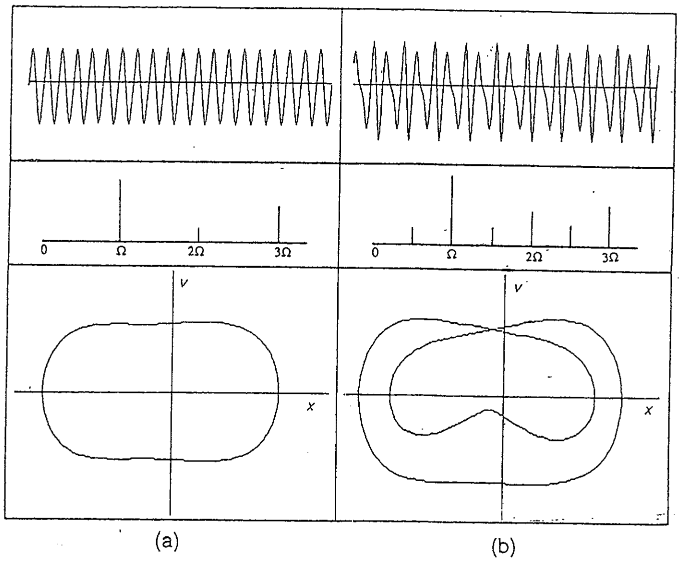

Figure 3: (a) Waveform, spectrum and orbit in phase space for a nearly sinusoidal solution of equation (3) corresponding to a slightly dished plate with $k = 0.02$, $\alpha = 0.3$, $\beta = 0.1$, $\Omega = 2$, $F = 12$; (b) bifurcation and generation of a subharmonic of order 2 for the same system with $F = 20$.

If we think of the motion of a simple loaded spring with stiffness given by (1), acted on by a sinusoidal force f sin ωt, then this motion is described by the equation

$$m\ddot{x} + r\dot{x} + ax + bx^3 = f \sin \omega t \qquad (2)$$

where m is the loading mass and r is the viscous damping. Dots signify differentiation with respect to time, so that $\dot{x}$ is velocity and $\ddot{x}$ is acceleration. If we plot the amplitude response of this system as the frequency ω is varied, then we get the distorted resonance curve shown in Figure 2, the small-amplitude resonance frequency being $\omega_0 = (a/m)^{1/2}$. The amount of distortion is proportional to the nonlinearity b/a. It is convenient to simplify equation (2) by dividing by m and changing the unit of time to $\tau \equiv \omega_0 t$ so that it can be written

$$\ddot{x} + k\dot{x} + x + \alpha x^2 + \beta x^3 = F \sin \Omega\tau \qquad (3)$$

where $k = r/m\omega_0$, $\beta = b/a$, $F = f/a$, and $\Omega = \omega/\omega_0$ is the ratio of the driving frequency to the small-amplitude resonance frequency. A quadratic term αx^2 has been added for generality. The parameter k is called the damping coefficient and is the reciprocal of the quality factor Q. Equation (3) is closely related to the Duffing equation, which has both the linear and quadratic terms omitted, so that the restoring force is simply βx^3. The Duffing equation is nonlinear at all amplitudes and has been extensively studied.

We can hear the effects of this nonlinearity quite easily with a rather loose metal string on a musical instrument such as a guitar. If we pluck the string to large amplitude, rather than exciting it with a sinusoidal force, then its oscillation decays along the spine of the curve, shown as a broken line in Figure 2, and the sound dies away with a twang as the pitch falls. Experiments with sinusoidal excitation of such a metal string show that we can have a sudden fall in amplitude from point A to point B if we slowly increase the frequency ω while keeping the force f constant. This is an elementary example of a catastrophe — a large change in some physical result (the amplitude) for a very small change in the excitation (the frequency in this case) near a critical point A. Catastrophe theory deals with more general features of this sort of behaviour.

3. ORBITS AND ATTRACTORS

We are used to looking at oscillatory phenomena in two complementary ways — either we examine the waveform with an oscilloscope, or we look at the frequency spectrum using, for example, an FFT (Fast Fourier Transform) analyser. These two approaches, provided we record the phases of the spectral components, give us exactly the same information, and one representation can be derived from the other mathematically.

For discussions of chaotic behaviour, it turns out that a rather different representation is also useful. The time behaviour of a vibrating system can be described by giving the value of its displacement x at all times t, but it can also be described if we know the displacement x and the velocity $v = \dot{x}$, which is just the slope of the x(t) waveform, at every point. We can then describe the behaviour by plotting the motion of the point representing the system on a graph in which the axes are x and v. If the waveform is repetitive, then the curve in (x,v) space, which is called phase space, is a closed orbit which repeats itself in every cycle of the motion. This is illustrated in Figure 3a for a nearly sinusoidal wave, such as would arise from solution of a particular case of equation (3) at rather small amplitude. If the wave were exactly sinusoidal then the curve would be an ellipse. There is actually a value of the time parameter t or τ associated with every point on the orbit, and we will need to know this later.

A simple repetitive wave represents a steady state but, if we apply a sinusoidal force to a system, it takes an appreciable time to settle down. This approach to a steady state can be represented in phase space, as well as in the (x,t) time domain seen on an oscilloscope. Figure 4a shows what happens for an arbitrary starting condition — the initial orbit can begin anywhere in phase space, but it is "attracted" towards the final stable orbit and eventually coincides with it. The stable orbit is then called an attractor for this particular motion. If the exciting force is zero, then the attractor is simply the point $x = 0$, $v = 0$, as in Figure 4b.

To further simplify the presentation it is useful to employ a device introduced by the French mathematician Poincaré, and hence called a Poincaré section. The easiest way to think of this is to recognise that we are dealing with a system driven by a regular sinusoidal force, according to equation (3). The time scale is thus fixed by this external force, and we can imagine taking a flash photograph of the phase space just once in each cycle, at a fixed phase of the external force, and plotting the position of the point representing the system response. Once we are on the stable orbit, this always shows up as a single point on the section, while the behaviour of the system in approaching the steady state shows up as a sequence of points steadily approaching this limit point.

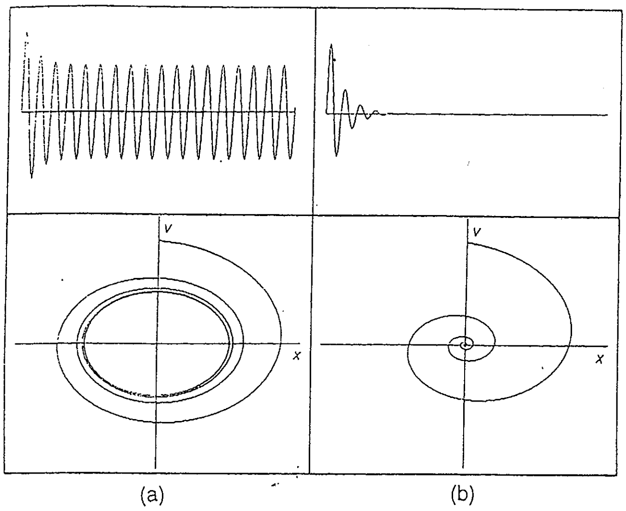

Figure 4: (a) Approach to the steady state, showing a periodic attractor orbit; (b) the same, showing a point attractor.

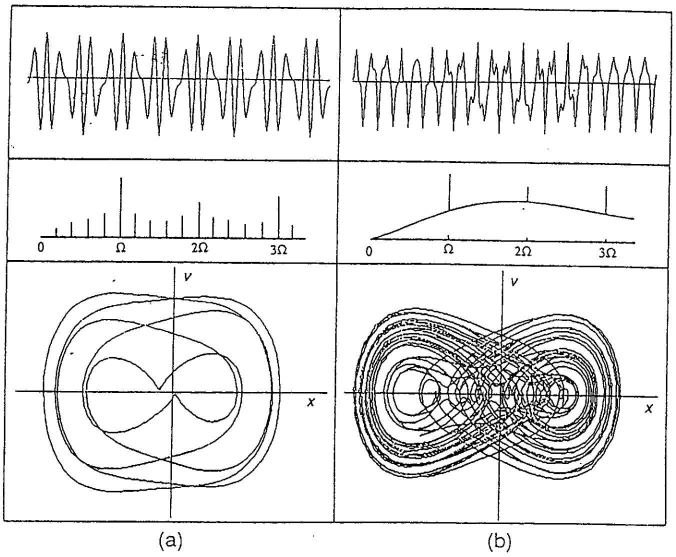

Figure 5: (a) Subharmonic splitting of order 5 for the system of Figure 3(a) with F = 12; (b) chaotic behaviour for the same system with Ω = 1, F = 23.

4. BIFURCATIONS AND STRANGE ATTRACTORS

All this is quite straightforward and introduces nothing unexpected. It is possible to calculate the approach of the system to its attractor by simply integrating the equation (3) from its given starting conditions. With a modern desk-top microcomputer this takes only a few seconds. However, playing around with such calculations soon turns up some very strange behaviour. Actually it was found first for even simpler equations, but the generalised Duffing equation (3) is most suited to our discussion here. The first thing to be discovered is that, for particular values of the relative frequency Ω and force amplitude F, the orbit doubles, or bifurcates. This shows up as a period doubling on the waveform display, a subharmonic of order 2 on the frequency spectrum, or a double orbit in phase space, as illustrated in Figure 3b. This phenomenon appears on the Poincaré section as two point attractors, which we have not bothered to illustrate.

Even this bifurcation behaviour is easy to accommodate among our usual ideas — it is simply the nonlinear driving of the mode at half the driving frequency, and occurs most easily when the driving frequency is about twice the free mode frequency. The other components in the spectrum in Figure 3b then simply arise as nonlinear distortion products. Rather surprisingly, however, an increase in the force amplitude or a decrease in the damping sometimes leads to further bifurcations, giving subharmonics of order 4, 8, and so on. Feigenbaum [2] has shown that this behaviour is governed by universal rules. For the particular equation we are studying, however, this does not appear to happen; if the force is increased outside a small range, then the system reverts to simple periodic behaviour. However, for other small ranges of frequency and force we find more complex behaviour such as 3rd or 5th order subharmonics. The fifth order case is illustrated in Figure 5a.

Further computer integration of the equations, however, shows up an entirely different and unexpected behaviour. For larger values of the driving force, the orbit simply never repeats! The orbits scribble over a large region of phase space when they are drawn in full as in Figure 5b, and the spectrum shows a large amount of wideband noise, with superposed peaks at the driving frequency and some of its harmonics or subharmonics. This behaviour is called chaotic — but it is

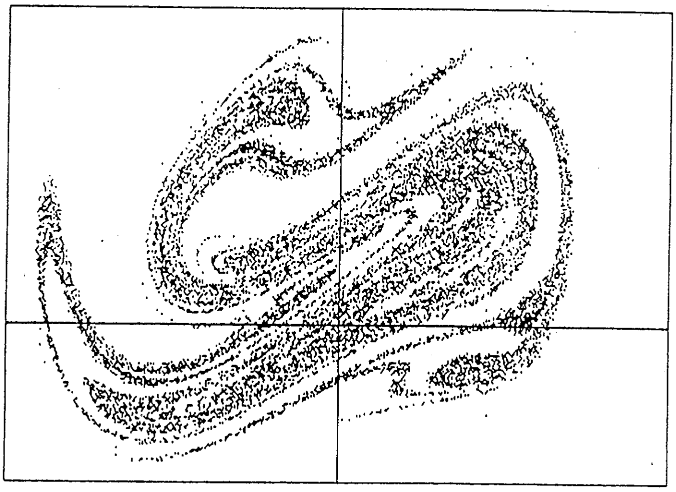

Figure 6: The strange attractor generated by the system of Figure 5(b).

deterministic chaos, in that it results from exact integration of the equation (3), and we get exactly the same result every time.

The beauty and unexpected structure of chaos appears when we examine the behaviour in the Poincaré section plane, plotting one point per orbit at a defined phase of the external force. After the initial transient has died down, the points on the section plane are not simply randomly placed, but all lie upon a complicated swirling figure of the type shown in Figure 6. It is clearly some sort of more complicated attractor for the chaotic motion and, with good reason, it is called a strange attractor. Its form is characteristic of the parameter values in the equation representing the physical system, together with the values of the external force amplitude and frequency. Transition to chaotic behaviour is again a catastrophic change — the system goes from a simple attractor to a chaotic attractor for a very small change in parameter values. In some cases there is a progression through sudden bifurcations of progressively higher order, as mentioned above.

If we watch the points, one per orbit, building up on the attractor, then we note that their placing is apparently random — though it is deterministically random in that the orbits can be calculated exactly. The essence of chaos, however, is that the exact sequence of orbits depends with the utmost sensitivity upon the initial conditions. For an ordinary attractor, two orbits or representative points that start off very close together remain close together, and indeed slowly approach one another. In a chaotic system, however, the points diverge exponentially, at least for a start, so that very soon their subsequent motion is quite uncorrelated. This behaviour in phase space and in its Poincaré section reflects what is occurring in real physical space — detailed behaviour is very sensitively dependent on initial conditions.

Examination of the geometry of strange attractors shows that they are much more complex objects even than they appear at first glance. The structure, indeed, remains equally complex if they are examined at higher and higher magnification — they are self-similar or fractal objects. Only a few attractors generated from differential equations have been examined in detail, but there is a wealth of beautiful pictorial information available on fractal objects, such as the Mandelbrot set, generated from simpler nonlinear algebraic equations [1,4,5].

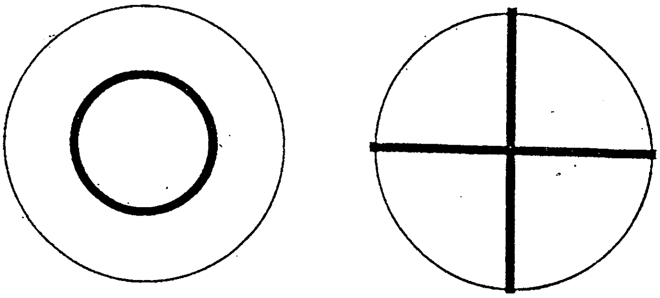

Figure 7: (0,1) and (2,0) modes of a freely suspended disc.

5. PHYSICAL EXAMPLES

Very many experimental studies of the occurrence of chaotic behaviour have been made for appropriate nonlinear systems. Many of the most convenient use electrical resonant circuits with nonlinear inductive elements, since these are easy to measure and are appropriately one dimensional, in the sense that the charge on the capacitor can be taken as the physical variable x, and the quantity $v = \dot{x}$ is then the current through the inductive element. More complex examples with larger numbers of variables abound.

Our experiments have concerned the vibration of a freely suspended metal plate, excited sinusoidally at its centre. The stiffness of the plate provides the linear part ax of the restoring force in (1), and the tension forces, which vary as the square of the amplitude and have a normal component additionally proportional to amplitude, provide the cubic restoring force term bx^3. The plate itself is an extended system and has an infinite number of normal vibration modes, but we can make an approximate separation of the motion so that each mode is described by a nonlinear equation of the form (3), with different value of the mode frequency ω_n, and with extra nonlinear terms linking the modes together. The mathematics is thus rather complicated and has not yet been explored in detail.

In the experiments, the plate was cut from steel sheet about 1 mm thick and had a diameter of about 40 cm. It was held vertically by light strings passing through holes near its edge, and was excited with a small B&K shaker attached to its centre. The displacement at any point could be measured with a B&K capacitive transducer — essentially the electrode of a condenser microphone with the plate forming the diaphragm — and the velocity by integrating the signal from a B&K subminiature accelerometer attached to the surface. Actually one could simply integrate this accelerometer signal once more to find the displacement, but a direct method has some advantages.

Exploration of the ordinary linear vibration modes showed that the two of lowest frequency were the (2,0) and (0,1) modes illustrated in Figure 7, the first number in the description giving the number of nodal diameters and the second the number of nodal circles. The (2,0) mode had a frequency of about 39 Hz and a Q value of 850 ($k \approx 0.001$) while the (0,1) mode had frequency 69 Hz and Q = 330 ($k \approx 0.003$). The (2,0) mode is actually a degenerate pair with the same frequency, the nodal lines of one being rotated by 45° relative to the other. The linear behaviour of these modes was quite unremarkable. The (0,1) mode was efficiently driven at the centre of the plate, but the (2,0) modes were nearly inactive, because their nodal lines cross there.

Interesting behaviour was found when the frequency was set at about 75 Hz, near to that of the (0,1) mode, and the driving force was increased. Quite suddenly, for an (0,1) amplitude of only a few tenths of a millimetre at the disc centre, the (2,0) mode became active at a frequency exactly half the driving frequency and reached an amplitude of about 1 mm at the disc edge. The orbit, as measured some distance from the disc centre, bifurcated, and a subharmonic of order 2 appeared on the FFT analyser. At a rather increased level of drive, giving an (0,1) mode amplitude of about 0.5 mm at the centre, the whole vibration became wildly chaotic in both modes, and the vibration amplitude at the disc edge exceeded 2 mm. Fortunately the low frequency and the small size of the disc meant that the radiated sound intensity was small! For other combinations of force and frequency near these values, subharmonics of other orders were observed, while if the shaker amplitude was increased much above that necessary for chaotic behaviour, the response again became simple.

One might be tempted to simply take this as a nice illustration of the general behaviour discussed above, except for one feature. This is that, while numerical integration of equation (3) leads one to expect a transition to chaos at an amplitude such that the nonlinear terms comfortably exceed the linear term ($\beta x^3 > 1$), the experimentally observed transition occurs for an amplitude nearly 10 times smaller, so that $\beta x^3 \sim 0.001$. The reason for this extreme sensitivity is not clear, but seems likely to be associated with the existence of two (or more) nonlinear modes, and the particular nature of the nonlinear coupling between them. It does not appear to be accounted for by the smaller damping of the experimental system. We discuss the significance of this behaviour in the final section.

Very similar behaviour was found for the case of an orchestral cymbal, which is essentially a shallow spherically-dished shell about 40 cm in diameter, and for a large Turkish gong, again a dished shell 50 cm in diameter and surrounded by a stiff conical flange [6]. The curvature of the shell adds a quadratic term αx^2 to equation (3). The conical flange on the gong reverses the frequency order of the two low-frequency modes of the gong and adds a nodal circle to (2,0), so that these modes are (0,1) at 96 Hz and (2,1) at about 180 Hz. There is also a mode (1,1) at 136 Hz, and many modes of higher frequency. The cymbal modes were not investigated in detail but, because of the high curvature of the shell, the (0,1) mode frequency was about 600 Hz.

For both these systems the behaviour when excited sinusoidally at the centre at a frequency close to that of the (0,1) mode was very similar to that of the simple plate. Subharmonics of various orders, particularly 2, 3 and 5, were observed, and the onset amplitude for chaotic behaviour was again of order 1 mm. The main difference was that, because of the higher frequencies involved, and the flange effectively baffling the gong, the radiated sound intensity was large, almost painfully so in the case of the cymbal! It was also noticeable that the timbre of the sound in the chaotic regime was very similar to that produced when the gong or cymbal was simply struck a heavy blow, as in normal playing.

6. CONCLUSION

It would be a mistake to regard nonlinear and chaotic behaviour as simply an interesting curiosity, for it has both profound basic significance and important practical consequences. The proliferation of current research literature attests to the former fact, and it is appropriate here to comment only briefly on the latter.

The behaviour of musical instruments such as gongs and cymbals is important to musicians, but is hardly seen as being significant in the larger world. It is often in musical instruments, however, that acoustic phenomena are most clearly exhibited, and for this reason their study can give valuable pointers in more practical fields. In this connection, it is perhaps the observation of chaotic behaviour and nonlinear mode coupling at force amplitudes several orders of magnitude smaller than expected from consideration of a simple Duffing equation that is most significant. Once chaotic behaviour has been initiated, the system then displays large vibration amplitudes in modes that might have been expected to be quiescent.

The most direct application of these ideas is to the vibration of panels, not necessarily of circular shape, under the influence of periodic exciting forces, generated for example by reciprocating machinery. If conditions are such that the response becomes chaotic, then panel amplitudes may greatly exceed those normally expected and may be in unexpected modes, leading to unpredicted and perhaps dangerous stresses on the structure. The same thing may apply to the flutter of panels under aerodynamic forces, where the initial vibration is to a large extent self-excited, rather than provided by an external force. Even the case of a plate or shell may be an unduly restricted model, for similar behaviour might well be expected of any extended system with multiple modes and appropriate nonlinearity. Certainly chaotic behaviour is a subject of which we will hear a great deal more in the future.

REFERENCES

1 J. Gleick: Chaos — Making a New Science, London: Cardinal (1988).

2 P. Cvitanovic (editor): Universality in Chaos, Bristol: Adam Hilger (1984).

3 N.H. Fletcher: Nonlinearities in musical acoustics. Acoustics Australia *14* 71-74 (1986).

4 B.B. Mandelbrot: The Fractal Geometry of Nature, New York: Freeman (1982).

5 H.O. Peitgen and P.H. Richter: The Beauty of Fractals, New York: Springer-Verlag (1986).

6 K.A. Legge and N.H. Fletcher: Nonlinearity, chaos and the sound of shallow gongs, J. Acoust. Soc. Am. *86* 2439-43 (1989).